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Hilbert-generated spaces ☆

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Abstract

We classify several classes of the subspaces of Banach spaces X for which there is a bounded linear operator from a Hilbert space onto a dense subset in X. Dually, we provide optimal affine homeomorphisms from weak star dual unit balls onto weakly compact sets in Hilbert spaces or in $c_0(\Gamma)$ spaces in their weak topology. The existence of such embeddings is characterized by the existence of certain uniformly Gâteaux smooth norms.

Keywords: Uniformly Gâteaux smooth norms; Hilbert generating; Uniformly Eberlein compact sets

1. Introduction

There is a connection between smoothness of norms and the weak compact generating of spaces. Lindenstrauss ([21, Problem 9]), asked whether smoothness of a Banach space X implies that some superspace of X is weakly compactly generated. A negative answer to this problem was then given in [19]. Shortly thereafter, Enflo [8] showed that spaces which have an equivalent uniformly Fréchet smooth norm are exactly the superreflexive spaces. Pisier then proved that every superreflexive space admits an equivalent norm with modulus of smoothness of power type [24].

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Troyanski [26,27] elucidated the structure of the nonseparable spaces which have an equivalent uniformly Gâteaux smooth norm (in short, UG smooth norm), when an unconditional basis is available.

Although spaces which have a Gâteaux (or even C^∞) smooth equivalent norm can be far from being subspaces of weakly compactly generated spaces (see [6,18, Chapter V and VI]), Lindenstrauss' problem mentioned earlier has a positive answer for spaces which have an equivalent *uniformly* Gâteaux smooth norm [10]. Indeed, if the norm of X is UG smooth, then the dual unit ball B_{X^*} with its weak* topology is a uniform Eberlein compact [10] (i.e. homeomorphic to a weakly compact set in a Hilbert space equipped with its weak topology). Thus there is a bounded linear operator from a Hilbert space onto a dense set in $C(B_{X^*})$ by [3]. If a Hilbert space is mapped by a bounded linear operator onto a dense set in a Banach space X, then X admits an equivalent UG smooth norm (cf. e.g. [6, Chapter II]). Hence *the spaces with* UG *smooth norms are exactly the subspaces of Hilbert-generated spaces*, according to the following notation.

Notation 1. We will say that the Banach space X is generated by a Banach space Y, or Y-*generated* for short if there is a bounded linear operator from Y onto a dense subset of X. Let $\mathscr{P}$ be a property on Banach spaces. We say that X *is* $\mathscr{P}$-*generated* if there is a Banach space Y with the property $\mathscr{P}$ such that X is Y-generated.

The classical interpolation theorem [5] asserts that a Banach space is weakly compactly generated if and only if it is reflexive generated.

In this paper, we investigate "uniform" versions of this interpolation result. Our work also blends the Enflo–Pisier renorming theorem with Troyanski's results on unconditional bases in nonseparable spaces.

The notion of strong UG smoothness is a weakening of uniform Fréchet smoothness, obtained by replacing the unit ball of the space by some bounded set M which spans a dense linear subspace (see Notation 2 below). We will show that strongly UG smooth spaces are weakly compactly generated, and moreover that they are superreflexive-generated (respectively Hilbert-generated) exactly when the relevant modulus of smoothness is of power type (respectively of power type 2). We characterize several classes of nonseparable spaces in terms of the existence of certain equivalent norms (see Theorems 2–4). In this way we display a chain of properties between "Hilbert-generated" and "subspace of a Hilbert-generated space". We refine on [10] by showing that the relevant implications are strict (see Theorem 1). We show that simple transfer formulas which go back to [27] always suffice for obtaining UG smooth norms (see Remark 1). A connection is made with the structure of uniformly Eberlein compact sets (see [1–3,14], cf. e.g. [12]). An appendix is devoted to a short proof of the known result [17] that weakly uniformly rotund spaces are Asplund spaces.

We refer to [6,9,12,15,28] for all unexplained terms used in this paper and for more information in this area.

Some results in this paper are proved under the restriction that the density character of the relevant space is ω_1 (that is, the first uncountable cardinal). We conjecture that Theorems 2–4 are true for spaces of arbitrary density character.

2. The results

Let $(X, \|\cdot\|)$ be a Banach space. Let B_X and S_X denote its closed unit ball and unit sphere, respectively. We say that the norm $\|\cdot\|$ is UG *smooth* if for every $h \in X$

$$\sup\{\|x+th\| + \|x-th\| - 2;\ x \in S_X\} = o(t) \quad \text{when} \quad t \to 0.$$

In this equation, the asymptotic behavior of the supremum depends upon h, and this cannot be avoided unless the space X is superreflexive. However, it may happen that this quantity is uniform on a bounded linearly dense subset. This motivates the following notation.

Notation 2. Let $M \subset X$ be a bounded set. We will say that the norm $\|\cdot\|$ is M-UG *smooth* if

$$\rho_M(t) := \sup\{\|x+th\| + \|x-th\| - 2;\ x \in S_X,\ h \in M\} = o(t) \quad \text{when} \quad t \to 0. \qquad (0)$$

We will say that the norm $\|\cdot\|$ is *strongly* UG *smooth* if it is M-UG smooth for some bounded and linearly dense subset $M \subset X$.

If X is a separable Banach space, it is clearly Hilbert-generated and thus it has an equivalent UG smooth norm (see [6, Theorem II.6.8]). More precisely, if $\|\cdot\|$ is a UG smooth norm and if (x_i) is a dense sequence in the unit sphere of X, then an elementary calculation which uses the compactness of $M := \{x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots\}$ and the Lipschitz property of the norm shows that the norm $\|\cdot\|$ is M-UG smooth and thus it is strongly UG smooth. Hence UG smoothness coincide with its strong version for separable spaces. We recall that there are reflexive (nonseparable) spaces that admit no equivalent UG smooth norm ([20], cf. e.g. [6, Chapter VI], or [12, Chapter 12]).

It is easy to check that the strong UG smoothness implies the UG smoothness. We note that the B_X-UG smoothness means the usual uniform Fréchet smoothness. By ([6, Theorem II.6.8]), if X is Y-generated and Y has an equivalent UG smooth norm, then X has such a norm, and the proof shows that it works as well for strong UG smoothness.

As we mentioned earlier, the main result of [10], when combined with [3], is that *a Banach space X is a subspace of a Hilbert-generated space if and only if it admits an equivalent* UG *smooth norm*. It is also observed in [10] that "subspace" is actually needed here, although for any compact space K, the Banach space $C(K)$ is Hilbert-generated if (and only if) it admits an equivalent UG smooth norm. Considering the strong UG smoothness yields a scale of distinct properties between being Hilbert-generated and being a subspace of a Hilbert-generated space.

