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Abstract. Here, we show that continuous set-valued maps which are generalized set contraction on noncompact topological spaces have a maximal invariant (fixed) set. As an application, we prove the existence and uniqueness of endpoints for topological contraction mappings. Also, we present fractal set results for system of continuous set-valued maps on regular topological spaces. As application of our result, we show how some fixed point theorems can be established from these results.
Keywords: Fixed set; Endpoint; Topological contraction; Measure of noncompactness; Generalized μ-set contraction; Fractal set; Fixed point.

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1. Introduction and preliminaries

Let X be a nonempty set and $T : X \multimap X$ be a set-valued map with nonempty values. A subset A of X is said to be a fixed set (or fractal or module set or invariant set) of T, if $T(A) = A$. Fixed set theory has significant applications in iterated functions systems, fractals, dynamical systems, game theory and fixed point theory; for more details see [5, 6, 11, 14, 16, 23, 24].
Most of the results of fixed sets are established by the well known order-theoretic fixed point theorems and metric fixed point theory. In this work, we first obtain a maximal compact fixed set for a continuous set-valued mapping in noncompact regular topological spaces. As applications of this result we establish a topological multivalued fractal, in the sense of Andres and Fišer [5] and some fixed point theorems.
Let us introduce some definitions and facts which will be used in the sequel. Let X and Y be topological spaces, a set-valued map $T : X \multimap Y$ is said to be:

(i) upper semicontinuous, if for each closed set $B \subseteq Y$, $T^{-}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is closed in X or equivalently for each open set $U \subseteq Y$, $T^{+}(U) = \{x \in X : T(x) \subseteq U\}$ is open in X,
(ii) lower semicontinuous if for each open set $V \subseteq Y$, $T^{-}(V) = \{x \in X : T(x) \cap V \neq \emptyset\}$ is open in X.
(iii) continuous if it is both upper and lower semicontinuous.

Let X be a topological space and $(\mathcal{C}, \tau)$ be a topological lattice with minimal element which is denoted by 0. Suppose that $\mathcal{B}$ is a collection of nonempty subsets of X such that $\overline{A}, A \cup B \in \mathcal{B}$ for any $A, B \in \mathcal{B}$. A measure of noncompactness on X with respect to $\mathcal{B}$ is simply any functional $\mu : \mathcal{B} \to \mathcal{C}$ such that:
(i) $\mu(\overline{A}) = \mu(A)$ for all $A \in \mathcal{B}$;
(ii) $\mu(A) = 0$ if and only if A is relatively compact;
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \mathcal{B}$.

It follows immediately that if $A \subseteq B$, then $\mu(A) \leq \mu(B)$. A sequence $(A_n)_{n=1}^{\infty}$ of nonempty subsets in $\mathcal{B}$ is called *μ-descending*, if A_n is closed, $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(A_n) = 0$. We say that μ has the Kuratowski property, if the intersection $A = \bigcap_{n\in\mathbb{N}} A_n$ is nonempty and compact for any μ-descending sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{B}$.

In the special case, the Kuratowski [resp. Hausdorff] measure of noncompactness on the complete metric spaces has the Kuratowski property [1]. For some other examples of measure of noncompactness with the Kuratowski property one can refer to [1, 3, 12, 21].

Definition 1.1. Let X be a topological space and $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0. Suppose that $\mathcal{B}$ is a collection of nonempty subsets of X such that $\overline{A}, A \cup B \in \mathcal{B}$ for any $A, B \in \mathcal{B}$. If $\mu : \mathcal{B} \to \mathcal{C}$ is a measure of noncompactness, then a set-valued map $T : X \multimap X$ is said to be a generalized μ-set contraction with respect to (w.r.t) $\mathcal{B}$, if $\lim_{n\to\infty} \mu(T^n(A)) = 0$ for every $A \in \mathcal{B}$.

Remark 1.2. Let (M, d) be a bounded metric space and γ be the Kuratowski or the Hausdorff measure of noncompactness. Assume that $T : M \multimap M$ is a set-valued map with nonempty values. We say that T is a condensing map if for every $A \subseteq M$ with $\gamma(A) > 0$,

$$\gamma(T(A)) < \gamma(A).$$

If T is upper semicontinuous and compact valued, then by a similar proof as that of Lemma 1.6.11 in [1], we get that

$$\lim_{n\to\infty} \gamma(T^n(M)) = 0.$$

Therefore, T is a generalized γ-set contraction w.r.t 2^M.
Example 2.4 in [3] shows the existence of a set-valued map which is generalized γ-set contraction w.r.t 2^M but it's not condensing map.

