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Generalized equilibrium states and behavior of averaging operators

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Abstract. We introduce a notion of generalized equilibrium states which include Riesz products, G-measures and classical equilibrium states on a subshift of finite type. We prove a rather precise estimate for $\|P_n f\|_\infty$ where P_n are the average operators (see Introduction). As a consequence we obtain an estimate for $\|L_g^n f\|_\infty$ where L_g is a transfer operator on a subshift of finite type. A second consequence concerns the almost everywhere convergence of some lacunary series; the obtained result is new in the case of G-measures studied by G. Brown and A.H. Dooley and improves to some extend a result of J. Peyrière in the case of Riesz products. © Académie des Sciences/Elsevier, Paris

États d'équilibre généralisés et comportement d'opérateurs de moyenne

Résumé. *Nous introduisons une notion d'état d'équilibre généralisé qui étend les produits de Riesz, les G-mesures et les états d'équilibre sur un « sous-shift » de type fini. Nous prouvons une estimation assez précise pour la norme $\|P_n f\|_\infty$ où les P_n sont les opérateurs de moyenne. Cette estimation est utilisée pour estimer $\|L_g^n f\|_\infty$ où L_g est un opérateur de transfert sur un « sous-shift » de type fini. Elle est aussi utilisée pour étudier la convergence presque partout d'une série lacunaire ; le résultat obtenu est nouveau dans le cas de G-mesures de G. Brown et A.H. Dooley et il améliore dans un certain sens un résultat de J. Peyrière dans le cas de produits de Riesz.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Soit $\{S_n\}_{n\geq 1}$ une suite d'ensembles finis et $X = \prod_{n=1}^{\infty} S_n$ le produit infini. Soit $A = \{A_n\}_{n\geq 1}$ une suite de matrices telles que $A_n \in M_{S_n \times S_{n+1}}$ soient d'éléments 0 ou 1. Nous définissons un

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sous-espace X_A de X, dit *espace symbolique (généralisé)*, par

$$X_A = \{x = (x_j)_{j\geq 1} \in X : A_n(x_n, x_{n+1}) = 1, \quad \forall n \geq 1\}.$$

Dans la suite, nous supposons toujours qu'il existe un $M \geq 0$ tel que la matrice $\prod_{j=n}^{n+M} A_j > 0$ pour tout $n \geq 1$. Alors X_A est dit *transitif.*

Une suite $G = \{g_n\}_{n\geq 1}$ de fonctions positives définies sur X_A s'appelle une *suite de potentiels* si, pour tout $n \geq 1$, $g_n(x)$ ne dépend pas des $n-1$ premières coordonnées de x (alors $g_n(x)$ s'écrit $g_n(x_n x_{n+1} \cdots)$). Si, de plus, pour tout $n \geq 1$,

$$\sum_{y_n : A_n(y_n, x_{n+1})=1} g_n(y_n x_{n+1} \cdots) = 1 \quad (\forall x = (x_n) \in X_A)$$

la suite de potentiels est dite *normalisée.* Soit $G_n(x) = \prod_{j=1}^{n} g_n(x)$ $(n \geq 1)$. Définissons les *opérateurs de moyenne* P_n sur l'espace $\mathrm{C}(X_A)$ des fonctions continues sur X_A par

$$P_n f(x) = \sum_{y_1, \ldots, y_n} G_n(y_1 \cdots y_n x_{n+1} \cdots) f(y_1 \cdots y_n x_{n+1} \cdots),$$

où la somme est prise sur toutes les suites $(y_1, \ldots, y_n)$ telles que $A_1(y_1, y_2) = 1$, $A_2(y_2, y_3), \ldots, A_n(y_n, x_{n+1}) = 1$. On peut vérifier que P_n est positif et $P_n 1 = 1$. Par conséquent, l'opérateur adjoint $P_n^* : M_1^+(X_A) \to M_1^+(X_A)$ admet des points fixes, où $M_1^+(X_A)$ désigne des probabilités boréliennes sur X_A. On dit qu'une probabilité $\mu \in M_1^+(X_A)$ est un *état d'équilibre (généralisé)* si $P_n^* \mu = \mu$ pour tout $n \geq 1$.