Theorem 1. *For a Banach space X consider the assertions*:

(i) *X is Hilbert-generated.*
(ii) *X is superreflexive-generated.*

(iii) *X is generated by the ℓ_2-sum of superreflexive spaces.*
(iv) *X admits an equivalent strongly UG smooth norm.*
(v) *X is weakly compactly generated and admits an equivalent UG smooth norm.*
(vi) *X is a subspace of a Hilbert-generated space.*

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (vi).

Moreover, no one of these implications can be reversed in general.

Let $1 < p \leqslant 2$ and let $M \subset X$ be a bounded set. We will say that the norm $\|\cdot\|$ is *p-M-UG smooth* if there is $c > 0$ such that

$$\rho_M(t) := \sup\{\|x + th\| + \|x - th\| - 2;\ x \in S_X,\ h \in M\} \leqslant ct^p \quad \text{for all} \quad t > 0.$$

Note that the p-B_X-UG smoothness means that the norm has modulus of smoothness of power type t^p. A Baire category argument (similar to that used in [4,7]) shows that if a norm is p-$\{h\}$-UG smooth for every $h \in S_X$ (with a constant c which may depend upon h), then the space X is superreflexive. We will say that the norm $\|\cdot\|$ is *p-strongly* UG *smooth* if it is p-M-UG smooth for some linearly dense and bounded set $M \subset X$. Scaling h, one can use a simple homogeneity argument to observe that a norm is p-strongly UG smooth if and only if it is p-$\{h\}$-UG smooth for every h from a dense subset of S_X (with a constant c which may depend upon h), that is, if and only if the modulus of uniform Gâteaux smoothness in the direction h is $O_h(t^p)$ for a dense set of directions.

Theorem 2. *Let X be a Banach space with* $\operatorname{dens} X = \omega_1$ *and let* $1 < p \leqslant 2$. *Then the following are equivalent*:

(a) *X is generated by a Banach space whose norm has modulus of smoothness of power type t^p.*
(b) *X admits an equivalent p-strongly UG smooth norm.*
(c) *X is $\ell_p(\Gamma)$-generated where $\#\Gamma = \omega_1$.*

Theorem 3. *Let X be a Banach space with* $\operatorname{dens} X = \omega_1$. *Then*

(i) *X is Hilbert-generated if and only if it satisfies the conditions of Theorem* 2 *for* $p = 2$.
(ii) *X is superreflexive-generated if and only if it satisfies the conditions of Theorem* 2 *for some* $1 < p \leqslant 2$.
(iii) *X is generated by the ℓ_2-sum of superreflexive spaces if and only if there are numbers $1 < p_n \leqslant 2$, $n \in \mathbb{N}$, such that X is generated by the ℓ_2-sum of $\ell_{p_n}(\Gamma)$, $n \in \mathbb{N}$, where $\#\Gamma = \omega_1$.*

Note that Theorems 2 and 3 imply that a space X, with $\operatorname{dens} X = \omega_1$, is superreflexive-generated if and only if it is $\ell_p(\Gamma)$-generated for some $1 < p \leqslant 2$. This equivalence relies on Pisier's renorming theorem [24].

If X is superreflexive and p-strongly UG smooth, it does not follow that X admits an equivalent norm with modulus of smoothness of power type t^p. Indeed, the (separable) space $\ell_{3/2}$ is Hilbert-generated, and hence 2-strongly UG smooth. Yet it does not admit any equivalent norm whose modulus of smoothness is of power type t^2, since it is not of type 2.

Theorem 4. *A Banach space* $(X, ||\cdot||)$ *with* dens $X = \omega_1$ *admits an equivalent strongly UG smooth norm if and only if* (B_{X^*}, w^*) *is a "homogeneous" uniform Eberlein compact, i.e., if there exist a set* Γ *with* $\#\Gamma = \omega_1$ *and a bounded linear one-to-one and* weak* *to weak continuous operator* $T : X^* \to c_0(\Gamma)$ *such that for every* $\varepsilon > 0$ *there is* $i \in \mathbb{N}$ *satisfying*

$$\#\{\gamma \in \Gamma;\ |Tx^*(\gamma)| > \varepsilon ||x^*||\} < i \quad \textit{for all} \quad x^* \in X^*.$$

The condition (iv) in Theorem 1 can be understood as follows: a space X has an equivalent strongly UG smooth norm if and only if it is "$\ell_1(\Gamma)$-generated", in the sense that there is a weak*-to-weak continuous linear operator T_* from $\ell_1(\Gamma)$ into X with dense range, and moreover there is a weak* uniformly rotund norm on X^* whose pointwise uniform rotundity is uniform on $M = T_*(B_{\ell_1(\Gamma)})$. The condition (v) in Theorem 1 is identical except that the uniformity condition on M is dropped; it is also equivalent to the assertion that X is generated by a reflexive space R which has an equivalent UG smooth norm (see [10, Remark 6]).

The condition (vi) in Theorem 1 is equivalent to the statement that (B_{X^*}, w^*) is a uniform Eberlein compact (see [10]). Condition (v) is equivalent to the statement that (B_{X^*}, w^*) can be continuously and *linearly* injected into a uniform Eberlein compact in $c_0(\Gamma)$ equipped with its weak topology. The word "homogeneous" used in Theorem 4 means that the Argyros–Farmaki decompositions ([1, Theorem 1.8]) do not depend upon ε when the stronger condition (iv) is satisfied. Note that by [26], a Banach space with a symmetric basis has an equivalent UG smooth norm if and only if it is not isomorphic to $\ell_1(\Gamma)$ for an uncountable set Γ. It follows from ([26, Lemma 2]) that the conditions (vi) and (iv) in Theorem 1 are equivalent for spaces which have a symmetric basis. Finally, conditions (v) and (vi) are equivalent for Asplund spaces (see [6, Corollary VI.4.4]), and thus a space which has an equivalent UG smooth norm and an equivalent Fréchet smooth norm is weakly compactly generated.

3. The proofs

The following lemma provides homogeneous inequalities that will be needed in this paper.

Lemma 0. *Let* $(X, ||\cdot||)$ *be a Banach space. Consider a nonempty set* $M \subset B_X$, *and let* $1 < p \leqslant 2$ *be given. Then there exists* $c > 0$ *such that*

$$||x + th|| + ||x - th|| - 2 \leqslant c|t|^p \quad \textit{for every } x \in S_X, \textit{ every } h \in M \textit{ and every } t \in \mathbb{R}$$

if and only if there exists $C>0$ *such that*

$$||x+th||^p + ||x-th||^p - 2||x||^p \leqslant C|t|^p$$

for every $x\in X$, *every* $h\in M$ *and every* $t\in\mathbb{R}$.