2. Fixed set Theorems

In this section, we present fixed set result for continuous set-valued maps on noncompact topological spaces. As application, we obtain an existence result of endpoint for the class of topological contraction set-valued mappings on regular topological spaces. Also, we present fractal set results for system of continuous set-valued maps on regular topological spaces.

Theorem 2.1. *Let X be a regular topological space, $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0 and $\mu : 2^X \to \mathcal{C}$ be a measure of noncompactness on X with the Kuratowski property. Suppose that $T : X \multimap X$ is a continuous set-valued map with closed values. If T is a a generalized μ-set contraction w.r.t 2^X, then T has maximal nonempty compact fixed set.*

Proof. Let $X_n := \overline{T^n(X)}$ for all $n \in \mathbb{N}$, then $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$. Since T is a generalized μ-set contraction w.r.t 2^X, then $\mu(X_n) = \mu(T^n(X)) \to 0$. Therefore, $K := \cap_{n=0}^{\infty} X_n$ is nonempty and compact. By lower semicontinuity of T, we get $T(\overline{T^n(X)}) \subseteq \overline{T(T^n(X))}$. Thus, $T(K) \subseteq K$. On the other hand by Theorem 17.10 of [2], $T(K)$ is closed. Therefore, $T(K)$ is compact. If $T(K) \neq K$, then there is an open neighborhood U of $T(K)$ such that $K \nsubseteq U$. Since X is regular and $T(K)$ is compact, then there is an open neighborhood U_1 of $T(K)$ such that $T(K) \subset U_1 \subset \overline{U_1} \subset U$. If $V = T^+(U_1)$, then $K \subseteq V$ and V is open (note that T is upper semicontinuous). We show that there is an integer $N_V = N$ such that $X_N \subset V$. Assume on the contrary that $X_n \cap V^c \neq \emptyset$, for every n. The sequence $(X_n \cap V^c)$ is decreasing and $\mu(X_n \cap V^c) \to 0$. So by the Kuratowski property of μ, $\cap_{n=0}^{\infty} X_n \cap V^c = K \cap V^c \neq \emptyset$. which is a contradiction.
If $n \geq N$, then $T(T^n(X)) \subset T(\overline{T^n(X)}) \subset T(V) \subset U_1$. Hence, $K \subset \overline{T^{n+1}(X)} \subset \overline{U_1} \subset U$, which is a contradiction. Therefore, $T(K) = K$.
To show that K is maximal fixed set, observe that if A is a compact set and $T(A) = A$, then $\overline{T^n(A)} = A$ so that $\cap_{n=0}^{\infty} \overline{T^n(A)} = A$. Since $\overline{T^n(A)} \subset \overline{T^n(X)}$, so $A \subset \cap_{n=0}^{\infty} \overline{T^n(X)} = K$. □

The hypothesis that T is a a generalized μ-set contraction w.r.t 2^X can not be relaxed as the following example shows:

Example 2.2. Let $X = c_0$ be the null sequences space equipped with its canonical norm

$$\|(x_n)\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\} \quad \forall (x_n) \in c_0.$$

Let B be the closed unit ball of c_0 and $T : X \multimap X$ be defined as $T(x) = B$. It's clear that T is continuous but it's not generalized γ-set contraction w.r.t 2^X, where γ is the Kuratowski measure of noncompactness. Here, B is a fixed set of T which is not compact.

A set-valued map T on a topological space X is said to be a topological contraction if for any compact subset A of X with $T(A) = A$ we have that A is a singleton; see [28]. An element $x \in X$ is called an endpoint of T, if $T(x) = \{x\}$. In [4, 7, 8, 9, 13, 26, 28, 29, 30, 31, 32, 33, 34, 35] some results for the existence of endpoints in complete metric spaces and complete uniform spaces are presented. As a consequence of the above theorem, we obtain the following result.

