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INVARIANTS UNDER STABLE EQUIVALENCES OF MORITA TYPE*

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Abstract The aim of this article is to study some invariants of associative algebras under stable equivalences of Morita type. First of all, we show that, if two finite-dimensional self-injective k-algebras are stably equivalent of Morita type, then their orbit algebras are isomorphic. Secondly, it is verified that the quasitilted property of an algebra is invariant under stable equivalences of Morita type. As an application of this result, it is obtained that if an algebra is of finite representation type, then its tilted property is invariant under stable equivalences of Morita type; the other application to partial tilting modules is given in Section 4. Finally, we prove that when two finite-dimensional k-algebras are stably equivalent of Morita type, their repetitive algebras are also stably equivalent of Morita type under certain conditions.

Key words Orbit algebra; repetitive algebra; stable equivalence of Morita type; quasitilted algebra

2000 MR Subject Classification 16G10; 16D90; 16E30

1 Introduction

Throughout this article, we always assume that k is a fixed field. In the representation theory of finite groups, the stable equivalence of Morita type introduced by Broné is of considerable importance due to its connection with the celebrated conjecture of Broné. It arises naturally for self-injective algebras, as was shown by a result of Rickard: if two self-injective algebras are derived equivalent, then they are stably equivalent of Morita type (see [16, Corollary 5.5]). Typical examples of stable equivalences of Morita type occur frequently in the block theory of finite groups. But it was shown that stable equivalences of Morita type are also of particular interest for general algebras. For example, they preserve many interesting properties of algebras, such as Hochschild (co) homology group [18], representation type [6], and Linckelmann's Theorem [8]. Therefore, it is an interesting work to get more new invariants. In [13,

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Theorem 1.1], Pogorzaly proved that, if two finite-dimensional self-injective k-algebras A, B are stably equivalent of Morita type, then their Hochschild cohomology algebras $\mathbb{A}(\Omega_{A^e}; A)$ and $\mathbb{A}(\Omega_{B^e}; B)$ are isomorphic. Inspired by this result, we are trying to set up a similar result for stable equivalences of Morita type between two self-injective k-algebras. Moreover, we prove that the quasitilted property of an algebra is also invariant under stable equivalences of Morita type.

In the previous references, all known examples of stable equivalences of Morita type are from the class of self-injective algebras. Recently, Liu and Xi offered some methods for constructing stable equivalences of Morita type from some finite-dimensional k-algebras and obtained some interesting results [9–11]. In this article, we will give a new method to construct stable equivalences of Morita type from two finite-dimensional k-algebras.

The main results of this article are as follows.

Theorem 1.1 Let A and B be two finite-dimensional self-injective k-algebras which are stably equivalent of Morita type induced by bimodules ${}_AM_B$ and ${}_BN_A$. Let ${}_AX_A$ and ${}_BY_B$ be bimodules, such that $N \otimes_A X \otimes_A M \cong Y$, where X is a left-right projective A-bimodule. Then, the algebras $\mathbb{A}(\nu_{A^e}(X); X)$ and $\mathbb{A}(\nu_{A^e}(Y); Y)$ are isomorphic.

The definition of $\mathbb{A}(\nu_{A^e}(X); X)$ will be given in Section 2.

For Morita equivalence as a special case of stable equivalences of Morita type, the above result is obvious.

Theorem 1.2 Let A and B be two finite-dimensional k-algebras without semisimple blocks, such that $A/\mathrm{rad}A$ and $B/\mathrm{rad}B$ are separable over k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. Then, A is a quasitilted algebra if and only if B is a quasitilted algebra.

Theorem 1.3 Let A and B be two finite-dimensional k-algebras which are stably equivalent of Morita type induced by bimodules ${}_AM_B$ and ${}_BN_A$. Assume that an A-bimodule I and a B-bimodule J satisfy the following conditions:

(1) $M \otimes_B N \otimes_A I \simeq I$ and $N \otimes_A M \otimes_B J \simeq J$ as bimodules,

(2) There is a B-A-bimodule isomorphism $\psi : J \otimes_B N \simeq N \otimes_A I$ and an A-B-bimodule isomorphism $\varphi : I \otimes_A M \simeq M \otimes_B J$, such that the following diagrams are commutative:

$$\begin{array}{ccc}
I\otimes_A M\otimes_B N & \xrightarrow{1_I\otimes\rho_1} & I\otimes_A A \\
\downarrow{\scriptstyle \varphi\otimes 1_N} & & \\
M\otimes_B J\otimes_B N & & \downarrow{\scriptstyle \mu} \\
\downarrow{\scriptstyle (\rho_1\otimes 1_I)(1_M\otimes\psi)} & & \\
A\otimes_A I & \xrightarrow{\mu'} & I
\end{array}$$

$$\begin{array}{ccc}
J\otimes_B N\otimes_A M & \xrightarrow{1_J\otimes\sigma_1} & J\otimes_B B \\
\downarrow{\scriptstyle \psi\otimes 1_M} & & \\
N\otimes_A I\otimes_A M & & \downarrow{\scriptstyle \tau} \\
\downarrow{\scriptstyle (\sigma_1\otimes 1_J)(1_N\otimes\varphi)} & & \\
B\otimes_B J & \xrightarrow{\tau'} & J
\end{array}$$

where μ, μ', τ, τ' are the corresponding actions of modules, $(\rho_1, \rho_2) : M \otimes_B N \longrightarrow A \oplus P$ and $(\sigma_1, \sigma_2) : N \otimes_A M \longrightarrow B \oplus Q$ define a stable equivalence of Morita type between A and B. Then, there is a stable equivalence of Morita type between the repetitive algebra of A by I and that of B by J.

This article is organized as follows. In Section 2, some basic facts and definitions are recalled. In Section 3, we show that if two finite-dimensional self-injective k-algebras are stably equivalent of Morita type, then their orbit algebras are isomorphic. Moreover, in Section 4 it is verified that the quasitilted property of an algebra is invariant under stable equivalences of Morita type. As an application of this result, it is obtained that if an algebra is of finite representation type, then its tilted property is invariant under stable equivalences of Morita type; the other application to partial tilting modules is given. Finally, it is proved that, when two finite-dimensional k-algebras are stably equivalent of Morita type, their repetitive algebras are also stably equivalent of Morita type under certain conditions.

2 Preliminaries

In this section, we shall fix some notations and recall the definitions and basic facts needed in this article.