Proof (*Necessity*). Fix $x\in X$, $h\in M$, and $t>0$. Owing to the homogeneity of the conclusion, we may assume that $||x||=1$. First, assume that $t\geqslant\frac{1}{2}$. Then, from the convexity of the function $u\mapsto u^p$, $u>0$, we have

$$||x+th||^p + ||x-th||^p - 2 \leqslant 2(1+t)^p - 2 \leqslant 2p(1+t)^{p-1}t$$
$$\leqslant 2p(3t)^{p-1}t = 2p3^{p-1}t^p \leqslant 12t^p.$$

Second, let $0<t<\frac{1}{2}$. Denote $\xi = \lim_{\tau\to 0}\frac{1}{\tau}(||x+\tau h||-1)$. Then, from the assumptions

$$||x\pm th|| - 1 \mp t\xi \leqslant ct^p,$$

and from the convexity of the function $u\mapsto u^p$, $u>0$, we have

$$||x\pm th||^p - 1 \leqslant (1\pm t\xi + ct^p)^p - 1 \leqslant p(1\pm t\xi + ct^p)^{p-1}(\pm t\xi + ct^p)$$
$$< p(1\pm t\xi + ct^p)^{p-1}(\pm t\xi) + p(2+c)^{p-1}ct^p.$$

Further, using the concavity of the function $u\mapsto u^{p-1}$, $u>0$, we get

$$|(1+t\xi+ct^p)^{p-1} - (1-t\xi+ct^p)^{p-1}|$$
$$\leqslant (p-1)(1-t|\xi|+ct^p)^{p-2}2t|\xi| < (p-1)(\tfrac{1}{2})^{p-2}2t = (p-1)2^{3-p}t.$$

Thus

$$||x+th||^p + ||x-th||^p - 2 \leqslant p(p-1)2^{3-p}t^2 + p(2+c)^{p-1}ct^p \leqslant [8+2c(2+c)]t^p.$$

Sufficiency: Fix $x\in X$, with $||x||=1$, $h\in M$, and $t>0$. Since the function $u\mapsto 2^{p-1}(1+u^p)-(1+u)^p$, $0\leqslant u\leqslant 1$, is decreasing, we have $(a+b)^p\leqslant 2^{p-1}(a^p+b^p)$ for all $a,b\geqslant 0$, and hence

$$||x+th|| + ||x-th||$$
$$\leqslant [2^{p-1}||x+th||^p + ||x-th||^p)]^{1/p}$$
$$\leqslant [2^{p-1}(2+Ct^p)]^{1/p} = 2\left(1+\frac{C}{2}t^p\right)^{1/p} < 2\left(1+\frac{C}{2}t^p\right) = 2+Ct^p. \qquad \square$$

For a set $M\subset X$, let $\bar{M}^*$ denote its weak* closure in the second dual X^{**}. As in [22,27], for $n,p\in\mathbb{N}$ put

$$S_{n,p} = \left\{ h\in X;\ \left\|x+\frac{1}{n}h\right\| + \left\|x-\frac{1}{n}h\right\| - 2\leqslant\frac{1}{np} \quad \text{whenever} \quad x\in S_X \right\}.$$

The following lemma will be crucial for showing that certain spaces are weakly compactly generated. Let us mention that this lemma provides a nontrivial information in the separable case as well. In fact, the Milman–Pettis theorem asserting that uniformly smooth Banach spaces are reflexive follows from it when we take $M = B_X$.

Lemma 1. *If the norm $\|\cdot\|$ on X is* UG *smooth, then for every $\sigma\in\mathbb{N}^{\mathbb{N}}$ the intersection $\bigcap_{p=1}^{\infty}\overline{S_{\sigma(p),p}}^*$ lies in X. In particular, if the norm $\|\cdot\|$ is M-*UG *smooth for some bounded set $M\subset X$, then the set M is relatively weakly compact in X.*

Proof. The first assertion is contained in the proof of Lemma 1 in [10]. Given such a set M, from the modulus ρ_M of uniform UG smoothness on M defined in Eq. (0), one obtains a sequence $\sigma\in\mathbb{N}^{\mathbb{N}}$ such that $M\subset\bigcap_{p=1}^{\infty} S_{\sigma(p),p}$. Thus M is relatively weakly compact. $\square$

A (long) sequence $(x_\alpha;\ \alpha<\omega_1)$ in a Banach space $(X,\|\cdot\|)$ is called *monotone* if

$$\left\|\sum_{\alpha\in F} a_\alpha x_\alpha\right\| \leqslant \left\|\sum_{\alpha\in F\cup H} a_\alpha x_\alpha\right\|$$

whenever F and H are finite sets in the interval $[0,\omega_1)$, with $\max F<\min H$, and $a_\alpha\in\mathbb{R}$, $\alpha\in F\cup H$. Given two vectors $x,h\in X$, we write $h\perp x$ whenever x,h is a monotone (two term) sequence.

Proof of Theorem 2. (a) $\Rightarrow$ (b). Let X be generated by some Banach space $(Y,\|\cdot\|)$ whose modulus of smoothness is of power type t^p. Let $|\cdot|$ be the Minkowski functional of the set

$$\mathrm{co}_p(B_X\cup T(B_Y)) \coloneqq \bigcup\{\alpha B_X + \beta T(B_Y);\ \alpha\geqslant 0, \beta\geqslant 0, \alpha^p+\beta^p\leqslant 1\}.$$

Note that this set is symmetric, convex, bounded, contains B_X, and is also closed since $T(B_Y)$ is weakly compact (Y is reflexive).

By Lemma 0, there is $C>0$ such that

$$\|y+h\|^p + \|y-h\|^p - 2\|y\|^p \leqslant C\|h\|^p \quad \text{whenever} \quad y\in Y \quad \text{and } h\in Y.$$

Now, take any $z\in X$, with $|z|=1$, any $h\in B_Y$, and any $t>0$. Find then $\alpha,\beta\geqslant 0$, with $\alpha^p+\beta^p\leqslant 1$, and $x\in B_X$, $y\in B_Y$ so that $z=\alpha x+\beta Ty$. Then $\alpha^p+\beta^p=1$ and

$||y|| = 1$. Since

$$z \pm tTh = \alpha x + ||\beta y \pm th|| T\left(\frac{\beta y \pm th}{||\beta y \pm th||}\right),$$

we have

$$\begin{aligned}|z + tTh|^p + |z - tTh|^p - 2 &\leqslant \alpha^p + ||\beta y + th||^p + \alpha^p + ||\beta y - th||^p - 2 \\ &\leqslant ||\beta y + th||^p + ||\beta y - th||^p - 2\beta^p \leqslant Ct^p.\end{aligned}$$

Thus the set $T(B_Y)$ witnesses that $|\cdot|$ is p-strongly UG smooth and so (b) is proved.

(b) $\Rightarrow$ (c). Find a bounded set $M \subset B_X$ that is linearly dense in X and $C \geqslant 1$ such that

$$||x + th||^p + ||x - th||^p - 2||x||^p \leqslant C|t|^p$$

for every $x \in X$, every $h \in M$, and every $t \in \mathbb{R}$.

