The Lefschetz fixed point theorem and its application to asymptotic fixed point theorem for set-valued mappings

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Abstract. On a subset of Banach space which is a locally finite union of closed, convex sets, we extend the Lefschetz fixed point theorem for set-valued mappings. As an application of this result we give a partial answer to Nussbaum's conjecture for set-valued mappings.

Mathematics Subject Classification (2010). Primary 47H10, 47H08, 54C60; Secondary 58C30, 55M20, 54C55.

Keywords. Fixed point, Condensing map, Lefschetz number.

1. Introduction

The famous Schauder fixed point theorem [22] has been generalized in various directions by using different methods, see [4, 5, 6, 7, 8, 10, 17, 14, 20, 21] and reference therein. Closely related with a generalization of that theorem there are long-standing conjectures in the fixed point theory which were formulated by R. D. Nussbaum in 1972 in [18] and which read as follows:

Conjecture 1.1. Let G be a closed, bounded convex set in a Banach space X and $f : G \to G$ a continuous map. Assume that there exists an integer $N \geq 2$ such that $f^N(G)$ is compact. Then f has a fixed point.

Conjecture 1.2. Let G be a closed, bounded convex set in a Banach space X and $f : G \to G$ a continuous map. Assume that there is an integer $N \geq 2$ such that $f^N(G)$ is a k-set-contraction with $k < 1$. Then f has a fixed point.

R. D. Nussbaum proved this conjecture with the additional assumption that the restriction of f on an appropriate open set is continuously Fréchet differentiable. Recently, R. Mallet-Paret and D. Nussbaum in [14] proved these conjectures in the presence of two other additional assumptions. Using methods of algebraic topology, they proved a series of asymptotic fixed point theorems for mappings defined on a subset of a Banach space that is locally

finite union of closed convex sets.
For set-valued mappings by using methods of algebraic topology, L. Górniewicz [9], extended Schauder fixed point theorem for class of non-compact set-valued mappings defined on metric absolute neighborhood retracts space X. Here, we extend the Lefschetz fixed point theorem for set-valued mappings on a subset of Banach space which is a locally finite union of closed convex sets. Also, we give a partial answer to Nussbaum's conjecture on a subset of Banach space which is a locally finite union of closed convex sets.

2. Topological and homological preliminaries

In this section, we introduce some definitions and facts which will be used in the sequel. All topological spaces are assumed to be metric. The set-valued map $\varphi : X \multimap Y$ is said to be:

(i) upper semicontinuous, if for each closed set $B \subseteq Y$, $\varphi^-(B) = \{x \in X : \varphi(x) \cap B \neq \emptyset\}$ is closed in X.
(ii) lower semicontinuous if for each open set $V \subseteq Y$, $\varphi^-(V) = \{x \in X : \varphi(x) \cap V \neq \emptyset\}$ is open in X.
(iii) continuous if it is both upper and lower semicontinuous.
(iv) uniformly Hausdorff upper semicontinuous if and only if

$$\forall A \text{ closed}, \ \forall \varepsilon > 0 \ \exists \delta > 0, \quad \varphi(N_\delta(A)) \subset N_\varepsilon(\varphi(A)),$$

where $N_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\}$.
Suppose that $(Y, \| \ \|)$ is a normed linear space and X is a Banach subset of Y. We shall say that $X \in \mathcal{F}$ if there exists a locally finite covering $\{C_j \mid j \in J\}$ of X by closed, convex sets $C_j \subset X$. Thus, $X = \cup_{j\in J} C_j$ and for each $x \in X$ there exists an open neighborhood W of $x \in X$ such that $W \cap C_j$ is empty except for a finitely many j. We shall write $X \in \mathcal{F}_0$ if there exist closed, convex sets $C_j \subset X$, for $1 \leq j \leq n$, with $n < \infty$, and $X = \cup_{j=1}^n C_j$.
A space X is an absolute neighborhood retract ANR if given any metric space M, a closed subset $A \subset M$ and a continuous map $f : A \to X$, there exists an open neighborhood U of A and a continuous map $F : U \to X$ such that $F(a) = f(a)$ for $a \in A$; X is called an absolute retract AR if F as above can defined on all of M. A theorem of Dugundji [8] asserts that any finite union of closed, metrizable convex sets in a locally convex space is an ANR. Also if there exist integers m and N such that $K_n = \bigcup_{j=1}^N C_{j,n}$ for all $n \geq m$, where $C_{j,n}$ is closed, metrizable convex set in a locally convex space with $C_{j,n} \supset C_{j,n+1}$ for all $n \geq m$ and $1 \leq j \leq N$. Then $K_\infty := \bigcap_{n\geq 1} K_n = \bigcup_{j=1}^N C_{j,\infty}$ is an ANR, where $C_{j,\infty} := \bigcap_{n\geq m} C_{j,n}$.
Let $f : E \to E$ be an endomorphism of an arbitrary vector space E. Denote by $f^{(n)} : E \to E$ the n-th iterate of f and observe that the kernels, $Kerf \subset Kerf^{(2)} \subset ... \subset Kerf^{(n)} \subset ...$, form an increasing sequence of subspaces of E. Let us now put $N(f) = \bigcup_n Kerf^{(n)}$ and $\tilde{E} = E/N(f)$. Clearly, f maps $N(f)$ into itself and therefore induces the endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$ on the factor space $\tilde{E} = E/N(f)$.