Throughout this article, all categories are k-categories and all functors are k-functors. For a finite-dimensional k-algebra A, denote by mod$A$ the category consisting of finite-dimensional left A-modules. The stable category $\underline{\text{mod}}A$ of mod$A$ is the quotient category mod$A/\mathcal{P}$, where $\mathcal{P}$ is the two-sided ideal in mod$A$ consisting of the morphisms which factorize through projective A-modules. For any two objects X, Y in $\underline{\text{mod}}A$, we denote the morphism space $\text{Hom}_{\underline{\text{modA}}}(\text{X}, \text{Y})$ by $\underline{\text{Hom}}_{\text{A}}(\text{X}, \text{Y})$. Every element of $\underline{\text{Hom}}_{\text{A}}(\text{X}, \text{Y})$ is a coset $\underline{f}$ of a morphism $f \in \text{Hom}_{\text{A}}(\text{X}, \text{Y})$ modulo $P(X, Y)$.

Definition 2.1 Two algebras A and B are said to be stably equivalent if there is an equivalence $F : \underline{\text{mod}}A \longrightarrow \underline{\text{mod}}B$ of the stable categories.

Two algebras A and B are said to be stably equivalent of Morita type if there exists an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that

(1) M and N are projective as left and right modules, respectively, and

(2) $M \otimes_B N \cong A \oplus P$ as A-bimodules for some projective A-bimodule P, $N \otimes_A M \cong B \oplus Q$ as B-bimodules for some projective B-bimodule Q.

Note that two algebras A and B are stably equivalent if they are stably equivalent of Morita type. In fact, the functor $N \otimes_A -$: mod$A \longrightarrow$ mod$B$ induces an equivalence: $\underline{\text{mod}}A \longrightarrow \underline{\text{mod}}B$ whose inverse is induced by $M \otimes_B -$: mod$B \longrightarrow$ mod$A$. If P and Q are zeros, we arrive at a Morita equivalence. Hence, we can say the notion of a stable equivalence of Morita type is a unification of a Morita equivalence and a stable equivalence.

Definition 2.2 [1] The Nakayama functor of mod$A$ is defined to be the endofunctor $\nu = D\text{Hom}_A(-, A) : \text{mod}A \to \text{mod}A$.

Now, suppose that $\mathcal{A}$ is an additive k-category. Then, for any k-linear endofunctor F : $\mathcal{A} \longrightarrow \mathcal{A}$ and any fixed object X, we define an algebra $\mathbb{A}(F(X); X)$ of F in X as follows. The algebra $\mathbb{A}(F(X); X) = \oplus_{n=0}^{\infty}\text{Hom}_{\mathcal{A}}(F^n(X), X)$ as k-linear spaces with the multiplication of a homogeneous element $u : F^n(X) \longrightarrow X$ of degree n by a homogeneous element $v : F^m(X) \longrightarrow$

X of degree m is given by the composition $vu := [F^{n+m}(X) \longrightarrow F^m(X) \longrightarrow X]$. Such algebra is called an orbit algebra, which was studied by Lenzing in [7] and Kerner in [5]. A particular example of such an algebra is the Hochschild cohomology algebra $HH(A)$ of a finite-dimensional self-injective k-algebra A. In fact, $HH(A) \cong \mathbb{A}(\Omega_{A^e}(A); A)$, where $A^e = A \otimes_k A^{op}$ is the enveloping algebra of A and $\Omega_{A^e} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the Heller's loop-space functor on the stable category of the finite-dimensional left A^e-modules.

Definition 2.3 [3] Let A be a finite-dimensional k-algebra and I be an A-bimodule. The repetitive algebra $\widehat{A}$ is given as follows: the underlying vector space is $\widehat{A} = (\oplus_{i\in\mathbb{Z}}A) \oplus (\oplus_{i\in\mathbb{Z}}I)$, we denote the elements of $\widehat{A}$ by $(a_i, \varphi_i)_i$, where $a_i \in A$, $\varphi_i \in I$, the multiplication is defined by $(a_i, \varphi_i)_i \cdot (b_i, \psi_i)_i = (a_ib_i, a_{i+1}\psi_i + \varphi_ib_i)_i$.

A more suggestive interpretation is to consider $\widehat{A}$ as the doubly infinite matrix algebra, without identity

$$\begin{pmatrix} \ddots & & & 0 & \\ \ddots & A_{i-1} & & & 0 \\ & Q_{i-1} & A_i & & \\ & & Q_i & A_{i+1} & \\ 0 & & & \ddots & \ddots \\ & 0 & & & \end{pmatrix}$$

in which matrices have only finitely many non-zero entries. $A_i = A$ is placed on the main diagonal, $Q_i = I$ for $i \in \mathbb{Z}$, all the remaining entries are zeros, and the multiplication is induced from the canonical map $A \otimes_A I \longrightarrow I$, $I \otimes_A A \longrightarrow I$ and the zero map $I \otimes_A I \longrightarrow 0$.

It is seen that $\hat{A}$-modules can be written in the following way: $M = (M_i, f_i)_{i\in\mathbb{Z}}$, where the M_i's are A-modules, all but finitely many being zero, the f_i's are A-linear maps: $f_i : I \otimes_A M_i \longrightarrow M_{i+1}$ such that $f_{i+1}(1 \otimes f_i) = 0$ for all $i \in \mathbb{Z}$.

Proposition 2.4 [1] (a) Let $\mathcal{T}$ be a full subcategory of mod$A$. The following statements are equivalent:

(1) $\mathcal{T}$ is the torsion class of torsion pairs $(\mathcal{T}, \mathcal{F})$ in mod$A$.

(2) $\mathcal{T}$ is closed under images, direct sums, and extensions.

(3) There exists an idempotent radical t, such that $\mathcal{T} = \{X \mid tX = X\}$.

(b) Let $\mathcal{F}$ be a full subcategory of mod$A$. The following statements are equivalent:

(1) $\mathcal{F}$ is the torsion-free class of some torsion pair $(\mathcal{T}, \mathcal{F})$ in mod$A$.

(2) $\mathcal{F}$ is closed under submodules, direct products, and extensions.

(3) There exists an idempotent radical t, such that $\mathcal{F} = \{Y \mid tY = 0\}$.

Definition 2.5 [4] A finite-dimensional k-algebra A is called a quasitilted algebra if the following conditions are satisfied:

(1) gl.dim$A \le 2$;

(2) If X is a finitely generated indecomposable A-module, then $\mathrm{pd}_AX \le 1$ or $\mathrm{id}_AX \le 1$.

Definition 2.6 [1] Let A be a finite-dimensional k-algebra. An A-module T is called a partial tilting module if the following conditions are satisfied:

(1) $\mathrm{pd}_AT \le 1$;

(2) $\mathrm{Ext}^1_A(T, T) = 0$.

Proposition 2.7 [14] Let A, B be two finite-dimensional k-algebras. Then, the following assertions hold:

(1) If P is a projective A-B-bimodule and X is a left-right projective A-bimodule, then $X \otimes_A P$ is a projective A-B-bimodule;

(2) If P is a projective A-B-bimodule and Y is a left-right projective B-bimodule, then $P \otimes_B Y$ is a projective A-B-bimodule.

The definition of left-right projective will be introduced later.

3 Proof of Theorem 1.1

We shall frequently use the following facts.