This can be done according to Lemma 0. We claim that

$$\left|\left|\sum_{i=1}^{n} a_i x_i\right|\right|^p \leqslant C \sum_{i=1}^{n} |a_i|^p$$

for every integer $n \in \mathbb{N}$, for every monotone sequence $x_1, \dots, x_n$ in M, and for every $a_1, \dots, a_n \in \mathbb{R}$. Indeed, if $n = 1$, there is nothing to prove. Assume that the inequality was verified for some $n \in \mathbb{N}$. Take any $a_1, \dots a_n, a_{n+1} \in \mathbb{R}$. Then

$$\begin{aligned}\left|\left|\sum_{i=1}^{n+1} a_i x_i\right|\right|^p &= \left|\left|\sum_{i=1}^{n} a_i x_i + a_{n+1} x_{n+1}\right|\right|^p \\ &\leqslant \left|\left|\sum_{i=1}^{n} a_i x_i\right|\right|^p + C|a_{n+1}|^p \leqslant C \sum_{i=1}^{n+1} |a_i|^p.\end{aligned}$$

This proves the claim.

We may and do assume that M is convex, symmetric, and weakly compact (Lemma 1). As $(X, ||\cdot||)$ is then weakly compactly generated, it has a PRI $(P_\alpha;\ \alpha \leqslant \omega_1)$. We may arrange things in such a way that in addition $P_\alpha(M) \subset M$ for every α, see, e.g. ([9, p. 109]). For every $\alpha < \omega_1$, the subspace $(P_{\alpha+1} - P_\alpha)X$ is separable. Find a countable set $\{x_n^\alpha;\ n \in \mathbb{N}\}$ which lies in M and is linearly dense in the space $(P_{\alpha+1} - P_\alpha)X$. For any element $(a_m^\alpha;\ \alpha < \omega_1,\ m \in \mathbb{N})$ of $\ell_p([0, \omega_1) \times \mathbb{N})$, with finite support, we put

$$T(a_m^\alpha) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\alpha < \omega_1} a_m^\alpha x_m^\alpha.$$

This is a linear mapping from a linear dense subset of $\ell_p([0,\omega_1)\times\mathbb{N})$ into X. Now, using the Hölder inequality and the claim, we can estimate

$$||T(a_m^\alpha)||\leqslant\left(\sum_{m=1}^{\infty}\frac{1}{m^q}\right)^{\frac{1}{q}}\left(\sum_{m=1}^{\infty}\left|\left|\sum_{\alpha<\omega_1}a_m^\alpha x_m^\alpha\right|\right|^p\right)^{\frac{1}{p}}$$
$$\leqslant K\left(\sum_{m=1}^{\infty}C\sum_{\alpha<\omega_1}|a_m^\alpha|^p\right)^{\frac{1}{p}}=KC^{\frac{1}{p}}||(a_m^\alpha)||_{\ell_p},$$

where we put $q=\frac{p}{p-1}$ and $K=(\sum_{m=1}^{\infty}m^{-q})^{\frac{1}{q}}$. We used the fact that the long "sequence" $(x_m^\alpha;\ \alpha<\omega_1)$ is monotone for every $m\in\mathbb{N}$. Hence T is a bounded mapping into X, and so it can be extended to a bounded linear mapping $\tilde{T}$, from all of $\ell_p([0,\omega_1)\times\mathbb{N})$ into X. As the range of $\tilde{T}$ contains the set $\{x_m^\alpha;\ \alpha<\omega_1,\ m\in\mathbb{N}\}$, the properties of (P_α) guarantee that the set $\bigcup_{\alpha<\omega_1}(P_{\alpha+1}-P_\alpha)M$ is linearly dense in X. Thus the range of $\tilde{T}$ is dense in X.

(c) $\Rightarrow$ (a). It is enough to recall that the canonical norm on $\ell_p(\Gamma)$, with $1<p\leqslant 2$, has modulus of smoothness of power type t^p, see e.g. ([6, Chapter V]). (A direct proof follows from Lemma 0 and from the inequality $|a+b|^p+|a-b|^p-2|a|^p\leqslant c_p|b|^p$ valid for all $1<p\leqslant 2$, all $a,b\in\mathbb{R}$, and a suitable constant $c_p>0$.) $\square$

Proof of Theorem 3. (i) is a consequence of Theorem 2, since the general form of Hilbert spaces is $\ell_2(\Gamma)$ for some nonempty set Γ.

(ii) *Necessity*: Consider $T: Y\to X$ where Y is superreflexive, T is linear bounded, and $T(Y)$ is dense in X. According to Pisier's renorming result, see, e.g., ([6, Proposition IV.5.2]), Y admits an equivalent norm whose modulus of smoothness is of power type t^p for some $1<p\leqslant 2$.

The *sufficiency* follows immediately from the fact that $\ell_p(\Gamma)$ is superreflexive once $1<p<+\infty$.

(iii) *Necessity*: Let X be generated by $(\sum_{\lambda\in\Lambda}Y_\lambda)_{\ell_2}$ where each Y_λ is superreflexive. According to Pisier's result ([6, Proposition IV.5.2]), we may assume that for every $\lambda\in\Lambda$, the space Y_λ has an equivalent norm $||\cdot||_\lambda$, whose modulus of smoothness is $\leqslant c_\lambda t^{p_\lambda}$ with suitable $c_\lambda>0$ and $1<p_\lambda\leqslant 2$. For $n=2,3,\dots$ put $\Lambda_n=\{\lambda\in\Lambda;\ c_\lambda\leqslant n,\ p_\lambda\geqslant 1+1/n\}$ and $Z_n=(\sum_{\lambda\in\Lambda_n}(Y_\lambda,||\cdot||_\lambda))_{\ell_{1+1/n}}$. We will check that the canonical norm of Z_n has modulus of smoothness of power type $t^{1+1/n}$. We know that

$$||y_\lambda+th_\lambda||_\lambda+||y_\lambda+th_\lambda||_\lambda-2\leqslant nt^{1+1/n}$$

whenever $y_\lambda\in S_{Y_\lambda}$, $h_\lambda\in B_{Y_\lambda}$, and $t\in\mathbb{R}$ (if $t>1$, then the left-hand side above is $\leqslant 2t\leqslant nt^{1+1/n}$). By Lemma 0, there is a constant $C_n>0$ such that for all $y_\lambda,h_\lambda\in Y_\lambda$ and all $t\in\mathbb{R}$,

$$||y_\lambda+th_\lambda||_\lambda^{1+1/n}+||y_\lambda+th_\lambda||_\lambda^{1+1/n}-2||y_\lambda||_\lambda^{1+1/n}\leqslant C_nt^{1+1/n}.$$