Supposons que $G = \{g_n\}_{n\geq 1}$ est une suite normalisée de potentiels définis sur un espace symbolique transitif X_A. Nous prouvons qu'il existe un et un seul état d'équilibre si $g_n(x) \geq c$ $(x \in X_A, n \geq 1)$ pour un certain $c > 0$ et

$$V := \sup \left\{ \frac{G_n(x)}{G_n(y)} : n \geq 1 \,;\, x_j = y_j,\ 1 \leq j \leq n \right\} < \infty.$$

Maintenant, soit μ l'état d'équilibre (unique). Si $\int f \mathrm{d}\mu = 0$, $P_n f$ tend vers zéro lorsque $n \to \infty$. Notre résultat principal est de fournir une vitesse de convergence. Pour $1 \leq n \leq m$, posons

$$V_{n,m} = -1 + \sup \left\{ \frac{G_n(x)}{G_n(y)} : x_j = y_j,\ 1 \leq j \leq n \right\}.$$

Pour $n \geq 1$ et $f \in \mathrm{C}(X_A)$, posons

$$\mathrm{var}_n(f) = \sup \{ |f(x) - f(y)| : x_j = y_j,\ 1 \leq j \leq n \}.$$

Voici le résultat principal : il existe une constante $0 < \gamma < 1$ ne dépendant que de G telle que pour toute $f \in \mathrm{C}(X_A)$, tout $N \geq 1$ et tout choix $1 \leq n_1 < n_2 < \cdots < n_k \leq N$, on ait

$$\|P_N f\|_\infty \leq \mathrm{var}_{n_1}(f) + \|f\|_\infty \sum_{j=1}^{k-1} V_{n_j, n_{j+1}} + \|f\|_\infty \gamma^k.$$

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1. Introduction and results

Let $\{S_n\}_{n\geq 1}$ be a sequence of finite sets and $X = \prod_{n=1}^{\infty} S_n$ be the infinite product space of S_n which are equipped with discrete topologies. Let $A = \{A_n\}_{n\geq 1}$ be a sequence of matrices such that A_n is of entries 0 or 1 and its rows are indexed by S_n and columns by S_{n+1}. We define the *(generalized) symbolic space* (restricted by $A = \{A_n\}$) as follows

$$X_A = \{x = (x_j)_{j\geq 1} \in X : A_n(x_n, x_{n+1}) = 1, \quad \forall n \geq 1\}.$$

In the sequel, we always suppose that there exists an $M \geq 0$ such that $\prod_{j=n}^{n+M} A_j > 0$ for all $n \geq 1$ ($A > 0$ means that the entiers of the matrix A are all strictly positive). In this case, X_A is said to be *transitive.*

A sequence $G = \{g_n\}_{n\geq 1}$ of non-negative functions defined on X_A is called a *sequence of potentials* if for any $n \geq 1$, $g_n(x)$ does not depend on the $n-1$ coordinates of x (then we may write $g_n(x) = g_n(x_n x_{n+1} \cdots)$). Furthermore, it is said to be *normalized* if for any $n \geq 1$, the following normalization condition is satisfied

$$\sum_{y_n : A_n(y_n, x_{n+1})=1} g_n(y_n x_{n+1} \cdots) = 1 \quad (\forall x = (x_n) \in X_A).$$

Let now $G_n(x) = \prod_{j=1}^{n} g_n(x)$ for $(n \geq 1)$. Then define a sequence of *averaging operators* P_n on the space $\mathrm{C}(X_A)$ of all continuous functions on X_A by

$$P_n f(x) = \sum_{y_1, \ldots, y_n} G_n(y_1 \cdots y_n x_{n+1} \cdots) f(y_1 \cdots y_n x_{n+1} \cdots)$$

where the sum is taken over all sequences $(y_1, \ldots, y_n)$ such that $A_1(y_1, y_2) = 1$, $A_2(y_2, y_3) = 1, \ldots, A_n(y_n, x_{n+1}) = 1$. It can be checked that P_n is positive and $P_n 1 = 1$. Hence the adjoint operator $P_n^* : M_1^+(X_A) \to M_1^+(X_A)$ admits fixed points, where $M_1^+(X_A)$ is the space of all Borel probability measures on X_A. A measure $\mu \in M_1^+(X_A)$ is called a (generalized) *equilibrium state* associated to $G = \{g_n\}$ if $P_n^* \mu = \mu$ for all $n \geq 1$.