If two finite-dimensional k-algebras A and B are self-injective, then, the tensor product algebra $A \otimes_k B^{op}$ is also self-injective. In particular, the enveloping algebra $A^e = A \otimes_k A^{op}$ of a self-injective algebra A is also self-injective.

Every finite-dimensional left $A \otimes_k B^{op}$-module can be interpreted as a finite-dimensional A-B-bimodule with a central action of k, and conversely, every such bimodule can be considered as a left $A \otimes_k B^{op}$-module. Hence, we shall not distinguish left $A \otimes_k B^{op}$-modules and A-B-bimodules.

A finite-dimensional A-B-bimodule X is said to be left-right projective if it is projective as a left A-module and as a right B-module. In general, there are lots of indecomposable A-B-bimodules which are left-right projective (see [14]). The A-bimodule A is an example, which is a left-right projective A-bimodule but is not a projective as A-bimodule.

Lemma 3.1 [2] Assume ${}_AM_B$ and ${}_BN_A$ are indecomposable bimodules that induce a stable equivalence of Morita type between A and B. Then, the following are true:

(1) There are bimodule isomorphisms $N \cong \mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_B(M, B)$ and $M \cong \mathrm{Hom}_A(N, A) \cong \mathrm{Hom}_B(N, B)$.

(2) The functor $- \otimes_A M_B$ is right and left adjoint to $- \otimes_B N_A$.

Lemma 3.2 [1] The Nakayama functor ν_A is right exact and is functorially isomorphic to $- \otimes_A D(A)$.

Lemma 3.3 [15] Let A, B be two finite-dimensional k-algebras. Then, for every finite-dimensional B-A-bimodule X, there is a B-A-bimodule isomorphism

$$X \otimes_{A\otimes_k B^{op}} D(A \otimes_K B^{op}) \cong X \otimes_A D(A).$$

Lemma 3.4 [17] Let A, B be two finite-dimensional k-algebras. Consider the modules $({}_AU, {}_AV_B, W_B)$. If U is a finite-dimensional projective A-module, then, there is a natural isomorphism

$$\mathrm{Hom}_B(V, W) \otimes_A U \cong \mathrm{Hom}_B(\mathrm{Hom}_A(U, V), W).$$

Lemma 3.5 Let A and B be two finite-dimensional self-injective k-algebras which are stably equivalent of Morita type induced by bimodules ${}_AM_B$ and ${}_BN_A$. Let ${}_AX_A$ and ${}_BY_B$ be bimodules, such that $N \otimes_A X \otimes_A M \cong Y$, where X is a left-right projective A-bimodule. Then, for any non-negative integer n, there is an isomorphism $N \otimes_A \nu^n_{A^e}(X) \otimes_A M \cong \nu^n_{B^e}(Y)$ in $\underline{\mathrm{mod}}(B^e)$.

Proof We shall show the Lemma by induction on n. First, observe that, for $n=0$, we have $N\otimes_A X\otimes_A M\cong Y$ in $\underline{\mathrm{mod}}(B^e)$.

Now, assume that, for some non-negative integer n, there is an isomorphism $N\otimes_A \nu^n_{A^e}(X)\otimes_A M\cong \nu^n_{B^e}(Y)$ in $\underline{\mathrm{mod}}(B^e)$. Consider

$$\begin{aligned}\nu^{n+1}_{B^e}(Y) &= \nu_{B^e}(\nu^n_{B^e}(Y)) = \nu_{B^e}(N\otimes_A \nu^n_{A^e}(X)\otimes_A M)\\ &\cong N\otimes_A \nu^n_{A^e}(X)\otimes_A M\otimes_{B^e} D(B^e)\\ &\cong N\otimes_A \nu^n_{A^e}(X)\otimes_A M\otimes_B D(B).\end{aligned}$$

By Lemma 3.1 and Lemma 3.4, we have

$$M\otimes_B D(B)\cong \mathrm{Hom}_k(\mathrm{Hom}_B(M,B),k)\cong \mathrm{Hom}_k(\mathrm{Hom}_A(M,A),k)\cong D(A)\otimes_A M.$$

Thus,

$$\begin{aligned}\nu^{n+1}_{B^e}(Y) &\cong N\otimes_A \nu^n_{A^e}(X)\otimes_A M\otimes_B D(B)\cong N\otimes_A \nu^n_{A^e}(X)\otimes_A D(A)\otimes_A M\\ &\cong N\otimes_A \nu^n_{A^e}(X)\otimes_{A^e} D(A^e)\otimes_A M\\ &\cong N\otimes_A \nu^{n+1}_{A^e}(X)\otimes_A M.\end{aligned}$$

Consequently, this lemma holds.

Lemma 3.6 Let A and B be two finite-dimensional self-injective k-algebras which are stably equivalent of Morita type induced by bimodules ${}_AM_B$ and ${}_BN_A$. Let ${}_AX_A$ and ${}_BY_B$ be bimodules, such that $N\otimes_A X\otimes_A M\cong Y$, where X is a left-right projective A-bimodule. Then, for any non-negative integer n, there is an isomorphism $M\otimes_B \nu^n_{B^e}(Y)\otimes_B N\cong \nu^n_{A^e}(X)$ in $\underline{\mathrm{mod}}(A^e)$.

Proof We shall show the Lemma by induction on n. First, observe that, for $n=0$, we have

$$\begin{aligned}M\otimes_B Y\otimes_B N &\cong M\otimes_B N\otimes_A X\otimes_A M\otimes_B N\cong (A\oplus P)\otimes_A X\otimes_A (A\oplus P)\\ &\cong {}_AX_A\oplus(A\otimes_A X\otimes_A P)\oplus(P\otimes_A X\otimes_A A)\oplus(P\otimes_A X\otimes_A P).\end{aligned}$$

Because P is a projective A-bimodule and X is a left-right projective A-bimodule, by Proposition 2.17, $A\otimes_A X\otimes_A P$, $P\otimes_A X\otimes_A A$, and $P\otimes_A X\otimes_A P$ are projective A-bimodules. Hence, we obtain $M\otimes_B Y\otimes_B N\cong_A X_A$ in $\underline{\mathrm{mod}}(A^e)$.