When adding the above inequalities for all $\lambda \in \Lambda_n$, we get

$$||y + th||_{1+1/n}^{1+1/n} + ||y - th||_{1+1/n}^{1+1/n} - 2||y||_{1+1/n}^{1+1/n} \leqslant C_n t^{1+1/n}$$

for all $y = (y_\lambda) \in Z_n$, $h = (h_\lambda) \in Z_n$, and all $t \in \mathbb{R}$. Then, Lemma 0 says that this norm on the space Z_n has modulus of smoothness of power type $t^{1+1/n}$. Clearly, we may assume that each Z_n has density at most ω_1 for otherwise we can go to a quotient of Z_n, still keeping the modulus of smoothness of power type $t^{1+1/n}$. Finally, Theorem 2 says that Z_n is $\ell_{1+1/n}(\Gamma)$-generated with $\#\Gamma = \omega_1$. Now, it remains to realize that the ℓ_2-sum of Z_n, $n \in \mathbb{N}$, embeds onto a dense subset of X, and hence so does the ℓ_2-sum of $\ell_{1+1/n}(\Gamma)$, $n \in \mathbb{N}$. This completes the proof of the necessity part.

The *sufficiency* part follows as in (ii). $\square$

In the proof of Theorem 4 we will need some more notation and the following four lemmas. Following [11], for $\varepsilon > 0$ and for $i \in \mathbb{N}$ greater than $2/\varepsilon$ we put

$$S_i^\varepsilon = \left\{ h \in S_X;\ ||x + th|| - 1 \leqslant \tfrac{1}{2}\varepsilon t \text{ whenever } x \in S_X, \quad 0 < t < \frac{2}{\varepsilon i} \text{ and } h \perp x \right\}.$$

Lemma 2 (Fabian et al. [11]). *If the norm* $||\cdot||$ *on* X *is* UG *smooth, and if* $x_1, \dots, x_i \in S_i^\varepsilon$ *is a monotone sequence, then*

$$||x_1 + \cdots + x_i|| < \varepsilon i.$$

Let Γ be an infinite set. We recall that Day's norm on $c_0(\Gamma)$, denoted here by $||\cdot||_{\mathscr{D}}$, is defined by

$$||u||_{\mathscr{D}}^2 = \sup \left\{ \sum_{k=1}^{n} 4^{-k} u(\gamma_j)^2;\ n \in \mathbb{N},\ \gamma_1, \dots, \gamma_n \in \Gamma,\ \gamma_k \neq \gamma_l \text{ if } k \neq l \right\},\ u \in c_0(\Gamma).$$

If $\beta \in \Gamma$, we define a canonical projection $\pi_\beta : c_0(\Gamma) \to c_0(\Gamma)$ by

$$\pi_\beta u(\gamma) = \begin{cases} u(\beta) & \text{if } \gamma = \beta, \\ 0 & \text{if } \gamma \in \Gamma \setminus \{\beta\}, \end{cases} \quad u \in c_0(\Gamma).$$

Lemma 3 (Troyanski [26]). *Let* $u \in c_0(\Gamma)$ *and* $\beta \in \Gamma$ *be such that* $u(\beta) \neq 0$. *Put*

$$i = \#\{\gamma \in \Gamma;\ |u(\gamma)| \geqslant 2^{-1/2} |u(\beta)|\}.$$

Then

$$||u||_{\mathscr{D}}^2 \geqslant ||u - \pi_\beta u||_{\mathscr{D}}^2 + 2^{-i-1} u(\beta)^2.$$

Lemma 4 (Troyanski [26,27]). *Let* $u, v \in c_0(\Gamma)$ *and* $\beta \in \Gamma$ *be such that* $u(\beta) + v(\beta) \neq 0$. *Put* $i = \#\{\gamma \in \Gamma;\ |u(\gamma) + v(\gamma)| \geqslant |u(\beta) + v(\beta)|\}$. *Then*

$$2||u||_{\mathscr{D}}^2 + 2||v||_{\mathscr{D}}^2 - ||u+v||_{\mathscr{D}}^2 \geqslant 2^{-i-1}(u(\beta) - v(\beta))^2.$$

The following lemma is a strengthening of ([26, Proposition 1]). Its proof follows Troyanski's argument.

Lemma 5. *Let* Z *be a linear subset of* $c_0(\Gamma)$, *with a norm* $|\cdot|$ *such that* $c|z| \geqslant ||z||_\infty$ *for every* $z \in Z$, *where* $c > 0$ *is a constant. Assume that for every* $\varepsilon > 0$ *there is* $i \in \mathbb{N}$ *such that*

$$\#\{\gamma \in \Gamma;\ |z(\gamma)| > \varepsilon |z|\} < i \quad \textit{for every } z \in Z.$$

If (u_n), (v_n) *are two sequences in* $B_{(Z,|\cdot|)}$, *and*

$$2||u_n||_{\mathscr{D}}^2 + 2||v_n||_{\mathscr{D}}^2 - ||u_n + v_n||_{\mathscr{D}}^2 \to 0 \quad \textit{as} \quad n \to \infty \tag{1}$$

then

$$||u_n - v_n||_\infty \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. Assume that the lemma is not true. Then we can find $s \in \mathbb{N}$, an infinite set $N \subset \mathbb{N}$, and $\gamma_n \in \Gamma$ for $n \in N$ such that

$$|(u_n - v_n)(\gamma_n)| > \frac{3}{s} \quad \text{for every} \quad n \in N.$$

We will first show that

$$\limsup_{n \in N}\ |u_n(\gamma_n) + v_n(\gamma_n)| > 0. \tag{2}$$

Assume that this is not true, that is, that

$$\lim_{n \in N}\ (u_n(\gamma_n) + v_n(\gamma_n)) = 0.$$

Then

$$\liminf_{n \in N}\ 2|u_n(\gamma_n)| \geqslant \liminf_{n \in N}(\ |u_n(\gamma_n) - v_n(\gamma_n)| - |u_n(\gamma_n) + v_n(\gamma_n)|) \geqslant \frac{3}{s},$$

and hence

$$\liminf_{n \in N}\ |u_n(\gamma_n)| > \frac{\sqrt{2}}{s}. \tag{3}$$

Also

$$||u_n + v_n||_{\mathscr{D}} - ||(u_n + v_n) - \pi_{\gamma_n}(u_n + v_n)||_{\mathscr{D}} \leqslant ||\pi_{\gamma_n}(u_n + v_n)||_{\mathscr{D}}$$
$$= \tfrac{1}{2}|u_n(\gamma_n) + v_n(\gamma_n)|| \to 0 \tag{4}$$

as $n \in N$ and $n \to \infty$. From the assumption, find $i \in \mathbb{N}$ such that for every $u \in Z$,

$$\#\left\{\gamma \in \Gamma;\ |u(\gamma)| > \frac{1}{s}|u|\right\} < i.$$

For any sufficiently large $n \in N$ we have from (3)

$$\begin{aligned}\#\{\gamma \in \Gamma;\ |u_n(\gamma)| \geqslant 2^{-1/2}|u_n(\gamma_n)|\} &\leqslant \#\left\{\gamma \in \Gamma;\ |u_n(\gamma)| > \frac{1}{s}\right\}\\ &\leqslant \#\left\{\gamma \in \Gamma;\ |u_n(\gamma)| > \frac{1}{s}|u_n|\right\} < i.\end{aligned}$$

