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2 authors:

M. Fakhar
University of Isfahan

Jafar Zafarani
Sheikhbahaee University and Uinversity of I…

MATCHING AND FIXED POINT THEOREMS IN L-CONVEX SPACES

M. FAKHAR AND J. ZAFARANI

ABSTRACT. The purpose of this paper is to obtain new Ky Fan type matching theorems and Fan-Browder type fixed point theorems for mappings with KKM property in L-convex spaces. Furthermore, application to these fixed point theorems for existence of solutions of generalized quasi-variational problem is given.

1. Introduction

Let X be nonempty sets, we shall denote by 2^X the family of all subsets of X, by $\mathcal{F}(X)$ family of all nonempty finite subsets of X and $|A|$ the cardinality of $A \in \mathcal{F}(X)$. Suppose that Y is a nonempty set and $F : X \to 2^Y$ is a multivalued mapping, fibers $F^-(y)$ for $y \in Y$ defined by $F^-(y) = \{x \in X : y \in F(x)\}$. For topological space X and Y, a multivalued mapping $F : X \to 2^Y$ is said to be compact if the closure $clF(X)$ of its range $F(X)$ is compact in Y. A multivalued mapping F is said to be upper semicontinuous (u.s.c.) if for each closed set $B \subseteq Y$, the set $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed subset of X.
A nonempty topological space is acyclic if all of its reduce homology groups over rational vanishes.
The following class $\mathbf{A}(X, Y)$ of approachable multivalued mappings was introduced by Ben-El-Mechaiekh *et al.* [2]. Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be uniform topological spaces with bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages for the uniformities on X and Y. For each $U \in \mathcal{U}$ and $V \in \mathcal{V}$, let

$$W = \{((x, y), (x', y')) \in (X \times Y) \times (X \times Y) : (x, x') \in U, (y, y') \in V\}.$$

Then the family $\mathcal{W} = (W)_{U\in\mathcal{U},V\in\mathcal{V}}$ is a base of symmetric entourages for the product uniformity, and the associated uniform topology on $X \times Y$ is the product of the uniform topologies on X and Y. Let $F : X \to 2^Y$ be a multivalued mapping. For given element $W \in \mathcal{W}$, a function $f : X \to Y$ is said to be a W-approximative selection of F if and only if $Gr(f) \subseteq W[Gr(F)]$, where $Gr(f)$ and $Gr(F)$ denote the graphs of f and F, respectively. A multivalued mapping F is said to be approachable if F admits a continuous W-approximative selection for each $W \in \mathcal{W}$. The class $\mathbf{A}(X, Y)$ of multivalued mappings is defined by

$$\mathbf{A}(X, Y) := \{F : X \to 2^Y : F \text{ is approachable}\}.$$

The notation of L-convex space was introduced by Ben-El-Mechaiekh *et al.*[2]. An *L-convex space* (X, Γ) consist of a topological space X and a multivalued map $\Gamma : \mathcal{F}(D) \to 2^X \setminus \{\emptyset\}$ such that for each $A \in \mathcal{F}(D)$ with the cardinality $|A| = n+1$, there exists a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that for each $J \in \mathcal{F}(A)$