Now, we assume that, for some non-negative integer n, there is an isomorphism $M\otimes_B \nu^n_{B^e}(Y)\otimes_B N\cong \nu^n_{A^e}(X)$ in $\underline{\mathrm{mod}}(A^e)$. Consider

$$\begin{aligned}\nu^{n+1}_{A^e}(X) &= \nu_{A^e}(\nu^n_{A^e}(X)) = \nu_{A^e}(M\otimes_B \nu^n_{B^e}(Y)\otimes_B N)\\ &\cong M\otimes_B \nu^n_{B^e}(Y)\otimes_B N\otimes_{A^e} D(A^e)\\ &\cong M\otimes_B \nu^n_{B^e}(Y)\otimes_B N\otimes_A D(A).\end{aligned}$$

By Lemma 3.1 and Lemma 3.4, we have

$$N\otimes_A D(A)\cong \mathrm{Hom}_k(\mathrm{Hom}_A(N,A),k)\cong \mathrm{Hom}_k(\mathrm{Hom}_B(N,B),k)\cong D(B)\otimes_B N.$$

Thus,

$$\begin{aligned}\nu_{A^e}^{n+1}(X) &\cong M \otimes_B \nu_{B^e}^{n}(Y) \otimes_B N \otimes_A D(A) \cong M \otimes_B \nu_{B^e}^{n}(Y) \otimes_B D(B) \otimes_B N \\ &\cong M \otimes_B \nu_{B^e}^{n}(Y) \otimes_{B^e} D(B^e) \otimes_B N \\ &\cong M \otimes_B \nu_{B^e}^{n+1}(Y) \otimes_B N.\end{aligned}$$

Consequently, this lemma follows.

Corollary 3.7 Let A and B be two finite-dimensional self-injective k-algebras which are stably equivalent of Morita type induced by bimodules ${}_AM_B$ and ${}_BN_A$. Let ${}_AX_A$ and ${}_BY_B$ be bimodules, such that $N \otimes_A X \otimes_A M \cong Y$, where X is a left-right projective A-bimodule. Then, for any non-negative integer n, there is an isomorphism $M \otimes_B N \otimes_A \nu_{A^e}^{n}(X) \otimes_A M \otimes_B N \cong \nu_{A^e}^{n}(X)$ in $\underline{\mathrm{mod}}(A^e)$.

Proof Apply Lemma 3.5 and Lemma 3.6.

Let A be a finite-dimensional self-injective k-algebra. It is well known that the enveloping algebra A^e is also self-injective. When X is a left-right projective A-bimodule, consider the full subcategory of $\underline{\mathrm{mod}}A^e$ which is formed by the finite direct sums of objects isomorphic to $\nu_{A^e}^{n}(X)$ for non-negative integers n. Denote this subcategory by $\underline{\mathrm{mod}}_X(A^e)$. It shall play a crucial role in the proof of Theorem 1.

Let A and B be two finite-dimensional self-injective k-algebras, which are stably equivalent of Morita type induced by bimodules ${}_BM_A$ and ${}_BN_A$. Now, we have to show the functor $N \otimes_A - \otimes_A M : \mathrm{mod}(A^e) \longrightarrow \mathrm{mod}(B^e)$ induces an equivalence between subcategories $\underline{\mathrm{mod}}_X(A^e)$ and $\underline{\mathrm{mod}}_Y(B^e)$, where X is a left-right projective A-bimodule and $Y \cong N \otimes_A X \otimes_A M$ as B-bimodules.

Proposition 3.8 There exists an equivalence $F : \underline{\mathrm{mod}}_X(A^e) \longrightarrow \underline{\mathrm{mod}}_Y(B^e)$ such that, for every non-negative integer n, it holds that $F(\nu_{A^e}^{n}(X)) \cong \nu_{B^e}^{n}(Y)$.

Proof Define the functor $F = N \otimes_A - \otimes_A M : \underline{\mathrm{mod}}_X(A^e) \longrightarrow \underline{\mathrm{mod}}_Y(B^e)$. For every object X_1 in $\underline{\mathrm{mod}}_X(A^e)$, put $F(X_1) = N \otimes_A X_1 \otimes_A M$. By Lemma 3.5, we know that $F(X_1)$ is in $\underline{\mathrm{mod}}_Y(B^e)$. For every morphism $\underline{f} : X_1 \longrightarrow X_2$ in $\underline{\mathrm{mod}}_X(A^e)$, we put $F(\underline{f}) = 1_N \otimes f \otimes 1_M$. It is known that a morphism $f : X_1 \longrightarrow X_2$ in $\mathrm{mod}(A^e)$ between objects from $\underline{\mathrm{mod}}_X(A^e)$ factors through a projective A^e-module if and only if the morphism $1_N \otimes f \otimes 1_M$ factors through a projective B^e-module. Thus, F is well defined.

Moreover, we can define a quasi-inverse functor G of F. We put $G(Y_1) = M \otimes_B Y_1 \otimes_B N$ for every object Y_1 in $\underline{\mathrm{mod}}_Y(B^e)$. By Lemma 3.6, $G(Y_1)$ is an object in $\underline{\mathrm{mod}}_X(A^e)$. For every morphism $\underline{g} : Y_1 \longrightarrow Y_2$ in $\underline{\mathrm{mod}}_Y(B^e)$, we put $G(\underline{g}) = 1_M \otimes g \otimes 1_N$. A similar argument shows that G is a quasi-inverse of F. Therefore, F is an equivalence of subcategories. Furthermore, we obtain from Lemma 3.5 $F(\nu_{A^e}^{n}(X)) \cong \nu_{B^e}^{n}(Y)$ for every non-negative integer n.

Proof of Theorem 1.1 From Proposition 3.8, we see that $\underline{\mathrm{End}}_{A^e}(X) \cong \underline{\mathrm{End}}_{B^e}(Y)$ and

$$\underline{\mathrm{Hom}}_{A^e}(\nu_{A^e}^{n}(X), X) \cong \underline{\mathrm{Hom}}_{B^e}(F(\nu_{A^e}^{n}(X)), F(X)) \cong \underline{\mathrm{Hom}}_{B^e}(\nu_{B^e}^{n}(Y), Y)$$

Thus, $\mathbb{A}(\nu_{A^e}(X); X) \cong \mathbb{A}(\nu_{B^e}(Y); Y)$ as k-vector spaces.

Next, we only need to show the following fact: for any morphism $\underline{g} : \nu_{A^e}^{n}(X) \longrightarrow \nu_{A^e}^{m}(X)$, it holds that $\underline{1_N \otimes \nu_{A^e}(g) \otimes 1_M} \cong \nu_{B^e}(\underline{1_N \otimes g \otimes 1_M})$, where n,m are some non-negative integers and $\nu_{A^e}(g)$ stands for a representative of the coset $\nu_{A^e}(\underline{g})$.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
N\otimes_A \nu^{n+1}_{A^e}(X)\otimes_A M & \xrightarrow{=} & N\otimes_A \nu_{A^e}(\nu^n_{A^e}(X))\otimes_A M & \xrightarrow{\cong} & N\otimes_A \nu^n_{A^e}(X)\otimes_{A^e} D(A^e)\otimes_A M \\
\Big\downarrow {\scriptstyle 1_N\otimes \nu_{A^e}(g)\otimes 1_M} & & \Big\downarrow {\scriptstyle 1_N\otimes \nu_{A^e}(g)\otimes 1_M} & & \Big\downarrow {\scriptstyle 1_N\otimes g\otimes 1_{D(A^e)}\otimes 1_M} \\
N\otimes_A \nu^{m+1}_{A^e}(X)\otimes_A M & \xrightarrow{=} & N\otimes_A \nu_{A^e}(\nu^m_{A^e}(X))\otimes_A M & \xrightarrow{\cong} & N\otimes_A \nu^m_{A^e}(X)\otimes_{A^e} D(A^e)\otimes_A M
\end{array}
$$