In this Note, we are interested in the properties of equilibrium states and the behavior of $P_n f$ as $n \to \infty$. Our first result concerns the existence and the uniqueness of equilibrium states and the Gibbs property (by which we mean Theorem 1 (d)).

THEOREM 1. – *Let $G = \{g_n\}_{n\geq 1}$ be a normalized sequence of potentials defined on a transitive symbolic space X_A.*

(a) *The set of all equilibrium states associated to G is a non-empty convex compact set.*

(b) *There is a unique equilibrium state if and only if for any $f \in \mathrm{C}(X_A)$, $P_n f(x)$ converges uniformly (on x) to a constant as $n \to \infty$.*

(c) *There is a unique equilibrium state μ if $g_n(x) \geq c$ ($x \in X_A$, $n \geq 1$) for some $c > 0$ and*

$$V := \sup\left\{\frac{G_n(x)}{G_n(y)} : n \geq 1\,;\ x_j = y_j,\ 1 \leq j \leq n\right\} < \infty.$$

(d) *Under the condition in* (c), *there exist constants D_1 and D_2 such that*

$$D_1 G_n(x) \leq \mu(I(x_n)) \leq D_2 G_n(x) \quad (\forall x \in X_A,\ \forall n \geq 1).$$

The constant in (b) is actually $\int f \mathrm{d}\mu$ where μ is the unique equilibrium state. When $\int f \mathrm{d}\mu = 0$, $P_n f$ converges uniformly to zero as $n \to \infty$. Our second theorem, the main result of this Note, provides a speed for this convergence. For $1 \leq n \leq m$, let

$$V_{n,m} = -1 + \sup\left\{\frac{G_n(x)}{G_n(y)} : x_j = y_j,\ 1 \leq j \leq m\right\}.$$

For a function f on X_A, define its variation of order n ($n \geq 1$) by $\mathrm{var}_n(f) = \sup|f(x) - f(y)|$, where the sup is taken over x, $y \in X_A$ such that $x_j = y_j$ for $1 \leq j \leq n$

THEOREM 2. – *Let $G = \{g_n\}_{n\geq 1}$ be a normalized sequence of potentials defined on a transitive symbolic space X_A. There exists a constant $0 < \gamma < 1$ such that for any $f \in C(X_A)$, any $N \geq 1$ and any choice $1 \leq n_1 < n_2 < \cdots < n_k \leq N$, we have*

$$\|P_N f\|_\infty \leq \mathrm{var}_{n_1}(f) + \|f\|_\infty \sum_{j=1}^{k-1} V_{n_j,n_{j+1}} + \|f\|_\infty \gamma^k.$$

2. Applications

We give here two applications of Theorem 2. The first one is related to a convergence problem which was studied for Riesz products [5] and the second one concerns with the classical equilibrium states and their corresponding transfer operators (*see* [1] for a general account of the subject).

Let μ be the unique equilibrium state as in Theorem 1. Let $\{\alpha_n\}$ be a sequence of complex numbers and $\{f_n\}$ be a sequence of μ-integrable functions on X_A. We would like to find conditions to guarantee the μ-almost everywhere convergence of the following series

$$\sum_{n=1}^{\infty} \alpha_n \left[f_n(x) - \int f_n \mathrm{d}\mu \right].$$

Our answer involves some regularity of f_n and g_n. By Theorem 2, we can show that the series is quasi-orthogonal, from which we get

THEOREM 3. – *Let $\{g_n\}_{n\geq 1}$ be a normalized sequence of potentials and let $\{f_n\}$ be a sequence of continuous functions. Suppose there are constants $A > 0$ and $p > 1$ such that for $m > n \geq 1$,*

$$\|f_n\|_\infty \leq A, \quad \mathrm{var}_m(f_n) \leq \frac{A}{(m-n)^p}, \quad \mathrm{var}_m(\log g_n) \leq \frac{A}{(m-n)^{p+1}}.$$

If $\sum_{n=1}^{\infty} |\alpha_n|^2 \log^2 n < \infty$, then the above series convergences μ-almost everywhere.