Corollary 2.3. *Let X be a regular topological space, $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0 and $\mu : 2^X \to \mathcal{C}$ be a measure of noncompactness on X with the Kuratowski property. Suppose that $T : X \multimap X$ is a continuous set-valued map with closed values. If T is a topological contraction and a generalized μ-set contraction w.r.t 2^X, then T has a unique endpoint x_0 and $\{x_0\} = \cap_{n=0}^{\infty} T^n(X)$.*

Proof. By the Theorem 2.1, there exists nonempty compact fixed set K i.e $T(K) = K$. Since T is a topological contraction, then K is singleton that is there exists $x_0 \in X$ such that $T(x_0) = \{x_0\} = K$ and so $\{x_0\} := \cap_{n=0}^{\infty} T^n(X)$. □

Now, we are going to obtain fractal set on regular topological space. In order to obtain our result, we need some more definitions.
Let X be a topological space and suppose that $\{T_i : X \multimap X,\ i = 1, \dots, n\}$ be a system of set-valued maps. We consider the Hutchinson-Barnsley map $F : X \multimap X$ generated by T_i as follows:

$$F(x) := \cup_{i=1}^{n} T_i(x) \quad x \in X$$

and the induced Hutchinson-Barnsley operator or the fractal operator $F^* : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X

$$F^*(A) := \overline{F(A)} = \overline{\cup_{i=1}^{n} T_i(A)} \quad A \in \mathcal{P}(X)$$

Fixed point of Hutchinson-Barnsley operator is called fractal set. Let X be a topological space and $\mu : 2^X \to [0, \infty]$ be a measure of noncompactness on X and $T : X \multimap X$ be a set-valued map. We say that $T : X \multimap X$ has the property (M-K) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq \mu(A) < \varepsilon + \delta$ implies that $\mu(T(A)) < \varepsilon$.

Theorem 2.4. *Let X be a regular topological space and $\mu : 2^X \to [0, \infty]$ be a measure of noncompactness on X with the Kuratowski property. Suppose that $\{T_i : X \multimap X,\ i = 1, \dots, n\}$ is a system of continuous set-valued maps with closed values such that $\mu(T_i(X)) < \infty$ for every $i = 1, \dots, n$. If T_i has the property (M-K) for each $i = 1, \dots, n$. Then there exists a maximal, nonempty, compact, invariant set $A^* \subset X$ w.r.t. the Hutchinson-Barnsley map F, i.e. a maximal topological multivalued fractal.*

Proof. Since T_i has the property (M-K) and $\mu(T_i(X)) < \infty$ for every $i = 1, \dots, n$, then it is obvious that F has the property (M-K) and $\mu(F(X)) < \infty$. Therefore, by Lemma 2.5 of [3] F is generalized μ-set contraction w.r.t 2^X. Also F is continuous with closed values. Hence, by Theorem 2.1, the Hutchinson-Barnsley map F, has a maximal, nonempty, compact, fixed set $A^* \subset X$.

□

3. FIXED POINT THEOREMS

It is clear that if f is a single-valued self map on a space X, then every endpoint of f is a fixed point for f. Inspired by this fact and using the results of the previous section, we present a new proof of Kirk's asymptotic fixed point theorem. Moreover, a new fixed point theorem in regular topological spaces is given. The following result is similar to Corollary 2.3 under a weaker condition.

Theorem 3.1. *Let X be a regular topological space, $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0 and $\mu : 2^X \to \mathcal{C}$ be a measure of noncompactness on X with the Kuratowski property. Suppose that $f : X \to X$ is a generalized μ-set contraction w.r.t 2^X. If there exists $p \in \mathbb{N}$ such that f^p is continuous and topological contraction, then f has a unique fixed point x_0 and $\{x_0\} = \cap_{n=0}^{\infty} f^n(X)$.*

Proof. Since f is a generalized μ-set contraction w.r.t 2^X, then $\mu(f^n(X)) \to 0$. Let $X_n := \overline{f^{pn}(X)}$ for all $n \in \mathbb{N}$, then $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$ and $\mu(X_n) \leq$