Let $f : E \to E$ be an endomorphism of a vector space E. Assume that $\dim \tilde{E} < \infty$, in this case we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$.
A graded vector space $E = \{E_q\}$ in the category $\mathcal{A}$ is said to be of finite type provided:

(i) $\dim E_q < \infty$, for all q and
(ii) $E_q = 0$, for almost all q,

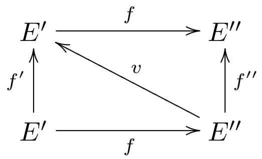
where the category $\mathcal{A}$ is the category of graded vector spaces over $\mathbb{Q}$ and linear maps of degree zero.
Let $f = \{f_q\}$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We say that f is a Leray endomorphism provided that the graded vector space $\tilde{E} = \{\tilde{E}_q\}$ is of finite type. For such an f we define the (generalized) Lefschetz number $\Lambda(f)$ of f by putting

$$\Lambda(f) = \sum_q (-1)^q Tr(f_q).$$

Definition 2.1. A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$, there exists n_x such that $f^{n_x}(x) = 0$.

Proposition 2.2. [12] *Any weakly nilpotent endomorphism $f : E \to E$ of a graded vector space E is a Leray endomorphism and $\Lambda(f) = 0$.*

Proposition 2.3. [9] *Assume that in the category of graded vector spaces the following diagram commutes*



If one of f' or f'' is a Leray endomorphism, then so is the other and $\Lambda(f') = \Lambda(f'')$.

Let H_* be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus, for a pair (X, X_0),

$$H_*(X, X_0) = \{H_q(X, X_0)\},$$

is a graded vector space and for $f : (X, X_0) \to (Y, Y_0)$, we have $H_*(f)$ to be the induced linear map:

$$H_*(f) = f_* = \{f_{*q}\} : H_*(X, X_0) \to H_*(Y, Y_0),$$

where $f_{*q} : H_q(X, X_0) \to H_q(Y, Y_0)$.
A space X is acyclic if

(i) X is nonempty,

(ii) $H_q(X) = 0$ for every $q \geq 1$,
(iii) $H_0(X) \approx \mathbb{Q}$.

A continuous mapping $f : X \to Y$ is called proper if for every nonempty compact set $K \subset Y$, the set $f^{-1}(K)$ is nonempty and compact. A proper map $p : X \to Y$ is called Vietoris provided that for every $y \in Y$ the set $p^{-1}(y)$ is acyclic.

Definition 2.4. A compact valued upper semicontinuous map $\varphi : X \multimap Y$ is called acyclic provided for every $x \in X$ the set $\varphi(x)$ is acyclic.

Proposition 2.5. [9] *If $\varphi : X \multimap Y$ is an acyclic map, then the natural projection $p_\varphi : \Omega = \mathrm{graph}(\varphi) \to \mathrm{X}$, defines as $p_\varphi(x, y) = x$, is a Vietoris map.*

The importance of the Čech homology functor with compact carriers is evident in the following Vietoris Mapping.

Theorem 2.6. [3] *(Vietoris Mapping Theorem). If $p : (X, X_0) \Rightarrow (Y, Y_0)$ is a Vietoris map, then $p_* : H_*(X, X_0) \to H_*(Y, Y_0)$ is an isomorphism.*

Assume $p, q : Y \to X$. are two mappings. We shall say that p and q have a coincidence provided there exists a point $x \in X$ such that $p(x) = q(x)$.

Theorem 2.7. [9] *Consider a diagram:*

$$X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X,$$

in which $X \in \mathrm{ANR}$, p is Vietoris and q is compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* p_*^{-1}) \neq 0$ implies that p and q have a coincidence, where $\Lambda(q_* p_*^{-1})$ denotes the (generalized) Lefschetz number of $q_* \circ p_*^{-1}$.

Definition 2.8. [9] A set-valued map $\varphi : X \multimap Y$ is called admissible (strongly admissible) provided there exist a (metric) space Ω and two mappings $p : \Omega \to X,\ q : \Omega \to Y$ such that:

(i) p is a Vietoris map,
(ii) $q(p^{-1}(x)) \subset \varphi(x)$ $(q(p^{-1}(x)) = \varphi(x))$ for every $x \in X$.

Note that any acyclic map $\varphi : X \multimap Y$ is strongly admissible. In fact, it is enough to take $\Omega = \mathrm{graph}(\varphi)$ and $p = p_\varphi,\ q = q_\varphi$, where $p_\varphi(x, y) = x$ and $q_\varphi(x, y) = y$.

Definition 2.9. [9] An admissible map $\varphi : X \multimap X$ is called a Lefschetz map provided the linear map $q_* \circ p_*^{-1} : H(X) \to H(X)$ is a Leray endomorphism for every selected pair $(p, q) \subset \varphi$.

For a Lefschetz map $\varphi : X \multimap X$, we define the Lefschetz set $\mathbf{\Lambda}(\varphi)$ of φ by putting:

$$\mathbf{\Lambda}(\varphi) = \{\Lambda(q_* p_*^{-1}) |\ (p, q) \subset \varphi\ \},$$

where $\Lambda(q_* p_*^{-1})$ denotes the generalized Lefschetz number of $q_* \circ p_*^{-1}$. Note that if φ is an acyclic Lefschetz map, then $\mathbf{\Lambda}(\varphi) = \{\Lambda(\varphi)\} = \{\Lambda(q_* p_*^{-1})\}$ is a singleton.

Proposition 2.10. [10] *Let* $\varphi : (X, X_0) \multimap (X, X_0)$ *be an admissible map of pairs and* $(p, q) \subset \varphi(X)$*. If any two of the endomorphisms* $q''_* p''^{-1}_* : H(X, X_0) \to H(X, X_0)$, $q_* p_*^{-1} : H(X) \to H(X)$, $\bar{q}_* \bar{p}_*^{-1} : H(X_0) \to H(X_0)$ *are Leray endomorphisms, then so is the third and*

$$\Lambda(q''_* p''^{-1}_*) = \Lambda(q_* p_*^{-1}) - \Lambda(\bar{q}_* \bar{p}_*^{-1}).$$

Definition 2.11. [10] An admissible map $\varphi : X \multimap X$ is called a compact absorbing contraction (written $\varphi \in CAC(X)$) provided there exists an open set $U \subset X$ such that:

(i) $\overline{\varphi(U)} \subset U)$ is a compact subset of U;
(ii) for every $x \in X$ there exists $n = n_x$ such that $\varphi^n(x) \subset U$.