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implies $\Phi_A(\Delta_J) \subset \Gamma(J)$, for which if $A = \{a_0, a_1, ..., a_n\}$ and $J = \{a_{i_0}, a_{i_1}, ..., a_{i_j}\}$, then $\Delta_J = co\{e_{i_0}, ..., e_{i_j}\}$. Suppose that (X, Γ) is a L-convex space, then $D \subseteq X$ is said to be L-convex if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subseteq D$. The *L-convex hull of D* denoted by *L-coD*, is the set $\bigcap\{B \subset X : B$ is an L-convex subset of X containing D$\}$. It is clear that each L-convex subset of L-convex space is also an L-convex space. A subset $D \subseteq X$ is said to be finitely L-closed(resp. L-open) in X if for each $A \in \mathcal{F}(X)$, $D \cap \Gamma(A)$ is closed(resp. open) in $\Gamma(A)$. D is called to be compactly closed (resp. open) in X if for any compact subset K of X, $D \cap K$ is closed(resp. open) in K.
If Γ as the above definition, verifies the additional condition: For each $A, B \in \mathcal{F}(X)$, $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$, then the pair (X, Γ) is what called by Park and Kim [13] , a G-convex spaces. Recently, Park [14] has removed the above condition and considered the G-convex space$(X, D; \Gamma)$, where D need not be a subset of X. If $D = X$, then a G-convex space in [14] is an L-convex space.
If for each $A, B \in \mathcal{F}(X)$, there exists $A_1 \subseteq A$ such that $A_1 \subseteq B$ implies $\Gamma(A_1) \subseteq \Gamma(B)$, then the pair (X, Γ) is called by Verma[15] a generalized H-space. It is clear that , the notion of L-convex spaces includes the G-convex spaces of Park and Kim [13], G-H spaces and H-spaces and therefore Hyperconvex spaces[7].
An L-convex space (X, Γ) is called a locally L-space, denoted by $(X, \Gamma, \mathcal{U})$ if X is a uniform topological space with a uniform structure $\mathcal{U}$ having a symmetric base such that for each L-convex subset $C \subseteq X$ and each $U \in \mathcal{U}$, $U[C] = \{x \in X : U[x] \cap C \neq \emptyset\}$ is L-convex, where $U[x] = \{y \in X : (x, y) \in U\}$.
Knaster, Kuratowski and Mazurkiewicz [8](simply KKM) first consider the closed valued multivalued mapping $T : D \to 2^{\Delta_n}$, where D is the set of vertices of an n-simplex Δ_n, satisfying $co(A) \subseteq T(A)$ for each $A \subseteq D$. The KKM principle implies that when T has closed(resp. open)-values, then $\cap_{x \in D} T(x) \neq \emptyset$ [14]. More generally if (X, Γ) is a L-convex space and Y is a set, then a multivalued map $T : X \to 2^Y$ is called a KKM map if for each $A \in \mathcal{F}(X)$, $\Gamma(A) \subset \bigcup_{x \in A} F(x)$. If $G : D \to 2^Y$, $F : X \to 2^Y$ are two multivalued maps such that for any $A \in \mathcal{F}(X)$ and $L \subseteq A$, $F(\Gamma(L)) \subseteq G(L)$, then G is said to be a KKM mapping respect to F. Let $F : X \to 2^Y$ be a multivalued mapping such that if $T : X \to 2^Y$ is a KKM mapping with respect to F, then the family $\{clG(x) : x \in D\}$ has the finite intersection property. In this case we say that F has the KKM property. We define

$$\mathfrak{K}(X, Y) := \{F : X \to 2^Y : F \text{ has the KKM property}\}.$$

When X is convex subset of topological vector space, the class $\mathfrak{K}(X, Y)$ was introduced and studied by Chang and Yen [3]. This concept is further extended for G-convex spaces by Lin *et al.*[10]. Ding[4] has shown that
$\mathfrak{A}_c^{\kappa}(X, Y) \subseteq \mathfrak{K}(X, Y)$, moreover, if $T \in \mathbf{A}(X, Y)$ which is u.s.c. and compact values, then $T \in \mathfrak{K}(X, Y)$.
In this paper, we obtain some new Ky Fan type matching theorems, fixed point theorems for mapping with KKM property and a generalized quasi-variational inequality as application.

2. Main results

Motivated by recent works on KKM maps and generalized KKM maps we introduce the following definition:

Definition 2.1 Let (X,Γ) be an L-convex space and Y be a subset of X, then the multivalued mapping $F : Y \to 2^X$ is called a Φ-KKM map if for each finite subset A of Y with $|A| = n+1$, we have $\Phi_A(\Delta_J) \subseteq F(J)$ for every $J \subseteq A$.
The following example shows that Φ-KKM map is more generalize than KKM map.