$\mu(f^{pn}(X)) \to 0$. Therefore, $K := \cap_{n=0}^{\infty} X_n$ is nonempty and compact. But continuity of f^p implies that

$$f^p(\overline{f^{pn}(X)}) \subseteq \overline{f^p(f^{pn}(X))}.$$

Thus, $f^p(K) \subseteq K$. If $f^p(K) \neq K$, there is an open neighborhood U of $f^p(K)$ such that $K \nsubseteq U$. Since X is regular and $f^p(K)$ is compact, then there is an open neighborhood U_1 of $f^p(K)$ such that $f^p(K) \subset U_1 \subset \overline{U_1} \subset U$. If $V := (f^p)^{-1}(U_1)$, then V is an open neighborhood of K and $f^p(V) \subset U_1$. Therefore, there is an integer $N(V)$ such that $X_{pn} \subset V$ for all $n \geq N(V)$. On the other hand $f^p(f^{pn}(X)) \subset f^p(\overline{f^{pn}(X)}) \subset f^p(V) \subset U_1$ for any $n \geq N(V)$. Then $K \subset \overline{f^{pn+p}(X)} \subset \overline{U_1} \subset U$, which is a contradiction. Hence, $f^p(K) = K$. Since f^p is a topological contraction, then K is singleton. Therefore, there exists $x_0 \in X$ such that $f^p(x_0) = \{x_0\}$ and $\{x_0\} = \cap_{n=0}^{\infty} X_n$. Hence,

$$f^p(\{x_0\} \cup f(x_0)) = f^p(x_0) \cup f^{p+1}(x_0) = \{x_0\} \cup f(x_0).$$

But f^p is a topological contraction, then $\{x_0\} \cup f(x_0)$ is singleton and so $f(x_0) = \{x_0\}$. Now, we show that f has precisely one fixed point. Suppose that $z \in X$ is another fixed point of f. Therefore, $z \in \cap_{n=0}^{\infty} f^n(X) \subseteq \cap_{n=0}^{\infty} \overline{f^n(X)} = K$ and so $z = x_0$. Hence, f has a unique fixed point x_0 and $\{x_0\} = \cap_{n=0}^{\infty} f^n(X)$. □

Remark 3.2. The above result was obtained by Pathak and Shahzad [25] in an arbitrary complete metric space in the case where f is continuous, closed and the diameter $\delta(f^n(M)) \to 0$ as $n \to \infty$. Since the condition $\delta(f^n(M)) \to 0$ as $n \to \infty$ implies that f is a generalized γ-set contraction, where γ is the Kuratowski measure of noncompactness on M, and for each $n \in \mathbb{N}$, f^n is topological contraction, then by Theorem 3.1, f has a unique fixed point. Therefore, Theorem 3.1 extends Theorem 3.1 of [25]. In order to justify our claim, we need the following result.

Theorem 3.3. [27] (*Basic Matrix Theorem*) *Let X be a topological vector space and $x_{ij} \in X$ for $i, j \in \mathbb{N}$. Suppose*

(I) $\lim_i x_{ij} = x_j$ *exists for each j and*
(II) *for each increasing sequence of positive integers (m_j), there is a subsequence (n_j) of (m_j) such that $(\sum_{j=1}^{\infty} x_{in_j})_{i=1}^{\infty}$ is Cauchy.*

Then $\lim_i x_{ij} = x_j$ uniformly for $j \in \mathbb{N}$. In particular,

$$\lim_i \lim_j x_{ij} = \lim_j \lim_i x_{ij} = 0 \text{ and } \lim_i x_{ii} = 0.$$

A matrix which satisfies condition (I) and (II) is called a $\mathcal{K}$-matrix.

Example 3.4. Let $B = \{x \in \ell^2 : \ \|x\| \leq 1\}$ and $f : B \to B$ be defined as $f(x) = (x_3, 0, x_5, 0, \ldots)$. Let μ be a weakly measure of noncompactness. Since B is weakly compact, then $\lim_n \mu(f^n(B)) = 0$. Therefore, f is a generalized μ-set contraction. Now, we show that f is weakly topological contraction. Suppose that A is weakly compact and $f^n(A) = A$. We have

$$f^n(A) = \{(x_{2n+1}, 0, x_{2n+3}, 0, \ldots); \ (x_1, x_2, \ldots) \in A\}$$

and $\lim_n f^n(A) = A = \{\lim_n y_n, \ y_n \in f^n(A)\}$.
For y_1, there exists $x_1 = (x_{11}, x_{12}, \ldots)$ such that $y_1 = (x_{13}, 0, x_{15}, 0, \ldots)$ and for y_2, there exists $x_2 = (x_{21}, x_{22}, \ldots)$ such that $y_2 = (x_{25}, 0, x_{27}, 0, \ldots)$. By continuing this process, for every y_n there exists $x_n = (x_{n1}, x_{n2}, \ldots)$ such that $y_n = (x_{n2n+1}, 0, x_{n2n+3}, 0, \ldots)$. The sequence (x_n) belongs to A and A is weakly