We shall also need some facts about the measure of noncompactness Kuratowski and Hausdorff. Let X be a Banach space and A be a bounded subset of X. Then the Kuratowski measure of non-compactness of A is defined as

$$\gamma(A) = \inf\{\delta > 0 | \ A = \cup_{i=1}^n A_i, \mathrm{diam(A_i)} \leq \delta\},$$

where $\mathrm{diam(A)} = \sup\{d(x, y) | x, y \in A\}$ and the Hausdorff measure of non-compactness of A is defined as

$$\chi(A) = \inf\{\delta > 0 | \ A = \cup_{i=1}^n B(x_j, \delta), x_j \in X\},$$

where $B(x_j, \delta)$ denotes the ball in X with center x_j and radius δ. We have $\gamma(A) = 0$ or $\chi(A) = 0$ if and only if $\overline{A}$ is compact. If (A_n) is a decreasing sequence of closed bounded nonempty subsets of a Banach space X and if $\lim_{n\to\infty} \gamma(A_n) = 0$, then Kuratowski [13] proved that $A = \bigcap_{n\in\mathbb{N}} A_n$ is compact and nonempty.
Suppose that $\varphi : X \multimap X$ is a set-valued map. Consider the following types of γ-set contraction:

- φ is said to be γ-k-set contraction if there exists $k \in [0, 1)$ such that $\gamma(\varphi(A)) \leq k\gamma(A)$ for each $A \subseteq X$.

- φ is said to be condensing if $\gamma(\varphi(A)) < \gamma(A)$ for all bounded subsets A of X for which $\gamma(A) > 0$.

Remark 2.12. It is clear if φ is γ-k-set contraction, then φ is condensing map. If φ is compact valued, upper semicontinuous and condensing map, then by a similar proof as that of Lemma 1.6.11 in [1], we get that

$$\lim_{n\to\infty} \gamma(\varphi^n(X)) = 0.$$

Example 2.4 in [2] shows the existence of a set-valued map which $\lim_{n\to\infty} \gamma(\varphi^n(X)) = 0$, but it's not a condensing map.

3. Main results

In this section, we extend the Lefschetz fixed point theorem for noncompact set-valued mappings on a subset of a Banach space which is a locally finite union of closed, convex sets. Moreover, a partial answer to Nussbaum's conjecture on a subset of Banach space which is a locally finite union of closed convex sets is given. We begin by recalling the following definition.

Definition 3.1. Let X be a Hausdorff topological space and $\varphi : X \multimap X$ be an upper semicontinuous map. We call a compact, nonempty set $\Gamma \subset X$ a compact attractor for φ if

(1) $\varphi(\Gamma) \subset \Gamma$; and
(2) given any open neighborhood U of Γ and any compact set $A \subset X$, there exists an integer $n = n(U, A)$ with $\varphi^m(A) \subset U$ for all $m \geq n$.

Lemma 3.2. *Let X be a Banach space and $\varphi : X \multimap X$ be a compact valued continuous map. Assume that there exists a compact set $\Gamma \subset X$ which is a compact attractor for φ. Then for any open neighborhood W of Γ there exists an open neighborhood G of Γ and $\overline{G} \subset W$ such that $\varphi(G) \subset G$. Furthermore, suppose that there exists a bounded open neighborhood V of Γ such that $\lim_{n\to\infty} \gamma(\varphi^n(V)) = 0$. Then given any open neighborhood O of Γ there exists an open neighborhood U of Γ and $\overline{U} \subset O$ such that $\overline{\varphi(U)} \subset U$.*

Proof. By some minor modifications in the proof of Lemma 5 and Lemma 9 in [17], we can obtain the conclusion for φ. □

The following result is an extension of the Lefschetz fixed point theorem.

Theorem 3.3. *Let $X \in \mathcal{F}$ and V be an open subset of X. Suppose that $\varphi : V \multimap V$ is an admissible continuous compact valued map which has a compact attractor $\Gamma \subset V$. Assume that there exist a bounded open neighborhood W of Γ in X with $\overline{W} \subset V$ and a decreasing sequence of sets $K_n \in \mathcal{F}_0$ with $K_n \subset X$ for $n \geq 1$, where $K_n = \bigcup_{j=1}^{N} D_{j,n}$ for all $n \geq m$ such that $D_{j,n}$ is closed convex sets of X with $D_{j,n} \supset D_{j,n+1}$ for all $n \geq m$ and $1 \leq j \leq N$ and*

(1) $W \subset K_1$;
(2) $\varphi(W \cap K_n) \subset K_{n+1}$ *for all* $n \geq 1$;
(3) $\lim_{n\to\infty} \gamma(K_n) = 0$, *where γ denotes the Kuratowski measure of noncompactness on X.*

Then, φ is Lefschetz and the condition $\mathbf{\Lambda}(\varphi) \neq 0$ implies φ has a fixed point.