Figure 1.1

By Lemma 3.3, we have

$$N \otimes_A \nu^n_{A^e}(X) \otimes_{A^e} D(A^e) \otimes_A M \cong N \otimes_A \nu^n_{A^e}(X) \otimes_A D(A) \otimes_A M,$$

$$N \otimes_A \nu^m_{A^e}(X) \otimes_{A^e} D(A^e) \otimes_A M \cong N \otimes_A \nu^m_{A^e}(X) \otimes_A D(A) \otimes_A M.$$

Combining the above two isomorphisms with the proof of Lemma 3.5, we have the following commutative diagram:

$$
\begin{array}{ccccc}
N\otimes_A \nu^n_{A^e}(X)\otimes_A D(A)\otimes_A M & \xrightarrow{\cong} & N\otimes_A \nu^n_{A^e}(X)\otimes_A M\otimes_B D(B) & \xrightarrow{\cong} & \nu_{B^e}(N\otimes_A \nu^n_{A^e}(X)\otimes_A M) \\
\Big\downarrow {\scriptstyle 1_N\otimes g\otimes 1_{D(A)}\otimes 1_M} & & \Big\downarrow {\scriptstyle 1_N\otimes g\otimes 1_M\otimes 1_{D(B)}} & & \Big\downarrow {\scriptstyle \nu_{B^e}(1_N\otimes g\otimes 1_M)} \\
N\otimes_A \nu^m_{A^e}(X)\otimes_A D(A)\otimes_A M & \xrightarrow{\cong} & N\otimes_A \nu^m_{A^e}(X)\otimes_A M\otimes_B D(B) & \xrightarrow{\cong} & \nu_{B^e}(N\otimes_A \nu^m_{A^e}(X)\otimes_A M)
\end{array}
$$

Figure 1.2

Because Figures 1.1 and 1.2 are commutative, we get the fact

$$\nu_{B^e}(\underline{1_N \otimes g \otimes 1_M}) = \underline{1_N \otimes \nu_{A^e}(g) \otimes 1_M}.$$

According to the above discussion and Proposition 3.8, we verify that, for any morphisms $g : \nu^n_{A^e}(X) \longrightarrow X$ and $h : \nu^m_{A^e}(X) \longrightarrow X$, it holds that $F(g * h) = F(g \circ \nu^n_{A^e}(h)) = F(\underline{g}) \circ \nu^n_{B^e}(F(h))$, where $F : \underline{\mathrm{mod}}_X(A^e) \longrightarrow \underline{\mathrm{mod}}_Y(B^e)$ is the equivalence induced by the functor $N \otimes_A - \otimes_A M : \mathrm{mod}(A^e) \longrightarrow \mathrm{mod}(B^e)$. Thus, the k-linear isomorphism of $\mathbb{A}(\nu_{A^e}(X); X)$ and $\mathbb{A}(\nu_{A^e}(Y); Y)$ is an algebraic isomorphism.

For a finite-dimensional k-algebra A, $D^b(\mathrm{mod}A)$ denotes the derived category of all bounded complexes of finite-dimensional left A-modules. Two k-algebras A and B are said to be derived equivalent if $D^b(\mathrm{mod}A)$ and $D^b(\mathrm{mod}B)$ are equivalent.

Remark 3.9 When two finite-dimensional self-injective k-algebras are derived equivalent, we can get the result of Theorem 1.1, because there is a stable equivalence of Morita type between self-injective algebras if they are derived equivalent (see [16, Corollary 5.5]).

4 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. The main tool is the fact that $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs under certain conditions.

Proposition 4.1 [2] Let A and B be two finite-dimensional k-algebras without semisimple blocks such that $A/\text{rad}A$ and $B/\text{rad}B$ are separable over k. If A and B are stably equivalent of Morita type induced by indecomposable bimodules ${}_AM_B$ and ${}_BA_A$, then $(M\otimes_B -, N\otimes_A -)$ and $(N\otimes_A -, M\otimes_B -)$ are adjoint pairs.

Proposition 4.2 [4] Let A be a finite-dimensional k-algebra of finite representation type. Then, A is a quasitilted algebra if and only if A is a tilted algebra.

Lemma 4.3 Let A and B be two finite-dimensional k-algebras without semisimple blocks, such that $A/\text{rad}A$ and $B/\text{rad}B$ are separable over k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B, and I is an injective A-module. Then, $N\otimes_A I$ is an injective B-module.

Proof By Proposition 4.1, we know that $(M\otimes_B -, N\otimes_A -)$ is an adjoint functor. Thus, $\text{Hom}_A(M\otimes_B -, I)\cong \text{Hom}_B(-, N\otimes_A I)$. Because M is a projective B-module and I is an injective A-module, $\text{Hom}_A(M\otimes_B -, I)$ and $\text{Hom}_B(-, N\otimes_A I)$ are exact functors. Hence, $N\otimes_A I$ is an injective B-module.

Proof of Theorem 1.2 1) By definition of quasitilted algebra, we only need to discuss the case that ${}_BX$ is neither a projective module nor an injective module. Suppose that X is an indecomposable non-projective B-module. Because there is a stable equivalence between A and B induced by two bimodules ${}_AM_B$ and ${}_BN_A$, there exists an indecomposable A-module ${}_AX'$, such that $N\otimes_A X'\cong_B X\oplus Q'$, where Q' is a projective B-module. Because A is a quasitilted algebra, $\text{pd}_AX'\le 1$ or $\text{id}_AX'\le 1$. If $\text{pd}_AX'\le 1$, there is a projective resolution of X', $0\longrightarrow P_1\longrightarrow P_0\longrightarrow_A X'\longrightarrow 0$. Because N_A is a projective module, we get a new exact sequence

$$0\longrightarrow N\otimes_A P_1\longrightarrow N\otimes_A P_0\longrightarrow N\otimes_A X'\longrightarrow 0,$$

where $N\otimes_A P_i, i=0,1$, are projective B-modules. Thus, $\text{pd}_B(N\otimes_A X')\le 1$, which implies $\text{pd}_BX\le 1$. If $\text{id}_AX'\le 1$, there is an injective resolution of ${}_AX'$, $0\longrightarrow_A X'\longrightarrow I_0\longrightarrow I_1\longrightarrow 0$. Because $N\otimes_A -$ is an exact functor, we get an exact sequence

$$0\longrightarrow_B X\oplus Q\longrightarrow N\otimes_A I_0\longrightarrow N\otimes_A I_1\longrightarrow 0.$$

By Lemma 4.3, we verify that $N\otimes_A I_i, i=0,1$, are also injective B-modules. So, $\text{id}_B(X\oplus Q)\le 1$, which implies that $\text{id}_BX\le 1$. According to the above proof, for any ${}_BX\in \text{mod}B$, we get $\text{pd}_BX\le 1$ or $\text{id}_BX\le 1$.