Example 2.2. Suppose that $Y = \mathbb{N}$, $X = [0,+\infty)$ and $\Gamma : \mathcal{F}(Y) \to 2^X \setminus \{\emptyset\}$ is defined by $\Gamma(A) = [a,+\infty)$ for which $a = \min A$. If $\Phi_A(\Delta_n) = coA$ for every $A \in \mathcal{F}(Y)$ with the cardinality $|A| = n+1$, then it is clear that $\Phi_A : \Delta_n \to \Gamma(A)$ is continuous and so (X,Γ) is a L-convex space. Also $\Phi_A(\Delta_n) \neq \Gamma(A)$ for every $A \in \mathcal{F}(Y)$. If $F(x) = [0,x]$ for all $x \in Y$, then F is a Φ-KKM map and is not a KKM map.
We introduce now the notation of finitely Φ-closed (resp. open) set in an arbitrary L-convex space.
Definition 2.3. Let D be a nonempty set of (X,Γ). D is called finitely Φ-closed (resp. open) if for every finite subset A of X with $|A| = n+1$ $D \bigcap \Phi_B(\Delta_n)$ is closed (resp. open) in $\Phi_B(\Delta_n)$.
Clearly, if D is closed (resp. open), then D is finitely Φ-closed (resp. open), and more generally if D is finitely L-closed(resp. open) valued [5], then F is finitely Φ-closed(resp. open) valued. Therefore the concept of finitely Φ-closed(resp. open) sets generalize that of finitely close (resp. open) in topological vector spaces of and in hyperconvex spaces of Kirk-Sims-Yuan [9]. Furthermore, the notion of finitely Φ-closed (resp. open) improves the notion of compactly closed(resp. open).
The following Ky Fan's type lemma is essential in our work.

Lemma 2.4. Let Y be a nonempty subset of (X,Γ). Assume that $F : Y \to 2^X$ has finitely Φ-closed (resp. Φ-open)valued and F is a Φ-KKM map, then the class $\{F(y) : y \in Y\}$ has the finite intersection property.
Proof. Suppose that $A = \{y_0, y_1, ...y_n\} \in \mathcal{F}(Y)$. Then by the definition of Φ-KKM, for each $\{i_0,...,i_j\}$ we have $\Phi_A(\Delta_J) \subseteq (\bigcup_{k=0}^{j} F(y_{i_k})) \bigcap \Phi_A(\Delta_n)$. Therefore, $co(e_{i_0},...e_{i_j}) \subseteq \bigcup_{k=0}^{j} \Phi_A^{-1}(F(y_{i_k}) \bigcap \Phi_A(\Delta_n))$ and $\Phi_A^{-1}(F(y_{i_k}) \bigcap \Phi_A(\Delta_n))$ is closed (resp. open) in Δ_n. Consequently, the map $e_i \to \Phi_B^{-1}(F(y_i) \bigcap \Phi_A(\Delta_n))$ from $\{e_0, e_1, ..., e_n\}$ to Δ_n is a KKM map, thus by the KKM principle, $\bigcap_{i=0}^{n} \Phi_A^{-1}(F(y_i) \bigcap \Phi_B(\Delta_n)) \neq \emptyset$, hence $\Phi_A(\Delta_n) \bigcap (\bigcap_{i=0}^{n} F(y_i)) \neq \emptyset$.

In the following theorem we unify and improve theorems 3.1 and 3.2 of Ding [5] for matching theorem.

Theorem 2.5. Let (X,Γ) be a L-convex space and $Y \subseteq X$. Suppose that $S : Y \to 2^X$ has finitely Φ closed(resp. open)-values such that for a finite subset $M = \{x_0,...,x_n\} \subset Y$ we have $X = \cup_{i \in M} S(x_i)$. Then there exists $\{x_{i_0},...,x_{i_j}\} \subset M$ such that:

$$\Gamma(\{x_{i_0},...,x_{i_j}\} \cap \bigcap_{k=0}^{j} S(x_{i_k}) \neq \emptyset.$$

Proof. Define a set-valued map $T : M \to 2^X$ by $T(x) = X \backslash S(x)$. Then T is finitely Φ-open (resp. Φ-closed) values. Our hypothesis on S implies that $\cap_{x \in M} T(x) = X \setminus \cup_{x \in M} S(x) = X \backslash X = \emptyset$ and therefore, $\{T(x) : x \in M\}$ has not the finite intersection property. Thus by lemma 2.4, T can not be a Φ-KKM map. Hence there exists a nonempty finite subset $J = \{x_{i_0}, ..., x_{i_j}\} \subseteq M$ such that $\Phi_A(\Delta_J) \not\subseteq \bigcup_{k=0}^{j} T(x_{i_k})$. Consequently,

$$\Gamma(\{x_{i_0}, ..., x_{i_j}\} \cap \bigcap_{k=0}^{j} S(x_{i_k}) \neq \emptyset.$$

As a consequence of theorem 2.5 we have a general form of the Fan-Browder theorem for L-convex spaces. This theorem improves theorem 3.3 of Ding [5] and theorem 4.1 of Park [11].