Proof. Since φ is an admissible map, then there exist a space Ω and two mappings p, q such that we have following diagram

$$V \overset{p}{\Longleftarrow} \Omega \overset{q}{\longrightarrow} V,$$

that p is a Vietoris map and $q(p^{-1})(x) \subset \varphi(x)$ for every $x \in V$.
Lemma 3.2 implies that there exists an open neighborhood G of Γ such that $\overline{G} \subset W$ and $\varphi(G) \subset G$. We have $\varphi^n(G) \subset K_{n+1} \cap G$ and since $\gamma(K_n) \longrightarrow 0$, so $\gamma(\varphi^n(G)) \longrightarrow 0$. Therefore, by Lemma 3.2 there exists an open neighborhood U of Γ, $\overline{U} \subset G$ such that $\overline{\varphi(U)} \subset U$. It follows from (2) that $\varphi(U \cap K_n) \subset$

$U \cap K_{n+1}$ for all $n \geq 1$.
Let $\varphi' : U \multimap U$ be the restriction of φ to U, then we define similarly, $q', p' : p^{-1}(U) \to U,\ p'(u) = p(u),\ q'(u) = q(u)$ for each $u \in U$. Observe that $(p', q') \subset \varphi'$.
We show that there exists a decreasing sequence of sets with $A_n \subset X$, for $n \geq 1$ and an integer $m \geq 1$ such that

(4) $U \subset A_1$;
(5) $\varphi'(U \cap A_n) \subset A_{n+1}$ for all $n \geq 1$;
(6) $\lim_{n\to\infty} \gamma(A_n) = 0$;
(7) $A_n \subset U$ for all $n \geq m+1$.

First since $\lim_{n\to\infty} \gamma(K_n) = 0$, so $\bigcap_{n=1}^{\infty} K_n = K_\infty$ is nonempty and compact. Since φ' is continuous and compact valued, then $\varphi'(\overline{U} \cap K_\infty)$ is a compact subset of $U \cap K_\infty$. Then, there exists $\delta > 0$ such that $N_\delta(N_\delta(\varphi'(\overline{U} \cap K_\infty))) \cap K_\infty \subset U \cap K_\infty$. Thus, there exists an integer m such that

$$N_\delta(\varphi'(U \cap K_n)) \cap K_n \subset U \cap K_n \tag{3.1}$$

for $n \geq m+1$ and $\gamma(K_n) < \frac{1}{2}\delta$.
As φ' is upper semicontinuous and compact valued, so the restriction of φ' to K_∞ is uniformly Hausdorff upper semicontinuous. Then, for $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\varphi'(N_\varepsilon(\overline{U} \cap K_\infty)) \subset N_\delta(\varphi'(\overline{U} \cap K_\infty)).$$

Also, there exists $m_0 \in \mathbb{N}$ such that for $n > m_0$, $\overline{U} \cap K_n \subset N_\varepsilon(\overline{U} \cap K_\infty)$, so

$$\begin{aligned} N_\delta(\varphi'(U \cap K_n)) \cap K_n &\subset N_\delta(\varphi'(\overline{U} \cap K_n)) \cap K_n \\ &\subset N_\delta(\varphi'(N_\varepsilon(\overline{U} \cap K_\infty))) \cap K_n \\ &\subset N_\delta(N_\varepsilon(\varphi'((\overline{U} \cap K_\infty)))) \cap K_n \\ &\subset N_\delta(N_\delta(\varphi'((\overline{U} \cap K_\infty)))) \cap K_n \\ &\subset U \cap K_n. \end{aligned}$$

Moreover, for $\delta > 0$ there exists m_1 such that for $n > m_1$ and $\gamma(K_n) < \frac{1}{2}\delta$. If $m = \max\{m_0, m_1\}$, define $A_j = K_j$ for $1 \leq j \leq m$. Since $\gamma(K_{m+1}) < \frac{1}{2}\delta$, cover $\varphi'(U \cap K_m)$ by a finite number of closed, convex sets $C_j,\ 1 \leq j \leq p$, such that (i) $C_j \subset K_{m+1}$ for $1 \leq j \leq p$, (ii) $\delta(C_j) < \frac{1}{2}\delta$ for $1 \leq j \leq p$ and (iii) each C_j has nonempty intersection with $\varphi'(U \cap K_m)$. By equation (3.1) and condition (ii) and (iii), we have $C_j \subset U$. Define $A_{m+1} = \cup_{j=1}^{p} C_j$ and notice that our construction implies that

$$\varphi'(U \cap K_m) \subset A_{m+1} \subset U \cap K_{m+1},$$

so that $\varphi'(A_{m+1}) \subset A_{m+1}$. For $n \geq m+1$, define $A_n = A_{m+1} \cap K_n$. It is easy to check that the A_n forms a decreasing sequence, $A_n \in \mathcal{F}_0$, $\varphi'(U \cap A_n) \subset A_{n+1}$, $A_n = U \cap A_n$, $Fix(p', q') \subset A_n$ for every n and $\lim_{n\to\infty} \gamma(A_n) = 0$, where $Fix(p', q') = \{x \in U :\ x \in q'(p'^{-1}(x))\}$. So $\cap_{n=1}^{\infty} A_n = A_\infty$, $\varphi'(A_\infty) \subset A_\infty$. We consider the pair $(\tilde{p}, \tilde{q})$:

$$A_\infty \overset{\tilde{p}}{\Longleftarrow} p^{-1}(A_\infty) \overset{\tilde{q}}{\longrightarrow} A_\infty,$$

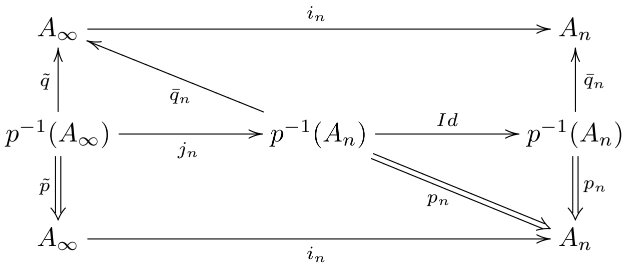
where $\tilde{p}(u) = p(u)$ and $\tilde{q}(u) = q(u)$. $\tilde{p}$ is a Vietoris map, $\tilde{q}$ is compact and A_∞ is ANR. Therefore, by Theorem 2.7 $\tilde{q}_* \circ \tilde{p}_*^{-1}$ is a Leray endomorphism. We show that $\Lambda(\tilde{q}_*\tilde{p}_*^{-1}) = \Lambda(q_*p_*^{-1})$.