When X_A is the full symbolic space $X = \prod_{n=1}^{\infty}\{1, 2, \ldots, \ell_n\}$, we get the G-measures studied by G. Browm and A.H. Dooley [2]. The result in the Theorem 3 for G-measures is totally new. The more special case of Riesz products corresponds to $g_n(x) = \ell_n^{-1}[1 + a_n \cos(2\pi \ell_1 \cdots \ell_{n-1} x)]$, where $-1 \leq a_n \leq 1$ (*see* [5]). The condition concerning $\log g_n$ is satisfied if $\sup |a_n| < 1$. The condition concerning f_n is implied by $\omega(f_n, t) = O(|\log t|^{-p})$, where $\omega(f_n, \cdot)$ is the usual modulus of continuity of f_n regarded as a function on the circle. This result improves to some extent a result of J. Peyrière in [5] where f_n is supposed to be analytic.

When $S_n = S = \{1, 2, \ldots, \ell\}$ $(\forall n \geq 1)$ and $A_n = A\,(\forall n \geq 1)$, where A is a $\ell \times \ell$ matrix of entrices 0 or 1, we recover the well-known subshift of finite type Σ_A. Note that Σ_A is transitive if and only if A is a primitive matrix. A specific property of Σ_A is that it admits a shift $T : \Sigma_A \to \Sigma_A$ which is defined by $x = (x_n)_{n\geq 1} \mapsto Tx = (x_{n+1})_{n\geq 1}$ (such a shift doesn't exists on generalized symbolic spaces). A function $g : \Sigma_A \to \mathbf{R}^+$ is called a *potential*. It is said to be *normalized* if $\sum_{y \in T^{-1}x} g(y) = 1$ $(\forall x \in \Sigma_A)$. Let $g_n(x) = g(T^{n-1}x)$. Then $\{g_n\}$ is a normalized sequence of potentials. The corresponding equilibrium states are the classical equilibrium states [1]. Associated to the potential g is the transfer operator $L_g : C(\Sigma_A) \to C(\Sigma_A)$ defined by

$$L_g f(x) = \sum_{y \in T^{-1}x} g(y) f(x).$$

In this case, Theorem 2 can be stated as follows: there exists $0 < \gamma < 1$ such that for any f and for any $1 \leq n \leq N$, we have

$$\|L_g^N f\|_\infty \leq \mathrm{var}_n(f) + \|f\|_\infty \sum_{j=n+1}^{N} \mathrm{var}_j(\log g) + \|f\|_\infty \gamma^{N-n}.$$

It is well known that there is a unique equilibrium state μ associated to g when $\log g$ is of summable variation, i.e. $\sum_{n=1}^{\infty} \mathrm{var}_n(\log g) < \infty$ ([1], [4], see also Theorem 1 (c)). For a function $f : \Sigma_A \to \mathbf{R}$, the *correlation function* of f (with respect to μ) is defined by

$$R_f(n) = \int f \circ T^n \cdot f \mathrm{d}\mu - \left(\int f \mathrm{d}\mu\right)^2 \quad (n \geq 1).$$

Suppose $\int f \mathrm{d}\mu = 0$. It is easy to see that $|R_f(n)| \leq \|L_g^n f\|_\infty$. So, the above the estimate for $\|L_g^n f\|_\infty$ gives us

THEOREM 4. – *We have:*

(1) *If* $\mathrm{var}_n(f) = \mathrm{O}(\alpha^n)$ *and* $\mathrm{var}_n(\log g) = \mathrm{O}(\beta^n)$ $(0 < \alpha,\ \beta < 1)$, *then* $R_f(n) = \mathrm{O}(\delta^n)$ *(for some* $0 < \delta < 1$*).*
(2) *If* $\mathrm{var}_n(f) = \mathrm{O}(n^{-s})$ *and* $\mathrm{var}_n(\log g) = \mathrm{O}(n^{-t})$ $(s > 0, t > 1)$, *then* $R_f(n) = \mathrm{O}(n^{-\min(s,t-1)})$.
(3) *If* $\mathrm{var}_n(f) = \mathrm{O}(\alpha^{(\log n)^p})$ *and* $\mathrm{var}_n(\log g) = \mathrm{O}(\alpha^{(\log n)^p})$ $(0 < \alpha < 1,\ p > 1)$, *then* $R_f(n) = \mathrm{O}(n(\log n)^{-p+1}\alpha^{(\log n)^p})$.
(4) *If* $\mathrm{var}_n(f) = \mathrm{O}(\alpha^{n^\beta})$ *and* $\mathrm{var}_n(\log g) = \mathrm{O}(\alpha^{n^\beta})$ $(0 < \alpha < 1,\ 0 < \beta < 1)$, *then* $R_f(n) = \mathrm{O}(n^{1-\beta}\alpha^{n^\beta})$.