2) By similar arguments, we can easily obtain $\text{gl.dim}B\le 2$.

From the above discussion, we verify that B is a quasitilted algebra.

Dually, it can be proved that A is a quasitilted algebra when B is so.

Corollary 4.4 Let A and B be two finite-dimensional k-algebras without semisimple blocks such that $A/\text{rad}A$ and $B/\text{rad}B$ are separable over k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. If A is a tilted algebra of finite representation type, then, B is also a tilted algebra.

Proof Because A and B are stably equivalent of Morita type, if A is an algebra of finite representation type, then B is an algebra of finite representation type (see [6]). A tilted algebra A is, of course, a quasitilted algebra. By Theorem 1.2, B is also a quasitilted algebra. Hence, B is a tilted algebra by Proposition 4.2.

Theorem 4.5 Let A and B be two finite-dimensional k-algebras, which are stably equivalent of Morita type induced by bimodules ${}_AM_B$ and ${}_BN_A$. If T is a partial tilting A-module, then $N\otimes_A T$ is a partial tilting B-module.

Proof By the fact that ${}_BN, N_A$ are projective modules and $\mathrm{pd}_A T\le 1$, we get $\mathrm{pd}_B(N\otimes_A T)\le 1$.

In contrast, for $\mathrm{pd}_A T\le 1$, we have a projective resolution of ${}_AT$:

$$0\longrightarrow P_1\longrightarrow P_0\longrightarrow T\longrightarrow 0.$$

Applying the functor $N\otimes_A -$, we have a projective resolution of $N\otimes_A T$:

$$0\longrightarrow N\otimes_A P_1\longrightarrow N\otimes_A P_0\longrightarrow N\otimes_A T\longrightarrow 0.$$

Furthermore, using the functor $\mathrm{Hom}_B(-,N\otimes_A T)$, we have the following complex:

$$\begin{aligned}0&\longrightarrow \mathrm{Hom}_B(N\otimes_A T,N\otimes_A T)\longrightarrow \mathrm{Hom}_B(N\otimes_A P_0,N\otimes_A T)\\&\longrightarrow \mathrm{Hom}_B(N\otimes_A P_1,N\otimes_A T)\longrightarrow 0.\end{aligned}$$

Because $(M\otimes_B -,N\otimes_A -)$ is an adjoint functor, we get the complex:

$$\begin{aligned}0&\longrightarrow \mathrm{Hom}_A(M\otimes_B N\otimes_A T,{}_AT)\longrightarrow \mathrm{Hom}_A(M\otimes_B N\otimes_A P_0,{}_AT)\\&\longrightarrow \mathrm{Hom}_A(M\otimes_B N\otimes_A P_1,{}_AT)\longrightarrow 0.\end{aligned}$$

In this case,

$$\begin{aligned}\mathrm{Ext}^1_B(N\otimes_A T,N\otimes_A T)&\cong \mathrm{Ext}^1_A(M\otimes_B N\otimes_A T,{}_AT)\\&\cong \mathrm{Ext}^1_A((A\oplus P)\otimes_A T,{}_AT)\\&\cong \mathrm{Ext}^1_A(T\oplus P\otimes_A T,{}_AT)\\&\cong \mathrm{Ext}^1_A(T,T)\oplus \mathrm{Ext}^1_A(P\otimes_A T,T)\\&\cong \mathrm{Ext}^1_A(P\otimes_A T,T).\end{aligned}$$

Because P is a projective A-bimodule, $P\otimes_A T$ is a projective A-module and $\mathrm{Ext}^1_B(N\otimes_A T,N\otimes_A T)\cong \mathrm{Ext}^1_A(P\otimes_A T,T)=0$. Thus, $N\otimes_A T$ is a partial tilting B-module.

5 Proof of Theorem 1.3

In this section, we always assume that A,B,M,N,P, and Q are fixed in Definition 2.1. Furthermore, we suppose that A and B are finite-dimensional k-algebras.

Now, we begin to prove Theorem 1.3.

Let $\hat{A}$ be the repetitive algebra of A by I, and $\hat{B}$ be the repetitive algebra of B by J.

By our assumption on I and J, $P\otimes_A I=I\otimes_A P=Q\otimes_B J=J\otimes_B Q=0$.

Define $\overline{M}=M\otimes_B\hat{B}$ and $\overline{N}=N\otimes_A\hat{A}$. Then,

$$\begin{aligned}\overline{N}&=N\otimes_A\hat{A}\\&=N\otimes_A(\oplus_{i\in\mathbb{Z}}A)\oplus(\oplus_{i\in\mathbb{Z}}I)=(\oplus_{i\in\mathbb{Z}}N)\oplus(\oplus_{i\in\mathbb{Z}}N\otimes_A I).\end{aligned}$$

Because $N \otimes_A I \cong J \otimes_B N$ as B-A-bimodules, the following k-linear space

$$\begin{aligned}\overline{N} &= (\oplus_{i\in\mathbb{Z}} N) \oplus (\oplus_{i\in\mathbb{Z}} N \otimes_A I)\\ &\cong (\oplus_{i\in\mathbb{Z}} B \otimes_B N) \oplus (\oplus_{i\in\mathbb{Z}} J \otimes_B N) \cong (\hat{B} \otimes_B N)\end{aligned}$$

also has left $\hat{B}$-module structure. Thus, $\overline{N}$ is a $\hat{B}$-$\hat{A}$-bimodule. Because N is a projective left B-module, there is an idempotent e_i, such that $N = Be_i$ and $\overline{N} \cong (\hat{B} \otimes_B N) = ((\oplus_{i\in\mathbb{Z}} B) \oplus (\oplus_{i\in\mathbb{Z}} J)) \otimes_B Be_i = \hat{B}\overline{e_i}$. Thus, $\overline{N}$ is a projective left $\hat{B}$-module. For the same reason, we can prove that $\overline{N}$ is a projective right $\hat{A}$-module. A similar assertion holds for $\overline{M}$.