compact, therefore there exists subsequence (x_{nk}) (we denote it again by (x_n)) such that weakly converges to x_0. Since f is weakly continuous, then $f(x_n)$ is weakly convergent to $f(x_0)$. Now, we consider following matrix

$$C = \begin{pmatrix} x_{13} & 0 & x_{15} & 0 & \dots \\ x_{23} & 0 & x_{25} & 0 & \dots \\ & & . & & \\ & & . & & \\ & & . & & \\ x_{n3} & 0 & x_{n5} & 0 & \dots \end{pmatrix}.$$

It is clear that the matrix C is a $\mathcal{K}$-matrix, therefore, by Theorem 3.3, the sequence (x_{nj}) is uniformly convergent to x_{0j} for $j = 3, 5, \dots$ and $\lim_n \lim_j x_{nj} = \lim_j \lim_n x_{nj} = 0$. Therefore, every column of following matrix convergent to 0,

$$D = \begin{pmatrix} x_{13} & 0 & x_{15} & 0 & \dots \\ x_{25} & 0 & x_{27} & 0 & \dots \\ & & . & & \\ & & . & & \\ & & . & & \\ x_{n2n+1} & 0 & x_{n2n+3} & 0 & \dots \end{pmatrix}.$$

Hence, the sequence (y_n) converges to 0, so $A = \{(0, 0, \dots)\}$. Thus, f is a weakly topological contraction.
On the other hand, if $\{e_n : n \in \mathbb{N}\}$ is the standard base of ℓ^2, $x = (0, 0, \dots)$ and $y = e_{2n+1}$, then

$$\|f^n(x) - f^n(y)\| = 1.$$

Therefore, $\delta(f^n(B)) \nrightarrow 0$.

Remark 3.5. In many metrically fixed point theorems, one can show that $\delta(f^n(M)) \to 0$, where (M, d) is a bounded complete metric space and f is selfmap on M with a contraction property; for example see [10, 22]. Since these mappings are topological contractions and generalized γ-set-contractions w.r.t 2^M, where γ is the Kuratowski measure of noncompactness on M, then the above example also shows that Theorem 3.1 extends and improves these results.

In 2003, Kirk [17] introduced the notion of asymptotic contraction on a metric space and proved a fixed-point theorem for such contractions.

Definition 3.6. [17] Let (M, d) be a metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then a mapping f on M is said to be an asymptotic φ-contraction if there exist a sequence $(\varphi_n)_{n=1}^\infty, \varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$, such that

(1) $$d(f^n(x), f^n(y)) \leq \varphi_n(d(x, y)) \text{ for all } x, y \in M \text{ and } n \in \mathbb{N},$$

$\varphi_n \to \varphi$ uniformly on $\mathbb{R}_+$.

Theorem 3.7. [17] *Let (M, d) be a complete metric space, $f : M \to M$ be a continuous asymptotic φ-contraction for which all functions φ_n in (1) are continuous and $\varphi(t) < t$ for all $t > 0$. Assume also that some orbit $(f^n(x_0))_{n=1}^\infty$ is bounded. Then f has a contractive fixed point.*

Jachymski and Jóźwik [15] proved a similar result to Theorem 3.7, by assuming φ is upper semicontinuous with $\varphi(t) < t$ for all $t > 0$, $\lim_{t\to\infty}(t - \varphi(t)) = \infty$, and f

is uniformly continuous. Now, we show that fixed points of asymptotic contractions can be obtained from Theorem 3.1.

Theorem 3.8. *Let (M, d) be a complete metric space, $f : M \to M$ and there exists n_0 such that $f^{n_0}(M)$ is bounded.*

Suppose for some $p \in \mathbb{N}$, f^p is continuous and asymptotic φ-contraction such that φ is upper semicontinuous and $\varphi(t) < t$ for all $t > 0$. Then f has a unique fixed point x_0 and $(f^n(x))$ converges to x_0, for each $x \in M$.