First for $n \geq m+1$, we consider the pair (p_n, q_n) :

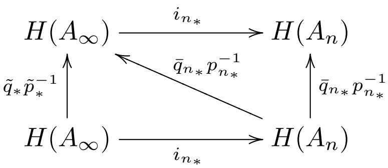
$$A_n \stackrel{p_n}{\Longleftarrow} p^{-1}(A_n) \stackrel{q_n}{\longrightarrow} A_n,$$

where $p_n(u) = p(u)$ and $q_n(u) = q(u)$.
Now, by definition of K_n and A_n, we have $A_n = \cup_{i=1}^m F_{i,n}$ is a union of m bounded closed convex subset of X such that $F_{i,n} \supset F_{i,n+1}$ for $1 \leq i \leq m$, $1 \leq n \leq \infty$ and $\lim_{n\to\infty}(\gamma(A_n)) = 0$, then Corollary 2 of [16] implies that there exists $n_0 \geq 1$ such that for $n \geq n_0$ there exists a retraction $R_n : A_n \to A_\infty$ such that $R_n(x) \in F_{i,n} \cap A_\infty$ if $x \in F_{i,n}$. For $n \geq \max\{n_0, m\}$, we define $\bar{q}_n$ on $Y = p^{-1}(A_n)$ by $\bar{q}_n = R_n(q_n)$. Since $R_n(x) \in F_{i,n} \cap A_\infty$, if $x \in F_{i,n}$, then $(1-t)\bar{q}_n + tq_n \in A_n$ for $0 \leq t \leq 1$. Therefore, $\bar{q}_n$ is homotopic to q_n ($\bar{q}_n \sim q_n$) and $\bar{q}_{n_*} = q_{n_*}$.

Let $i_n : A_\infty \to A_n$ be the inclusion map. Of course the following diagram is commutative:



Consequently, its image under H is also a commutative diagram:



Since $\tilde{q}_* \circ \tilde{p}_*^{-1}$ is a Leray endomorphism, then by Proposition 2.3 $\bar{q}_{n_*} \circ p_{n_*}^{-1}$ is Leray endomorphism and because $\bar{q}_{n_*} = q_{n_*}$ we obtain

$$\Lambda(\tilde{q}_*\tilde{p}_*^{-1}) = \Lambda(q_{n_*}p_{n_*}^{-1}).$$

On the other hand, Γ is a compact attractor and $\Gamma \subset U$, so for any compact set K of U there exists an integer q such that $[\varphi']^q(K) \subset U$. Since $\varphi'(U) \subset U$, $\varphi'(U) \subset U \cap A_2$ and $\varphi'(U \cap A_2) \subset U \cap A_2$, then for any compact set K of U, $[\varphi']^{q+2}(K) \subset U \cap A_2$. Repeating this argument and using the fact that $\varphi'(U \cap A_n) \subset U \cap A_{n+1}$ and $\varphi'(U \cap A_{n+1}) \subset U \cap A_{n+1}$, we see that for $n \geq 1$ and for any compact set K of U

$$[\varphi']^{q+n+1}(K) \subset U \cap A_{n+1}.$$

However, if $n \geq m+1$, $U \cap A_n = A_n$, then for any compact set K of U

$$[\varphi']^{q+n+1}(K) \subset A_{n+1}, \quad \text{for } n \geq m+1. \tag{3.2}$$

Let $n \geq N$, we consider the homeomorphism

$$q''_* \circ p''^{-1}_* : H(U, A_n) \to H(U, A_n)$$

(where $p'', q'' : (\Omega, p^{-1}(A_n)) \to (U, A_n)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$). Since we consider Čech homology functor with compact carriers, hence equation (3.2) implies that $q''_* \circ p''^{-1}_*$ is weakly nilpotent. Then, by Proposition 2.2 $q''_* \circ p''^{-1}_*$ is a Leray endomorphism and $\Lambda(q''_* p''^{-1}_*) = 0$. Since $q_{n_*} \circ p^{-1}_{n_*}$ is Leray endomorphism, then Proposition 2.10 implies that $q_* \circ p^{-1}_*$ is Leray endomorphism and

$$\Lambda(q_* p^{-1}_*) = \Lambda(q_{n_*} p^{-1}_{n_*}) = \Lambda(\tilde{q}_* \tilde{p}^{-1}_*).$$

If $\mathbf{\Lambda}(\varphi) \neq 0$, then $\Lambda(q_* p^{-1}_*) \neq 0$. Therefore, $\Lambda(\tilde{q}_* \tilde{p}^{-1}_*) \neq 0$ and Theorem 2.7 implies that $\tilde{p}$ and $\tilde{q}$ have a coincidence point u, i.e. $\tilde{p}(u) = \tilde{q}(u)$. Thus, $x = \tilde{p}(u)$ is a fixed point for pair $(\tilde{p}, \tilde{q})$. Since $\tilde{q} \circ \tilde{p}^{-1}(A_\infty) \subset A_\infty$, so we get $Fix(p, q) = Fix(\tilde{p}, \tilde{q})$.