Theorem 2.6. Let (X, Γ) be an L-convex space and $Y \subseteq X$, $S : Y \to 2^X$, $T : Y \to 2^X$ two multivalued maps satisfying:

(1) S is finitely Φ-closed (resp. open) values,
(2) $X = S(M)$ for some finite subset M of Y,
(3) for each $x \in Y$, any finite subset $A \subseteq S^-(x)$, we have $\Gamma(A) \subseteq T^-(x)$.

Then T has a fixed point.

Proof. By theorem 2.5 there exist a nonempty finite subset $A = \{x_0, ..., x_n\} \subseteq M$ such that $\Gamma(A) \cap \bigcap_{k=0}^{n} S(x_k) \neq \emptyset$. Now take any $x \in \Gamma(A) \cap \bigcap_{k=0}^{n} S(x_k)$, then $x_k \in S^-(x)$, for each $k = 0, ..., n$. Hence by condition (3), $x \in \Gamma(A) \subseteq T^-(x)$ and therefore, x is a fixed point for T.

Motivated by the concept of Ψ-condensing map for topological vector spaces, we define this notion in the similar way for L-convex spaces. Let (X, Γ) be a L-convex space and C be a lattice with least element which is denoted by 0. A function $\Psi : 2^X \to C$ is called a measure of compactness on X provided that the following conditions hold for any $Z_1, Z_2 \in 2^X$:

(1) $\Psi(Z_1) = 0$ if and only if Z_1 is relatively compact.
(2) $\Psi(cl(\text{L-con}Z_1)) = \Psi(Z_1)$, where $cl(\text{L-con}Z_1)$ denotes the closed L-convex hull of Z_1.
(3) $\Psi(Z_1 \cap Z_2) = \max\{\Psi(Z_1), \Psi(Z_2)\}$.

If $F : X \to 2^X$, then F is called Ψ-condensing if, whenever $\Psi(Z) \leq \Psi(F(Z))$ for $Z \subset X$, then Z is relatively compact.

In order to obtain our result for Ψ-condensing map, we need the following lemma. The proof of this lemma is similar to that for topological vector spaces.

Lemma 2.7. Let(X, Γ) be an L-convex space and Z be a closed L-convex subset of X. If $T : Z \to 2^Z$ is Ψ-condensing map, then there exists a nonempty compact L-convex subset $K \subset Z$ such that $T(K) \subset K$.

Proof. Let x_0 be an element of Z and consider the family $\mathfrak{C}$ of all closed L-convex subsets C of Z such that $x_0 \in C$ and $T(x) \subseteq C$ for each $x \in C$. Clearly, $\mathfrak{C}$ is nonempty. Let $C_0 = \cap_{C \in \mathfrak{C}} C$. Then C_0 is nonempty, closed, L-convex and $x_0 \in C_0$. If $x \in C_0$, then $T(x) \subseteq C$ for all C so $T : C_0 \to 2^{C_0}$. Now we prove C_0 is also compact. Suppose that C_0 were not compact. Since T is Ψ-condensing,

$\Psi(T(C_0)) \not\geq \Psi(C_0)$. Let $C_1 = cl(\text{L-co}(\{x_0\} \bigcup T(C_0))$. Then $C_1 \subseteq C_0$, which implies that $T(C_1) \subseteq T(C_0) \subseteq C_1$. Hence $C_1 \in \mathfrak{C}$ so that $C_0 \subseteq C_1$. Therefore, $C_0 = C_1$. But then $\Psi(C_0) = \Psi(C_1) = \Psi[cl(\text{L-co}(\{x_0\} \bigcup T(C_0))] = \Psi(T(C_0))$ which is a contradiction. Thus, C_0 must be compact.

Theorem 2.8. Let (X, Γ) be a L-convex space provided uniform topology $\mathcal{U}$ such that for every $U \in \mathcal{U}$ and every $x \in X$, $U[x]$ be L-convex. Then any u.s.c. closed valued Ψ-condensing map $T \in \mathfrak{K}(X, X)$ has a fixed point.

Proof. By lemma 2.7, there is a nonempty compact L-convex subset K of X such that $T(K) \subset K$. Since $T \in \mathfrak{K}(X, X)$, it is easy to see that $T|_K \in \mathfrak{K}(K, K)$ is u.s.c and closed valued. Hence an appeal to theorem 3.2 of Ding [4] completes the proof.