Hence, according to Lemma 3, (3) and (4), we get

$$\begin{aligned}&\liminf_{n \in N}(2||u_n||_{\mathscr{D}}^2 + 2||v_n||_{\mathscr{D}}^2 - ||u_n + v_n||_{\mathscr{D}}^2)\\ &\quad \geqslant \liminf_{n \in N}\ (2||u_n - \pi_{\gamma_n} u_n||_{\mathscr{D}}^2 + 2 \cdot 2^{-i-1} u_n(\gamma_n)^2\\ &\qquad +\ 2||v_n - \pi_{\gamma_n} v_n||_{\mathscr{D}}^2 + 2 \cdot 2^{-i-1} v_n(\gamma_n)^2 - ||u_n + v_n||_{\mathscr{D}}^2)\\ &\quad \geqslant \liminf_{n \in N}(2^{-i} u_n(\gamma_n)^2 + 2^{-i} v_n(\gamma_n)^2\\ &\qquad - ||u_n + v_n||_{\mathscr{D}}^2 + ||(u_n + v_n) - \pi_{\gamma_n}(u_n + v_n)||_{\mathscr{D}}^2)\\ &\quad \geqslant 4 \cdot 2^{-i-1}\left(\frac{\sqrt{2}}{s}\right)^2 = 2^{-i+2} s^{-2}\ (>0),\end{aligned}$$

a contradiction with (1). This proves (2).

From (2) we find an infinite set $N' \subset \mathbb{N}$ and $r \in \mathbb{N}$ such that

$$|u_n(\gamma_n) + v_n(\gamma_n)| > \frac{2}{r} \quad \text{for every} \quad n \in N'.$$

For every $n \in N'$ we then have

$$|u_n(\gamma_n) + v_n(\gamma_n)| > \frac{2}{r} \geqslant \frac{1}{r}(\ |u_n| + |v_n|) \geqslant \frac{1}{r}|u_n + v_n|.$$

Now, find $k \in \mathbb{N}$ so that for every $u \in Z$

$$\#\left\{\gamma \in \Gamma;\ |u(\gamma)| > \frac{1}{r}|u|\right\} < k.$$

Hence for every $n \in N'$

$$\begin{aligned}&\#\{\gamma \in \Gamma;\ |u_n(\gamma) + v_n(\gamma)| \geqslant |u_n(\gamma_n) + v_n(\gamma_n)|\}\\ &\quad \leqslant \#\left\{\gamma \in \Gamma;\ |u_n(\gamma) + v_n(\gamma)| > \frac{1}{r}|u_n + v_n|\right\} < k.\end{aligned}$$

Then, by Lemma 4,

$$2\|u_n\|_{\mathscr{D}}^2 + 2\|v_n\|_{\mathscr{D}}^2 - \|u_n + v_n\|_{\mathscr{D}}^2$$

$$\geqslant 2^{-k-1}(u_n(\gamma_n) - v_n(\gamma_n))^2 > 2^{-k-1} \cdot \left(\frac{3}{s}\right)^2 \ (>0)$$

for every $n \in N'$. This contradicts (1). $\square$

Proof of Theorem 4. *Necessity*: Assume that the original norm $\|\cdot\|$ of X is strongly UG smooth. Let M be the set witnessing this fact. Without loss of generality we assume that M is closed convex symmetric and bounded. Since $\|\cdot\|$ is M-UG smooth, there is $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $\bigcap_{p=1}^{\infty} S_{\sigma(p),p} \supset M$. By Lemma 1, we can then conclude that the set M lies in a weakly compact set in X. As M is linearly dense in X, the space X is weakly compactly generated. We shall use some ideas from [11]. According to a known technique, see, e.g., ([9, p. 109]), $(X, \|\cdot\|)$ admits a PRI $(P_\alpha;\ \alpha \leqslant \omega_1)$ such that $P_\alpha(M) \subset M$ for every $\alpha \leqslant \omega_1$. For every $m \in \mathbb{N}$ and for every $\alpha < \omega_1$, we find a dense set $\{x_{m,j}^{\alpha};\ j \in \mathbb{N}\}$ in $(P_{\alpha+1} - P_\alpha)(mM) \cap S_X$ (Note that each $P_\alpha X$ is separable). Put

$$Tx^*(m,j,\alpha) = \frac{1}{mj}\langle x^*, x_{m,j}^{\alpha}\rangle, \quad x^* \in X^*, \quad \alpha < \omega_1, \quad m,j \in \mathbb{N}.$$

Clearly, T is a linear, bounded, and weak* to pointwise continuous mapping from X^* into $\ell_\infty(\mathbb{N}^2 \times [0, \omega_1))$. T is also injective since M is linearly dense in X.

It remains to check the proclaimed property of T. In order to do so, fix any $\varepsilon > 0$. Fix any $(m,j) \in \mathbb{N}^2$. Find $i_m \in \mathbb{N}$ so large that

$$\|x + th\| + \|x - th\| - 2 \leqslant \frac{\varepsilon t}{2} \quad \text{whenever } x \in S_X,\ h \in 2mM, \quad \text{and} \quad t \in \left(0, \frac{2}{\varepsilon i_m}\right).$$

Take any $x^* \in X^*$. We claim that

$$\#\{\alpha \in [0, \omega_1);\ mj|Tx^*(m,j,\alpha)| > \varepsilon\|x^*\|\} < i_m.$$

Assume, by contradiction, that there exist $\alpha_1 < \alpha_2 < \cdots < \alpha_{i_m} < \omega_1$ such that

$$mj|Tx^*(m,j,\alpha_k)| > \varepsilon\|x^*\| \quad \text{for every} \quad k = 1, 2, \ldots, i_m.$$