□

Proposition 3.4. *Suppose that $X \in \mathcal{F}$ and $\varphi \in CAC(X)$, then φ satisfies in condition Theorem* 3.3. *Therefore, φ is Lefschetz and the condition $\mathbf{\Lambda}(\varphi) \neq 0$ implies φ has a fixed point.*

Proof. Let $U \subset X$ be an open set as Definition 2.11. First we prove that $\overline{\varphi(U)}$ is compact attractor of φ. It's clear $\varphi(\overline{\varphi(U)}) \subset \overline{\varphi(U)}$. Suppose that A is a compact subset of X. From Remark 3.2 of [19], there exists $n \in \mathbb{N}$ such that $\varphi^n(A) \subset U$. Therefore, $\varphi^m(A) \subset U$ for all $m \geq n$. If V is an open neighborhood of $\overline{\varphi(U)}$, then

$$\varphi^{m+1}(A) \subset \varphi(U) \subset \overline{\varphi(U)} \subset V,$$

for all $m \geq n$. Hence, $\overline{\varphi(U)}$ is a compact attractor of φ.
Since $\overline{\varphi(U)} \subset U$ is compact, there exists $\varepsilon > 0$ such that $W = N_\varepsilon(\overline{\varphi(U)})$ is a bounded open neighborhood of $\overline{\varphi(U)}$ and $\overline{W} \subset U$. We define a decreasing sequence $\{K_n\}$ of closed and convex subsets of X by putting $K_1 = \overline{co}(W)$, $K_2 = \overline{co}(\varphi(W))$ and $K_n = \overline{co}(\varphi(W \cap K_{n-1}))$ for $n \geq 3$. It is clear that $W \subset K_1$ and $\varphi(W \cap K_n) \subset K_{n+1}$ for all $n \geq 1$. Since φ_U is compact, $\lim_{n\to\infty} \gamma(K_n) = 0$. Thus, φ satisfies the conditions of Theorem 3.3. Therefore, φ is Lefschetz and the condition $\mathbf{\Lambda}(\varphi) \neq 0$ implies φ has a fixed point. □

Recently, D. O'regan in [18] presented new Lefschetz fixed point results for compact absorbing contraction in extension type spaces with respect to the map. Also, M. Ślosarski generalized the class of admissible mapping and proved Lefschetz fixed point theorem for compact absorbing contraction in new class of admissible mapping; see [23, 24]. Here, we obtain a Lefschetz fixed point theorem for admissible maps, where $X \in \mathcal{F}$.

Theorem 3.5. *Suppose that $X \in \mathcal{F}$, U is an open subset of X and $\varphi : U \multimap U$ is an admissible continuous compact valued map. Assume that there exists a sequence (K_j) of subsets of X such that*

(1) *$K_j \subset U$ and $K_j \in \mathcal{F}_0$ for all $j \geq 1$;*
(2) *$\varphi(K_j) \subset K_j$ for all $j \geq 1$;*
(3) *for every compact set $A \subset U$ and every $j \geq 1$, there exists an integer $\nu = \nu(A, j)$ with $\varphi^{\nu}(A) \subset K_j$;*
(4) *$\lim_{j\to\infty} \gamma(K_j) = 0$.*

Then $\Gamma := \cap_{j\geq 1} K_j$ is nonempty compact and a compact attractor for φ. Moreover, φ is Lefschetz and if $\mathbf{\Lambda}(\varphi) \neq 0$, then φ has a fixed point.

Proof. Since φ is an admissible map, then there exist a space Ω and two mappings p, q such that we have following diagram

$$V \overset{p}{\Longleftarrow} \Omega \overset{q}{\longrightarrow} V,$$

that p is a Vietoris map and $q(p^{-1})(x) \subset \varphi(x)$ for every $x \in V$.
By a similar proof as that of Theorem 4.18 of [14], Γ is a compact attractor for φ. For $k \geq 1$, we define $C_k := \cap_{j=1}^{k} K_j$ so $C_k \supset \Gamma$ and (C_k) is a decreasing sequence of closed bounded subsets of U. Since $\gamma(C_k) \leq \gamma(K_k)$, then $\lim_{k\to\infty} \gamma(C_k) = 0$. Also, we have $C_k \in \mathcal{F}_0$ and $\varphi(C_k) \subset C_k$ for all $k \geq 1$.
We consider the pair (p_k, q_k) as:

$$C_k \overset{p_k}{\Longleftarrow} p_k^{-1}(C_k) \overset{q_k}{\longrightarrow} C_k,$$

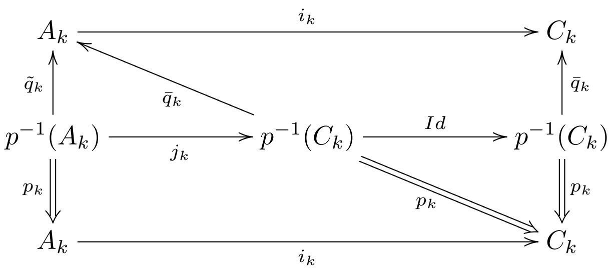
where $p_k(u) = p(u)$ and $q_k(u) = q(u)$.
Let $\gamma(C_k) < \delta_k$, $\lim_{k\to\infty} \delta_k = 0$ and $C_k = \cup_{i=1}^{n} D_{i,k} \in \mathcal{F}_0$. We consider map $q_k : p_k^{-1}(C_k) \to C_k$. Since $\gamma(C_k) < \delta_k$, we can write $q_k p_k^{-1}(C_k) = \cup_{j=1}^{m} T_{j,k}$ where $\mathrm{diam}(T_j) < \delta_k$ for $1 \leq j \leq m$. We have the inclusion of $q_k p_k^{-1}(C_k) \subset E$, where $E \in \mathcal{F}_0$ and