3. Sketch of proofs

To prove Theorem 1, it suffices to follow [4] (*see* also [3]). The proof of Theorem 2 consists of several lemmas. The first one, which is a key point, is an elementary inequality.

LEMMA 1. – *Let* $0 < a < b < \infty$ *be two constants. There exists a constant* $0 < \gamma = \gamma(a, b) < 1$ *such that the inequality*

$$\left|\sum_{j=1}^{n} \alpha_j x_j\right| \leq \gamma \sum_{j=1}^{n} |\alpha_j| x_j$$

holds for any two sequences $\{\alpha_j\}$ *and* $\{x_j\}$ *of real numbers satisfying the following conditions* $\sum_{j=1}^{n} \alpha_j = 0$, $a \leq x_j \leq b (1 \leq j \leq n)$.

Let $\mathcal{B}_n$ be the σ-algebra on X_A generated by the cylinders of length n. Let $E_n = \mathbf{E}(\cdot|\mathcal{B}_n)$ be the conditional expectation with respect to $\mathcal{B}_n$ on the probability space (X_A, μ).

LEMMA 2. – *There exists a constant* $0 < \gamma < 1$ *depending on potentials such that for any* $f \in \mathrm{L}^\infty(\mu)$ *with* $\int f(x)\mathrm{d}\mu(x) = 0$, *we have*

$$\|P_n E_n f\|_\infty \leq \gamma \|f\|_\infty.$$

Proof. – Let $x|^n = x_{n+1}x_{n+2}\cdots$. Write

$$P_n E_n f(x) = \sum_c G_n(cx|^n) \frac{1}{\mu(I(c))} \int_{I(c)} f \mathrm{d}\mu$$

where c are sequences in $S_1 \times \cdots \times S_n$ such that $cx|^n \in X_A$, $I(c)$ is the cylinder determined by c. Then it suffices to use Lemma 1 and Theorem 1 (d). □

The following lemma is obtained by a direct estimation and Lemma 4 is obvious.

LEMMA 3. – *If* $m > n$, *we have*

$$\mathrm{var}_m(P_n f) \leq \|f\|_\infty V_{n,m} + \mathrm{var}_m(f).$$

In particular, $\mathrm{var}_m(P_n f) \leq \|f\|_\infty V_{n,m}$ *if* f *is* $\mathcal{B}_n$*-measurable.*

LEMMA 4. – *For any* $1 \le p \le \infty$ *we have* $\|(I - E_n)f\|_p \le \mathrm{var}_n(f)$.

Prove now Theorem 2. It can be checked that the sequence of averaging operators $\{P_n\}$ satisfies the relation $P_m P_n = P_n P_m = P_m$ for $1 \le n \le m$. It follows that, if Q_n denotes $P_n E_n$, we have for $N \ge n$,

$$P_N = P_N[(I - E_n) + Q_n].$$

By induction, we have

$$P_N = P_N\left[(I - E_{n_1}) + \sum_{j=2}^{k-1}(I - E_{n_j})\prod_{i=1}^{j-1} Q_{n_i} + \prod_{i=1}^{k} Q_{n_i}\right].$$

By using the obvious fact that P_N is a contraction on $\mathrm{L}^\infty(\mu)$, Lemma 4, Lemma 3 and Lemma 2 we have

$$\begin{aligned}
\|P_N f\|_\infty &\le \|(I - E_{n_1})f\|_\infty + \sum_{j=2}^{k-1}\left\| (I - E_{n_j}) + \prod_{i=1}^{j-1} Q_{n_i} f \right\|_\infty + \left\| \prod_{i=1}^{k} Q_{n_i} f \right\|_\infty \\
&\le \mathrm{var}_{n_1}(f) + \|f\|_\infty \sum_{j=2}^{k-1} \mathrm{var}_{n_j}\left(\prod_{i=1}^{k} Q_{n_i} f\right) + \gamma^k \|f\|_\infty \\
&\le \mathrm{var}_{n_1}(f) + \|f\|_\infty \left[\sum_{j=2}^{k-1} V_{n_{j-1},n_j} + \gamma^k\right]. \qquad \square
\end{aligned}$$

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