Next, we prove the following $\hat{A}$-bimodule isomorphisms:

$$\begin{aligned}\overline{M} \otimes_{\hat{B}} \overline{N} &\cong (M \otimes_B \hat{B}) \otimes_{\hat{B}} (\hat{B} \otimes_B N) = M \otimes_B \hat{B} \otimes_B N\\ &\cong M \otimes_B ((\oplus_{i\in\in\mathbb{Z}} B) \oplus (\oplus_{i\in\mathbb{Z}} J)) \otimes_B N\\ &\cong (\oplus_{i\in\mathbb{Z}} M \otimes_B N) \oplus (\oplus_{i\in\mathbb{Z}} M \otimes_B J \otimes_B N)\\ &\cong (\oplus_{i\in\mathbb{Z}} (A \oplus P)) \oplus (\oplus_{i\in\mathbb{Z}} I)\\ &\cong (\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} I) \oplus (\oplus_{i\in\mathbb{Z}} P),\end{aligned}$$

the last but one isomorphism holds because $M \otimes_B N \cong A \oplus P$ as A-bimodules, and

$$\begin{aligned}M \otimes_B J \otimes_B N &\cong M \otimes_B N \otimes_A I\\ &\cong (A \oplus P) \otimes_A I = (A \otimes_A I) \oplus (P \otimes_A I)\\ &\cong A \otimes_A I \cong I\end{aligned}$$

as A-bimodules by Condition (2). Thus, $\overline{M} \otimes_{\hat{B}} \overline{N}$ is isomorphic to $\hat{A} \oplus (\oplus_{i\in\mathbb{Z}} P)$ as A-bimodules. Next, we only need to prove that $(\oplus_{i\in\mathbb{Z}} M \otimes_B N) \oplus (\oplus_{i\in\mathbb{Z}} M \otimes_B J \otimes_B N)$ and $(\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} I) \oplus (\oplus_{i\in\mathbb{Z}} P)$ possess the same $\hat{A}$-bimodule structure. Now, we define

$$f = (f_i)_{i\in Z} = \begin{pmatrix} \oplus_{i\in\mathbb{Z}} \rho_1 & 0 & \oplus_{i\in\mathbb{Z}} \rho_2 \\ 0 & \oplus_{i\in\mathbb{Z}} \mu(1_I \otimes \rho_1)(\varphi^{-1} \otimes 1_N) & 0 \end{pmatrix}$$

from $(\oplus_{i\in\mathbb{Z}} M \otimes_B N) \oplus (\oplus_{i\in\mathbb{Z}} M \otimes_B J \otimes_B N)$ to $(\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} I) \oplus (\oplus_{i\in\mathbb{Z}} P)$, and show that this is an $\hat{A}$-bimodule isomorphism. Clearly, this is an A-bimodule isomorphism. Moreover, for $\overline{\varphi} = \begin{pmatrix} 0 & \varphi \otimes 1_N \\ 0 & 0 \end{pmatrix}$ and $\overline{\psi} = \begin{pmatrix} 0 & \mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the commutative diagram

$$\begin{array}{ccc} I \otimes_A M \otimes_B N \oplus I \otimes_A M \otimes_B J \otimes_B N & \xrightarrow{\overline{\varphi}} & M \otimes_B N \oplus M \otimes_B J \otimes_B N \\ \downarrow{\scriptstyle 1_I \otimes f_i} & & \downarrow{\scriptstyle f_i} \\ I \otimes_A A \oplus I \otimes_A I \oplus I \otimes_A P & \xrightarrow{\overline{\psi}} & A \oplus I \oplus P \end{array}$$

shows that f is a left $\hat{A}$-bimodule morphism, and, for $\overline{\eta} = \begin{pmatrix} 0 & 1_M \otimes \psi^{-1} \\ 0 & 0 \end{pmatrix}$ and $\overline{\tau} = \begin{pmatrix} 0 & \mu' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the commutative diagram

$$\begin{array}{ccc}
M \otimes_B N \otimes_A I \oplus M \otimes_B J \otimes_B N \otimes_A I & \xrightarrow{\overline{\eta}} & M \otimes_B N \oplus M \otimes_B J \otimes_B N \\
\Big\downarrow {\scriptstyle f_i \otimes 1_I} & & \Big\downarrow {\scriptstyle f_i} \\
A \otimes_A I \oplus I \otimes_A I \oplus P \otimes_A I & \xrightarrow{\overline{\tau}} & A \oplus I \oplus P
\end{array}$$

shows that f is a right $\hat{A}$-module morphism. Thus, we have

$$\overline{M} \otimes_{\hat{B}} \overline{N} \cong (\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} I) \oplus (\oplus_{i\in\mathbb{Z}} P)$$

as $\hat{A}$-bimodules. Similarly, we have

$$\overline{N} \otimes_{\hat{A}} \overline{M} \cong (\oplus_{i\in\mathbb{Z}} B) \oplus (\oplus_{i\in\mathbb{Z}} J) \oplus (\oplus_{i\in\mathbb{Z}} Q)$$

as $\hat{B}$-bimodules. Because ${}_AP_A$ and ${}_BQ_B$ are projective bimodules, $\oplus_{i\in\mathbb{Z}} P \cong \widehat{A} \otimes_A P \otimes_A \widehat{A}$ and $\oplus_{i\in\mathbb{Z}} Q \cong \widehat{B} \otimes_B Q \otimes_B \widehat{B}$ are projective bimodules. Thus, the above isomorphisms mean that $\overline{M}$ and $\overline{N}$ define a stable equivalence of Morita type between $\hat{A}$ and $\hat{B}$.

6 Torsion Pair Under Stable Equivalences of Morita Type

Definition 6.1 [1] A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of mod$A$ is called a torsion pair if the following conditions are satisfied:

(1) $\mathrm{Hom}_A(X, Y) = 0$ for all $X \in \mathcal{T}, Y \in \mathcal{F}$.

(2) $\mathrm{Hom}_A(X, -) \mid_{\mathcal{F}} = 0$ implies $X \in \mathcal{T}$.

(3) $\mathrm{Hom}_A(-, Y) \mid_{\mathcal{T}} = 0$ implies $Y \in \mathcal{F}$.

The first condition in the definition says that there is no nonzero homomorphism from an object in $\mathcal{T}$ to one in $\mathcal{F}$, and the other two conditions say that these two subcategories are maximal for this property. The subcategory $\mathcal{T}$ is called the torsion class, and its objects are called torsion objects, while the subcategory $\mathcal{F}$ is called the torsion-free class, and its objects are called torsion-free objects.

It follows directly from the definition that the torsion class and the torsion-free class determine uniquely each other.

In this section, we denote the functor $N \otimes_A -$ by F and the functor $M \otimes_B -$ by G.

Lemma 6.2 If ${}_AX \in \mathrm{mod}A$ satisfies $FGF(X) \cong F(X)$, then, so does any submodule and quotient module of X. Conversely, if X has a submodule X' such that $FGF(X') \cong F(X')$ and $FGF(X/X') \cong X/X'$, then, $FGF(X) \cong F(X)$.