$$E = \cup_{i=1}^{n} \cup_{j=1}^{m} \overline{co}(D_{i,k} \cap T_{j,k}).$$

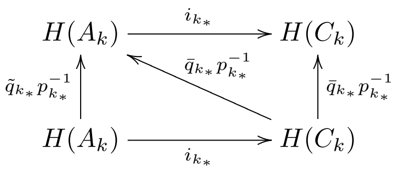
Now, Corollary 2.4 of [14] implies that there exists a finite dimensional linear subspace $X_{0,k}$ of X, a compact set $A_k \in \mathcal{F}_0$ with $A_k \subset E \cap X_{0,k}$ and a continuous retraction $R_k : E \to A_k$ of E onto A_k such that for all i, j, we have $R_k(y) \in \overline{co}(D_{i,k} \cap T_{j,k})$ for all $y \in \overline{co}(D_{i,k} \cap T_{j,k})$. We define the continuous map $\bar{q_k} : p_k^{-1}(C_k) \to A_k$ as $\bar{q_k}(x) = R_k(q_k)(x)$ for all $x \in p_k^{-1}(C_k)$. So we have

(5) $\|q_k(x) - \bar{q_k}(x)\| \leq \delta_k$ for all $x \in p^{-1}(C_k)$;
(6) if $q_k(x) \in D_{i,k}$ for some $x \in p^{-1}(C_k)$ and some i, then $\bar{q_k} \in D_{i,k}$.

By condition (6), $(1-t)\bar{q}_k + t q_k \in C_k$ for $0 \leq t \leq 1$. Therefore, $\bar{q}_k$ is homotopic to q_k ($\bar{q}_k \sim q_k$) and $\bar{q}_{k_*} = q_{k_*}$.
Let $i_k : A_k \to C_k$ be the inclusion map. Of course the following diagram is commutative:



where $\tilde{q}_k(x) = \bar{q}_k(x), p_k(x) = p(x)$.
Consequently, its image under H is also a commutative diagram:



Now, it follows from Theorem 2.7 that $\tilde{q}_{k_*} \circ p_{k_*}^{-1}$ is a Leray endomorphism. So, by Proposition 2.3, $\bar{q}_{k_*} \circ p_{k_*}^{-1}$ is a Leray endomorphism and because $\bar{q}_{k_*} = q_{k_*}$, we obtain:

$$\Lambda(\tilde{q}_{k_*} p_{k_*}^{-1}) = \Lambda(\bar{q}_{k_*} p_{k_*}^{-1}) = \Lambda(q_{k_*} p_{k_*}^{-1}). \tag{3.3}$$

On the other hand, condition (3) imolies that if B is a compact subset of U and $k \geq 1$, then there exists an integer $n = n(B, k)$ such that

$$\varphi^n(B) \subset C_k. \tag{3.4}$$

Let $k \geq 1$, we consider the homeomorphism

$$q''_* \circ p''^{-1}_* : H(U, C_k) \to H(U, C_k)$$

(here $p'', q'' : (\Omega, p^{-1}(C_k)) \to (U, C_k)$ is given by $p''(u) = p(u)$ and $q''(u) = q(u)$. Since we consider Čech homology functor with compact carriers and equation (3.4) implies that $q''_* \circ p''^{-1}_*$ is weakly nilpotent. Then, by Proposition 2.2, $q''_* \circ p''^{-1}_*$ is a Leray endomorphism and $\Lambda(q''_* p''^{-1}_*) = 0$. Since $q_{k_*} \circ p_{k_*}^{-1}$ is Leray endomorphism, then by Proposition 2.10, $q_* \circ p_*^{-1}$ is Leray endomorphism and

$$\Lambda(q_{k_*} p_{k_*}^{-1}) = \Lambda(q_* p_*^{-1}). \tag{3.5}$$

Now, let us assume that $\mathbf{\Lambda}(\varphi) \neq 0$ so $\Lambda(q_* p_*^{-1}) \neq 0$. Then by (3.3) and (3.5), $\Lambda(\tilde{q}_{k_*} p_{k_*}^{-1}) \neq 0$, therefore, Theorem 2.7 implies that we deduce that

$$p_k(y_k) = \tilde{q}_k(y_k).$$

Let $x_k = p_k(y_k) = \tilde{q}_k(y_k)$. We put $q(y_k) = \bar{x}_k$. Since $x_k \in C_k$ and $\gamma(C_k) \leq \gamma(K_k) \to 0$, then Kuratowski Theorem [13] implies that (x_k) has a convergent subsequence $x_{k_i} \to x \in \Gamma$. We have $\|x_{k_i} - \bar{x}_{k_i}\| = \|\tilde{q}_k(y_k) - q(y_k)\| < \delta_k$ and since $\lim_{k\to\infty} \delta_k = 0$, hence $\lim_{k\to\infty} x_{k_i} = x$. Therefore, $x \in q(p^{-1}(x))$. □

As a corollary of Theorem 3.5, we obtain a version of Frum-Ketkov type Theorem in our context similar to Corollary 4.19 in [14].

Corollary 3.6. *Suppose that $X \in \mathcal{F}$, U is an open subset of X and $\varphi : U \multimap U$ is an admissible continuous map. Assume that there exists a compact, nonempty set $\Gamma \subset U$ which is a compact attractor for φ. Assume that there exist sequences r_k and s_k, for $k \geq 1$, with $0 \leq s_k < r_k$ for all $k \geq 1$, with $\lim_{k\to\infty} r_k = 0$ and $\varphi(N_{r_k}(\Gamma)) \subset N_{s_k}(\Gamma)$ and $N_{r_k}(\Gamma) \subset U$ for all $k \geq 1$. Then there exists a sequence K_j, for $j \geq 1$, of subspaces of X which meet conditions (1)-(4) of Theorem 3.5 and also satisfy $N_{s_j}(\Gamma) \subset K_j \subset N_{r_j}(\Gamma)$ for $j \geq 1$. It follows that φ satisfies all of the conclusions of Theorem 3.5. In particular, if $\mathbf{\Lambda}(\varphi) \neq 0$, then φ has a fixed point. Hence, if U is acyclic, then $\mathbf{\Lambda}(\varphi) = 1$ and φ has a fixed point.*

Proof. By a similar proof as that of Corollary 4.19 of [14], we obtain the conclusion for compact valued continuous maps. □

We need the following result on the sequence.