Proof. Since $\varphi_n \to \varphi$ uniformly on $\mathbb{R}_+$, then for every $k \in \mathbb{N}$, there exists N_k such that

$$\varphi_n(t) \leq \varphi(t) + \frac{1}{k} \quad \forall n \geq N_k, \quad \forall t \in \mathbb{R}_+. \tag{2}$$

If $A \in B(M)$, where $B(M)$ is the set of all nonempty bounded subsets of M, then by (1) we get that

$$\begin{aligned} \delta(f^{pn}(A)) &\leq \sup_{x,y\in A} \varphi_n(d(x,y)) \leq \sup_{x,y\in A} \varphi(d(x,y)) + \frac{1}{2} \\ &\leq \delta(A) + \frac{1}{2} < \infty; \quad \forall n \geq N_2. \end{aligned}$$

Therefore, $f^{pn}(A) \in B(M)$ for sufficiently large n. Now, we prove that

$$\lim_{n\to\infty} \delta(f^{pn}(A)) = 0 \quad \forall A \in B(M). \tag{3}$$

Assume that (3) does not hold. Thus, without loss of generality, we have

$$\exists A_0 \in B(M)\ \exists \varepsilon_0 > 0\ \exists (n_k) : n_k \geq N_k\ \forall k, \quad \lim_k \delta(f^{pn_k}(A_0)) = \varepsilon_0. \tag{4}$$

Now, we show that

$$\exists k_0, \quad \sup\{\varphi(d(x,y)),\ x,y \in f^{pn_{k_0}}(A_0)\} < \varepsilon_0. \tag{5}$$

Indeed, otherwise

$$\forall k, \quad \sup\{\varphi(d(x,y)),\ x,y \in f^{pn_k}(A_0)\} \geq \varepsilon_0. \tag{6}$$

Since φ is upper semicontinuous, by Lemma 1 in [18], there exists a continuous and strictly increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(t) \leq \psi(t) < t \text{ for all } t > 0. \tag{7}$$

Therefore, by (4), (6) and (7), we have

$$\begin{aligned} \varepsilon_0 &\leq \lim_k \sup\{\varphi(d(x,y)),\ x,y \in f^{pn_k}(A_0)\} \\ &\leq \lim_k \sup\{\psi(d(x,y)),\ x,y \in f^{pn_k}(A_0)\} \\ &\leq \lim_k \psi(\delta(f^{pn_k}(A_0))) \\ &= \psi(\lim_k \delta(f^{pn_k}(A_0))) = \psi(\varepsilon_0) < \varepsilon_0, \end{aligned}$$

which is impossible. Thus, (5) holds. Now, by using (2), (3) and (5), we get that

$$\begin{aligned}\varepsilon_0 &= \lim_k \delta(f^{pn_k}(A_0)) \\ &= \lim_k \delta(f^{p(n_k-n_{k_0})}(f^{pn_{k_0}}(A_0))) = \lim_k \delta(f^{pn_k}(f^{pn_{k_0}}(A_0))) \\ &\leq \lim_k \sup\{\varphi_{n_k}(d(x,y)),\ x,y\in f^{pn_{k_0}}(A_0)\} \\ &\leq \lim_k (\sup\{\varphi(d(x,y)),\ x,y\in f^{pn_{k_0}}(A_0)\} + \frac{1}{k}) \\ &= \sup\{\varphi(d(x,y)),\ x,y\in f^{pn_{k_0}}(A_0)\} < \varepsilon_0,\end{aligned}$$

which is also impossible. Thus, (3) holds. So f^p is a topological contraction and $\lim \gamma(f^n(A)) = 0$, where γ is the Kuratowski measure of noncompactness on M. Since there exists n_0 such that $f^{n_0}(M)$ is bounded so f is a generalized γ-set contraction w.r.t 2^M. Hence, by Theorem 3.1, f has a unique fixed point x_0 and $\cap_{n=1}^{\infty} f^n(M) = \{x_0\}$. Furthermore, $d(f^n(x), x_0) \leq \delta(f^n(M))$ for all $n \geq 1$. Therefore, since $\lim_{n\to\infty} \delta(f^n(M)) = 0$ so $(f^n(x))$ converges to x_0, for any $x \in M$. □

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