Here we obtain a matching theorem similar to theorem 7 of Chang and Yen [3] for locally L-convex spaces.

Theorem 2.9. Let $(X; \Gamma, \mathcal{U})$ be a compact locally L-convex space such that $\Gamma(A)$=$L\text{-}coA$ and $\Gamma(A)$ is compact for each $A \in \mathcal{F}(X)$.Suppose that $S : X \to 2^X$ has closed valued such that for a finite subset $M = \{x_0, ..., x_n\} \subset X$, we have $X = \cup_{i \in M} S(x_i)$ and $T \in \mathfrak{K}(X, X)$ is u.s.c. and closed values. Then there exists $\{x_{i_0}, ..., x_{i_j}\} \subset M$ such that:

$$T(\Gamma(\{x_{i_0}, ..., x_{i_j}\}) \cap \bigcap_{k=0}^{j} S(x_{i_k}) \neq \emptyset.$$

Proof. For each $x \in X$, let $I(x) = \{k : x \in S(x_k)\}$, then $I(x) \neq \emptyset$ for each $x \in X$ since $X = \cup_{i \in M} S(x_i)$. Define $F : X \to 2^X$ by $F(x) = \Gamma(I(x))$. One can show that F is an u.s.c. map. Hence F is u.s.c. map which has L-convex compact values. Hence by proposition 3.9 of [2] $F \in \mathbf{A}(X, X)$. Now by a similar method as it is given in the proof of corollary 6 of [3], we can show that $TF \in \mathfrak{K}(X, X)$ which is u.s.c. and closed values. Now by theorem 2.8, TF has a fixed point x_0. Therefore,

$$x_0 \in T(\Gamma(I(x_0)) \cap (\bigcap_{k \in I(x_0)} S(x_k).$$

As an application of our results, here we obtain a solution for generalized quasi-variational inequality in L-convex spaces.

Theorem 2.10. Let $(X; \Gamma)$ be a locally L-convex space such that $\Gamma(\{x\}) = \{x\}$ for each $x \in X$. Suppose that Y is a Hausdorff topological space, $T : X \to 2^Y$ is an u.s.c. multivalued mapping with nonempty compact values and $S : X \to 2^X$ is an u.s.c. with closed values and compact. Assume that $\phi : X \times Y \times X \to \mathbb{R}$ is a l.s.c. function such that:

(1) for each $A \in \mathcal{F}(X)$ and $x \in \Gamma(A)$, there exist $z \in A$ and $y \in T(x)$ such that $\phi(x, y, z) \leq 0$,
(2) for each $x \in X$, there exists $M_x \in \mathcal{F}(X)$ such that for each $y \in X \setminus S(x)$ there exists a point $z \in M_x$ such that $\inf_{x' \in T(y)} \phi(y, x', z) > 0$,
(3) for each $x \in X$, the set

$$\{x^* \in S(x) : \inf_{y \in T(x^*)} \phi(x^*, y, z) \leq 0\}$$

is L-convex,

(4) for each $(x,z) \in X \times X$, the set $\{y \in T(x) : \phi(x,y,z) \leq 0\}$ is acyclic.

(5) for each $(x,y) \in X \times Y$, the set $\{z \in X : \phi(x,y,z) > 0\}$ is L-convex.

Then there exist $x_0 \in S(x_0)$, $y_0 \in T(x_0)$ such that $\phi(x_0, y_0, x) \leq 0$ for all $x \in X$.

Proof. Assume that $x, z \in X$ and $f(z,x) = \inf_{y \in T(x)} \phi(x,y,z)$, then by proposition 3.1.21 of [1] $f(z,x)$ is l.s.c. in x. If $G : X \to 2^X$ defined by $G(z) = \{x : f(z,x) \leq 0\}$, then G has closed valued and condition (1) implies that G is a Φ-KKM. Moreover condition (2) implies that for each $x \in X$ there exists $M_x \in \mathcal{F}(X)$ such that $\cap_{z \in M_x} G(z) \subseteq S(x)$. Therefore by lemma 2.4 we have $\cap_{z \in X} G(z) \neq \emptyset$ for any $x \in X$. Hence for each $x \in X$, there is a point $x^* \in S(x)$ such that