Lemma 3.7. [1] *Let R_n, $n \in \mathbb{N}$, be a subset of the bounded set Q in the metric space (X, ρ) and let $\mathcal{U}$ denote the family of the set A that are representable in the form $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is finite and $A_n \subset R_n$ for all n. Then there is an $A^* \in \mathcal{U}$ such that*

$$\chi(A^*) = sup\{\chi(A) : A \in \mathcal{U}\}. \tag{3.6}$$

Proposition 3.8. *Let X be a Banach space and G be a open bounded subset of X. Suppose that $\varphi : G \multimap G$ is condensing, continuous and compact valued and there exists an open subset U of G such that $\overline{\varphi(U)} \subset U$. We define a decreasing sequence $\{K_n\}$ of closed, bounded and convex subsets of G by putting $K_1 = \overline{co}(U)$, $K_2 = \overline{co}(\varphi(U))$ and $K_n = \overline{co}(\varphi(U \cap K_{n-1}))$ for $n \geq 3$. Then $\lim_{n\to\infty} \chi(K_n) = 0$.*

Proof. We set $R_n = K_n$, $n \in \mathbb{N}$ and $Q = \overline{co}(U) \cup \overline{co}\varphi(U)$. Clearly, $R_n \subset \overline{co}\varphi(U)$ and since φ is condensing so Q is bounded. Thus, all of the conditions of Lemma 3.7 are satisfied, so there is a set $A^* \in \mathcal{U}$ such that (3.6) holds. By some modifications in the proof Theorem 1.6.12 [1], we have $\chi(A^*) = 0$. Suppose now that

$$\chi(R_n) \nrightarrow 0. \tag{3.7}$$

Then there are a positive ε_0 and an infinite increasing sequence of positive integers n_k, $k \in \mathbb{N}$, such that $\chi(R_{n_k}) > 0$ for each $k \in \mathbb{N}$. Fix an arbitrary element $x_1 \in R_{n_1}$ and choose $x_2 \in R_{n_2}$ such that $\|x_1 - x_2\| \geq \varepsilon_0$. The existence of such an element x_2 follows from the fact that equation (3.7) means, in particular, that the set R_{n_2} has no finite ε_0-net in X. For the same reason, $\{x_1, x_2\}$ cannot be an ε_0-net for R_{n_3}, hence, there is an $x_3 \in R_{n_3}$ whose distance to $\{x_1, x_2\}$ is not smaller than ε_0. Continuing in this manner, we produce a sequence $\{x_{n_k}\}$ which, is necessarily totally bounded (since $A = \{x_{n_k} : k \in \mathbb{N}\}$ belong to $\mathcal{U}$) and on the other hand, is not totally bounded since distance between any its element is not smaller than ε_0. Thus, $\chi(R_n) \to 0$ so $\chi(K_n) \to 0$. □

Corollary 3.9. *Let X be a Banach space and G be bounded closed convex subset of X. Suppose that $\varphi : G \multimap G$ is a compact convex valued continuous map. Assume that there exists an integer $p \geq 1$ such that φ^p is condensing map. Then set $\Gamma := \cap_{j\geq 1}\overline{\varphi^j(G)}$ is a compact attractor for φ. Furthermore, if φ satisfies one of the following hypotheses:*

(i) *there is an open neighborhood W of Γ in X such that $\varphi|_{W\cap G}$ is condensing map;*

(ii) *there is a compact set M with $M \subset \Gamma \subset G$ and sequences of reals r_k and s_k, for $k \geq 1$, with $0 \leq s_k < r_k$ for all $k \geq 1$, with $\lim_{k\to\infty} r_k = 0$ and $\varphi(N_{r_k}(M)) \subset N_{s_k}(M)$ and $N_{r_k}(M) \subset G$ for all $k \geq 1$.*

Then φ has a fixed point.

Proof. First, since φ is compact convex valued then φ is acyclic and so φ is strongly admissible. Also G is acyclic, then $\mathbf{\Lambda}(\varphi) = 1$. By a similar proof as that of Proposition 4.4 of [14], $\Gamma := \cap_{j\geq 1}\overline{\varphi^j(G)}$ is a compact attractor for φ. If (i) holds, by Remark 2.12 and Lemma 3.2, there exists an open neighborhood U of Γ such that $U \subset W \cap G$ and $\overline{\varphi(U)} \subset U$. Now, we define a decreasing sequence $\{K_n\}$ of closed bounded and convex subsets of G by putting $K_1 = \overline{co}(U)$, $K_2 = \overline{co}(\varphi(U))$ and $K_n = \overline{co}(\varphi(U \cap K_{n-1}))$ for $n \geq 3$. It is clear that $U \subset K_1$ and $\varphi(U \cap K_n) \subset K_{n+1}$ for all $n \geq 1$. Also, by Proposition 3.8, $\lim_{n\to\infty} \gamma(K_n) = 0$. Therefore, φ satisfies the conditions of Theorem 3.3, so φ has a fixed point.
If (ii) holds, Corollary 3.6 implies that φ has a fixed point. □

Acknowledgment

The authors would like to thank Professor Górniewicz for his helpful comments concerning this paper. The first author was partially supported by a grant from IPM (No. 91550414).

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