Proof If ${}_AX \in \mathrm{mod}A$ satisfies $FGF(X) \cong F(X)$, the condition $FGF(X) \cong F(X)$ is equivalent to $Q \otimes_B N \otimes_A X = 0$. Let X' be a submodule of X. Then, we have a natural exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X/X' \longrightarrow 0$. Because $Q \otimes_B N$ is a projective right A-module, applying the functor $Q \otimes_B N \otimes_A -$ to the above exact sequence, we have the following exact sequence:

$$0 \longrightarrow Q \otimes_B N \otimes_A X' \longrightarrow Q \otimes_B N \otimes_A X \longrightarrow Q \otimes_B N \otimes_A X/X' \longrightarrow 0.$$

It is seen that $Q \otimes_B N \otimes_A X = 0$ if and only if $Q \otimes_B N \otimes_A X' = Q \otimes_B N \otimes_A X/X' = 0$. This completes the proof.

Let $\mathcal{C}$ be the full subcategory of mod$A$ consisting of all A-module X satisfying $FGF(X) \cong F(X)$. Let $\mathcal{C}_0$ be the full subcategory of mod$A$ consisting of all A-modules T, such that each composition factor S of T satisfies $FGF(S) \cong F(S)$. Then, it follows from Lemma 6.2 that $\mathcal{C}_0 = \mathcal{C}$.

Let $\mathcal{D}$ be the full subcategory of mod$B$ consisting of all B-modules Y, such that $GFG(Y) \cong G(Y)$. Similarly, we can see that $\mathcal{D}$ is a full subcategory of mod$B$ consisting of all B-modules T' such that each composition factor S of T' satisfies $GFG(S) \cong F(S)$.

Proposition 6.3 $\mathcal{C}$ and $\mathcal{D}$ are both torsion classes and torsion-free classes in mod$A$ and mod$B$, respectively.

Proof By the definition of $\mathcal{C}$ and $\mathcal{D}$ and Lemma 6.2, we know that $\mathcal{C}$ and $\mathcal{D}$ are closed under submodules, factor modules and extensions. Note that direct sum coincides with direct product in mod$A$ and mod$B$. Thus, we only need to prove that $\mathcal{C}$ and $\mathcal{D}$ are closed under images and direct sums. Direct sum is a special case of extension. Hence, $\mathcal{C}$ and $\mathcal{D}$ are closed under direct sum. Let $f : X \longrightarrow Y$ be an epimorphism in $\mathcal{C}$ (or $\mathcal{D}$), then we have an exact sequence $0 \longrightarrow \mathrm{Kerf} \longrightarrow \mathrm{X} \longrightarrow \mathrm{Y} \longrightarrow 0$ in $\mathcal{C}$ (or $\mathcal{D}$). Because Kerf $\in \mathcal{C}$ (or $\mathcal{D}$), $Y \in \mathcal{C}$ (or $\mathcal{D}$).

It is well known that each simple module is isomorphic to $Ae_i/\mathrm{rad}(Ae_i)$ for some primitive idempotent $e_i \in A$. Let I be the ideal of A generated by those primitive idempotent e_i such that $FGF(Ae_i/\mathrm{rad}(Ae_i))$ is not isomorphic to $F(Ae_i/\mathrm{rad}(Ae_i))$. Then, mod$A/I$ can be regarded as a full subcategory of mod$A$ consisting of all those A-modules X with $IX = 0$. Then, mod$A/I = \mathcal{C}$. Similarly, there exists an ideal J in B, such that mod$B/J = \mathcal{D}$.

We define $\mathcal{C}'$ to be the full subcategory of mod$A$ consisting of all A-modules Y with $IY = Y$, and define $\mathcal{D}'$ to be the full subcategory of mod$B$ consisting of all B-modules Z with $JZ = Z$.

We remark that, if ${}_AX$ is in $\mathcal{C}$, then $F(X)$ is in $\mathcal{D}$, because $GFGF(X) \cong GF(X)$. So, F is a functor from $\mathcal{C}$ to $\mathcal{D}$. Similarly, G is a functor from $\mathcal{D}$ to $\mathcal{C}$.

Definition 6.4 Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. If there exist two additive functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, such that $FGF \cong F, GFG \cong G$, then, we say that $\mathcal{A}$ and $\mathcal{B}$ are weakly equivalent.

Theorem 6.5 F and G induce a weak equivalence between $\mathcal{C}$ and $\mathcal{D}$.

Proof For any ${}_AX$ in $\mathcal{C}$, $FGF({}_AX) \cong F({}_AX)$, hence, $Q \otimes_B N \otimes_A X = 0$. For any $f :{}_A X_1 \longrightarrow_A X_2$ in $\mathcal{C}$, the following diagram

$$\begin{array}{ccc}
N \otimes_A X_1 & \xrightarrow{\quad F(f) \quad} & N \otimes_A X_2 \\
\downarrow{\scriptstyle \varphi_{X_1}} & & \downarrow{\scriptstyle \varphi_{X_2}} \\
N \otimes_A M \otimes_B N \otimes_A X_1 & \xrightarrow{FGF(f)} & N \otimes_A M \otimes_B N \otimes_A X_2
\end{array}$$

commutes. Thus, $FGF \cong F$. Similarly, $GFG \cong G$.

Theorem 6.6 (1) The pair $(\mathcal{C}', \mathcal{C})$ is a torsion pair in mod$A$, and the pair $(\mathcal{D}', \mathcal{D})$ is a torsion pair in mod$B$.

(2) Let A and B be two finite-dimensional indecomposable k-algebras, such that $A/\mathrm{rad}A$ and $B/\mathrm{rad}B$ are separable over k. Then, $F(\mathcal{C}') \subseteq \mathcal{D}'$.

Proof (1) Because the proofs of the two assertions are similar, we only show that the first one is true, that is, the pair $(\mathcal{C}', \mathcal{C})$ is a torsion pair in mod$A$. For any $f \in \mathrm{Hom}_A(X_1, X_2)$, where

$X_1 \in \mathcal{C}', X_2 \in \mathcal{C}$. Because $f(X_1) = f(IX_1) = If(X_1) = 0$, $f = 0$. When $\mathrm{Hom}_A(X_1, -) \mid_{\mathcal{C}} = 0$, if X_1 is not in $\mathcal{C}'$, then $IX_1 \neq X_1$, in this case we can get $(A/I)X_1 \neq 0$, and $0 \neq f \in \mathrm{Hom}_A(X_1, -) \mid_{\mathcal{C}}$, thus, $X_1 \in \mathcal{C}'$. Similarly, we can prove that $\mathrm{Hom}_A(-, X_2) \mid_{\mathcal{C}'} = 0$ implies $X_2 \in \mathcal{C}$.

(2) Because the functor F is left adjoint to G, we have

$$\mathrm{Hom}_B(F\mathcal{C}', \mathcal{D}) \cong \mathrm{Hom}_A(\mathcal{C}', G\mathcal{D}) \subseteq \mathrm{Hom}_A(\mathcal{C}', \mathcal{C}) = 0.$$

Thus, $F(\mathcal{C}') \subseteq \mathcal{D}'$.

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