Choosing appropriate $\delta_k \in \{-1, 1\}$, $k = 1, \ldots, i_m$, we can estimate

$$i_m\varepsilon\|x^*\| < \sum_{k=1}^{i_m} mj|Tx^*(m,j,\alpha_k)| = \left\langle x^*, \sum_{k=1}^{i_m} \delta_k x_{m,j}^{\alpha_k}\right\rangle$$

$$\leqslant \|x^*\| \left\| \sum_{k=1}^{i_m} \delta_k x_{m,j}^{\alpha_k} \right\| \leqslant \|x^*\| \varepsilon i_m,$$

a contradiction. Here, the most right inequality was guaranteed by Lemma 2. Indeed, in the setting of this lemma, each $\delta_k x_{m,j}^{\alpha_k}$ lies in $2mM \cap S_X \subset S_{i_m}^{\varepsilon}$, and the sequence $x_{m,j}^{\alpha_1}, \ldots, x_{m,j}^{\alpha_{i_m}}$ is monotone owing to the properties of P_α's. We have thus proved our claim. Find $m_0 \in \mathbb{N}$ such that $m_0 > 1/\varepsilon$. Then for every $x^* \in X^*$,

$$|Tx^*(m,j,\alpha)| \leqslant \frac{1}{mj} \|x^*\| \leqslant \varepsilon \|x^*\|$$

whenever $\max\{m,j\} > m_0$ and $\alpha \in [0, \omega_1)$. Hence, for every $x^* \in X^*$,

$$\#\{(m,j,\alpha) \in \mathbb{N}^2 \times [0,\omega_1);\ |Tx^*(m,j,\alpha)| > \varepsilon \|x^*\|\} \leqslant m_0^2 \cdot \max\{i_1, \ldots, i_{m_0}\}\ (< +\infty).$$

Finally, observing that $\#(\mathbb{N}^2 \times [0,\omega_1)) = \omega_1$, we can put $\Gamma = \mathbb{N}^2 \times [0,\omega_1)$ and the necessity is proved.

Sufficiency: Put

$$|||x^*|||^2 = \|x^*\|^2 + \|Tx^*\|_{\mathscr{D}}^2, \quad x^* \in X^*, \tag{5}$$

where $\|\cdot\|_{\mathscr{D}}$ is Day's norm. Clearly, $|||\cdot|||$ is an equivalent norm on X^*, which is moreover weak* lower semicontinuous. We will use the symbol $|||\cdot|||$ also for denoting the norm on X that is predual to $|||\cdot|||$. We will show that the norm $|||\cdot|||$ on X is strongly UG smooth.

For $\gamma \in \Gamma$ put

$$x_\gamma(x^*) = Tx^*(\gamma), \quad x^* \in X^*.$$

This is a linear weak* continuous functional on X^*. Hence each x_γ is an element of X (see e.g. [12, Chapter 3]). Put then

$$M = \{x_\gamma;\ \gamma \in \Gamma\}.$$

This is a bounded set in X. Since T is injective, M is linearly dense in X.

It remains to prove that the norm $|||\cdot|||$ on X is M-UG smooth, which means, by the Šmulyan test, that the norm $|||\cdot|||$ on X^* is "M-uniformly rotund" (the proof of this fact follows the lines of the proof of [6, Theorem II.6.7]). Indeed, assume that (x_n^*) and (y_n^*) are two bounded sequences in X^* which satisfy

$$2|||x_n^*|||^2 + 2|||y_n^*|||^2 - |||x_n^* + y_n^*|||^2 \to 0 \quad \text{as} \quad n \to 0.$$

We have to show that

$$\sup\langle x_n^* - y_n^*, M\rangle \to 0 \quad \text{as} \quad n \to 0,$$

that is,

$$\sup\{T(x_n^* - y_n^*)(\gamma);\ \gamma \in \Gamma\} \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

From the convexity we have

$$2\|Tx_n^*\|_{\mathscr{D}}^2 + 2\|Ty_n^*\|_{\mathscr{D}}^2 - 2\|T(x_n^* + y_n^*)\|_{\mathscr{D}}^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Now, putting in Lemma 5, $Z := T(X^*)$, $|z| := \|T^{-1}z\|$, $z \in Z$, $u_n := Tx_n^*$, and $v_n := Ty_n^*$ we get (6). $\square$

Remark 1. We are now able to claim that UG smooth renormings, when possible, can always be obtained through the same methods as in Theorem 4. Indeed, the proof of Theorem 4 shows that if X is strongly UG smoothly renormable, a strongly UG smooth norm can always be obtained by the formula (5). If X is any UG renormable Banach space, then by [10] the dual unit ball (B_{X^*}, w^*) is a uniform Eberlein compact, hence the space $C((B_{X^*}, w^*))$ is Hilbert-generated according to ([3, Theorem 3.2]) and a UG smooth norm on this superspace is obtained through (5) with T taking values in a Hilbert space equipped with its natural norm. Finally, if X is weakly compactly generated and UG renormable, according to the Amir–Lindenstrauss theorem (cf. e.g. [6, Chapter VI] or [12, Chapter 11]), there are a set Γ and a linear bounded injective and weak* to weak continuous mapping T from X^* into $c_0(\Gamma)$. Since by [11] $(T(B_{X^*}), w)$ is a uniform Eberlein compact, Farmaki's result ([14, Theorem 2.9]) gives for every $\varepsilon > 0$ a splitting $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i^{\varepsilon}$ such that

$$\sup_{x^* \in X^*} \#\{\gamma \in \Gamma_i^{\varepsilon};\ |Tx^*(\gamma)| > \varepsilon\|x^*\|\} < i$$

for every $i \in \mathbb{N}$. Then we are exactly in the setting of ([27, Proposition 1]). Hence, putting

$$|\,\|x^*\|\,|^2 = \|x^*\|^2 + \sum_{i,k=1}^{\infty} 2^{-i-k} \|Tx^*_{|\Gamma_i^{1/k}}\|_{\mathscr{D}}^2, \quad x^* \in X^*, \tag{7}$$

where $\|\cdot\|_{\mathscr{D}}$ is Day's norm, we get $|Tx_n^*(\gamma) - Ty_n^*(\gamma)| \to 0$ as $n \to \infty$ for every $\gamma \in \Gamma$ provided that $|\,\|x_n^*\|\,| \leqslant 1$, $|\,\|y_n^*\|\,| \leqslant 1$, and

$$2|\,\|x_n^*\|\,|^2 + 2|\,\|y_n^*\|\,|^2 - |\,\|x_n^* + y_n^*\|\,|^2 \to 0.$$

Then the norm on X, predual to $|\,\|\cdot\|\,|$, will be, by Šmulyan's test ([6, Theorem II.6.7]), uniformly Gâteaux smooth with respect to the directions from a dense set. This clearly implies the UG smoothness. It follows that (7) suffices for constructing UG smooth norms on any weakly compactly generated space on which such norms exist.