$$f(z, x^*) = \inf_{y \in T(x^*)} \phi(x^*, y, z) \leq 0 \ \forall z \in X.$$

Let $F : X \to 2^X$ is defined as the following

$$F(x) = \{x^* \in S(x) : f(z, x^*) \leq 0 \text{ for all } z \in X\},$$

then $F(x) \neq \emptyset$ for all $x \in X$. It is clear that F is u.s.c., closed valued and compact. Also by condition (3) $F(x)$ is L-convex for all $x \in X$. Hence by the same argument as that in the proof of theorem 2.9 $F \in \mathbf{A}(X,X)$. Thus by lemma 3.2 of Ding[4], $F \in \mathfrak{K}(X,X)$. Hence by theorem 2.8, F has a fixed point $x_0 \in F(x_0)$, i.e. $x_0 \in S(x_0)$ and $f(z, x_0) \leq 0$ for each $z \in X$. Now suppose that $H(x) = \{y \in T(x_0) : \phi(x_0, y, x) \leq 0\}$, then it is clear that H is u.s.c. with closed and acyclic valued thus by lemma 3.1 (2) $H \in \mathfrak{K}(X, T(x_0))$. If the conclusion is false, then for each $y \in T(x_0)$, there exists $z \in X$ such that $\phi(x_0, y, z) > 0$. Let $H'(y) = \{x \in X : \phi(x_0, y, x) > 0\}$, then $H' : T(x_0) \to 2^X$ is multivalued with nonempty L-convex values and for each $x \in X$, $H'^{-}(x)$ is open in $T(x_0)$. Hence by theorem 2.1 of [6] H' has a continuous selection h. Since $hH \in \mathfrak{K}(X,X)$ and hH is u.s.c, closed values and compact, then by theorem 2.8 hH has a fixed point $x^* \in X$. Thus there exists a point $y_0 \in T(x_0)$ such that $\phi(x_0, y_0, x^*) \leq 0$ and $x^* = h(y_0) \in H'(y_0)$. So $\phi(x_0, y_0, x^*) > 0$. This is a contradiction, therefore the conclusion of theorem holds.

References

[1] J. P. Aubin and I. Ekeland, *Applied nonlinear analysis*, Wiely, New York, 1984.

[2] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J-L. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–150.

[3] T. H. Chang and C. L. Yen, *KKM proprty and fixed point theorems*, J. Math. Anal. Appl. **203** (1996), 224–235.

[4] X. P. Ding, *Abstract convexity and generalizations of Himmelberg type fixed-point theorems*, J. computers Math. Applic. **41** (2001), 497–504.

[5] X. P. Ding, *Generalized L-KKM type theorems in L-convex spaces with applications*, Computers Math. Appl. **43** (2002), 1249–1256.

[6] X. P. Ding and J. Y. Park, *Continuous selection theorem, coincidence theorem, and generalized equilibrium in L-convex spaces*, J. computers Math. Applic. **44** (2002), 95–103.

[7] C. Horvath, *Extension and selection theorems in topological spaces with generalized convexity structure*, Annal. Fac. Sci. Toulouse **2** (1993), 93–101.

[8] B. Kanaster, C. Kuratowski, and S. Mazurkiewicz, *Ein bewies des Fixpunktsatzes für n-dimensional simplexe*, Fund. Math. **XIV** (1929), 132–137.

[9] W. A. Kirk, B. Sims, and X. Z. Yuan, *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications*, Nonlinear Anal. T. M. A. **39** (2000), 611–627.

[10] L.-J. Lin, C.-J. Ko, and S. Park, Coincidence theorems for set-valued maps with G-KKM property on generalized convex spaces, *Discuss. Math. Differential Incl.* **18**(1998), 69-85.

[11] S. Park, *New topological version of the Fan-Browder fixed point theorem*, Nonlinear Anal. T. M. A. **47** (2001), 595–606.

[13] S. Park and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. SNU **18** (1993), 1–21.

[14] S. Park and W. Lee, *A unified approach to generalized KKM maps in generalized convex spaces*, J. Nonlinear and Convex Anal. **2** (2001), 157–166.

[15] R. U. Verma, *G-H-KKM type theorems and theire applications to new classes of minimax inequalities*, Computers Math. Applic. **37** (1999), 45-48.

Department of Mathematics, University of Isfahan, Isfahan 81745-163 Iran.
E-mail address: fakhar@sci.ui.ac.ir
E-mail address: jzaf@sci.ui.ac.ir