Proof of Theorem 1. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (iv)

Condition (iii) and the proof of Theorem 3(iii) imply that there exist uniformly Fréchet smooth spaces Y_n, $n \in \mathbb{N}$, and a linear bounded mapping T from $(\sum_{n=1}^{\infty} Y_n)_{\ell_2}$ onto a dense set in X. For simplicity we will assume that each Y_n

coincides with the subspace $\{(y_m)\in(\sum_{m=1}^{\infty} Y_m)_{\ell_2};\ y_m=0 \text{ if } m\neq n\}$. Put

$$|\,||x^*||\,|^2 = ||x^*||^2 + \sum_{n=1}^{\infty} 2^{-n}||T^*|x^*_{|Y_n}||^2, \quad x^*\in X^*.$$

Clearly, $|\,||\cdot||\,|$ is an equivalent dual norm on X^*. Put $M=\bigcup_{n=1}^{\infty}\frac{1}{n}T(B_{Y_n})$. It is clear that M is linearly dense in X. We will show that the norm $|\,||\cdot||\,|$ is "M-uniformly rotund". Then the corresponding predual norm on X will be M-UG smooth (see the proof of [6, Theorem II.6.7]). Let (x_i^*) and (y_i^*) be bounded sequences in X^* such that

$$2|\,||x_i^*||\,|^2 + 2|\,||y_i^*||\,|^2 - |\,||x_i^*+y_i^*||\,|^2 \to 0 \quad \text{as} \quad i\to\infty.$$

Then, from the convexity, for every $n\in\mathbb{N}$,

$$2||T^*x^*_{i|Y_n}||^2 + 2||T^*y^*_{i|Y_n} - ||T^*x^*_{i|Y_n} + T^*y^*_{i|Y_n}||^2 \to 0 \quad \text{as} \quad i\to\infty.$$

Since the norm on Y_n^* is uniformly rotund, we get

$$\sup(\langle x_i^*-y_i^*, T(B_{Y_n})\rangle) = \ ||T^*x^*_{i|Y_n} - T^*y^*_{i|Y_n}|| \to 0 \quad \text{as} \quad i\to\infty.$$

Thus $\sup\langle x_i^*-y_i^*, M\rangle \to 0$ as $i\to\infty$. Hence the corresponding norm $|\,||\cdot||\,|$ on X is M-UG smooth.

(iv) $\Rightarrow$ (v)

Assume that $M\subset X$ is a bounded, linearly dense set in X, and that the norm $||\cdot||$ on X is M-UG smooth. Then by Lemma 1 the set M lies in a weakly compact subset of X. Therefore X is weakly compactly generated.

(v) $\Rightarrow$ (vi) can be found in ([11, Theorem 1]). □

4. Counterexamples to the converse implications in Theorem 1

(ii) does not imply (i)

Indeed, if X is Hilbert-generated, then it admits an equivalent 2-strongly UG smooth by Theorem 3(i). So, considering $X=\ell_{3/2}(\Gamma)$, with Γ uncountable, Lemma 6 below gives a contradiction. Thus $\ell_{3/2}(\Gamma)$ is an example of a subspace of a Hilbert-generated space which is not Hilbert-generated. Note that if a Banach space X satisfies (ii) but not (i), then B_{X^*} embeds homeomorphically into a Hilbert space but the embedding cannot be affine. □

(iii) does not imply (ii)

This *in general* can be demonstrated on the space $X=(\sum_{m=1}^{\infty}\ell_{p_m}(\Gamma))_{\ell_2}$ where Γ is uncountable and (p_m) is a sequence of all rational numbers in $(1,\infty)$. Assume that X satisfies (ii). Then, according to Theorem 3(ii), X is p-strongly UG smooth for some $1<p\leqslant 2$. Find $m\in\mathbb{N}$ so that $p_m<p$. Then, using a canonical projection, we can easily check that $\ell_{p_m}(\Gamma)$ is also p-strongly UG smooth. This contradicts Lemma 6 below.

Note that if X satisfies (iii) but not (ii), the modulus $\rho_M(t)$ defined in Eq. (0) is $o(t)$ but cannot be $O(t^p)$ for any $p>1$. □

Lemma 6. *Given* $2\geqslant q>p>1$, *and an uncountable set* Γ, *then* $\ell_p(\Gamma)$ *does not admit any equivalent* q*-strongly* UG *smooth norm* (*and hence* $\ell_p(\Gamma)$ *is not* $\ell_q(\Gamma)$ *generated*).

Proof. Assume the statement is false. By Lemma 0 there exist an equivalent norm $\|\cdot\|$ on $\ell_p(\Gamma)$, a bounded linearly dense set $M\subset\ell_p(\Gamma)$, and $C>0$ such that

$$\|x+th\|^q+\|x-th\|^q-2\|x\|^q\leqslant Ct^q \text{ whenever } x\in X,\ h\in M,\ \text{and } t>0.$$

Let $\|\cdot\|_p$ denote the canonical norm on $\ell_p(\Gamma)$. Apply Stegall's variational principle) ([23, Corollary 5.22]), [13] or ([12, Chapter 10]) to the function $x\mapsto\|x\|^q-\|x\|_p^p$. Thus we get $x\in\ell_p(\Gamma)$ such that

$$\|x+h\|^q+\|x-h\|^q-2\|x\|^q\geqslant\|x+h\|_p^p+\|x-h\|_p^p-2\|x\|_p^p$$

for all $h\in\ell_p(\Gamma)$ (the linear term gets cancelled). Then we have

$$Ct^q\geqslant\|x+th\|_p^p+\|x-th\|_p^p-2\|x\|_p^p \quad \text{whenever } h\in M,\ \text{and } t>0.$$

Find $\gamma\in\Gamma$ so that $x(\gamma)=0$. Surely, there exists $h\in M$ so that $h(\gamma)\neq 0$. Then for all $t>0$,

$$Ct^q\geqslant|(x+th)(\gamma)|^p+|(x-th)(\gamma)|^p-2|x(\gamma)|^p=2t^p|h(\gamma)|^p,$$

which is impossible. We used here the fact that $|a+b|^p+|a-b|^p\geqslant 2|a|^p$ whenever $a,b\in\mathbb{R}$. □

(iv) does not imply (iii)

This can be shown on the following example. Let X be the Banach space such that its dual X^* is a Tsirelson like space $\mathscr{T}^*$ constructed in ([16, p. 43]). We recall that $\mathscr{T}^*$ is nonseparable, reflexive, admits a 1-unconditional symmetric Schauder basis $\{(e_\gamma,f_\gamma)\}_{\gamma<\omega_1}$, and has the property that for every nonseparable subspace Y of X^*, for every $1<p<+\infty$ and for every uncountable set Γ there is no linear bounded injection from Y into $\ell_p(\Gamma)$.

According to Theorem 4, in order to show that this X satisfies (iv), it is enough to prove that for every $\varepsilon>0$ there is $m\in\mathbb{N}$ such that

$$\#\{\gamma\in[1,\omega_1);\ |f_\gamma(x^*)|>\varepsilon\|x^*\|\}<m \quad \text{for every} \quad x^*\in X^*.$$

Since X is reflexive and has a symmetric basis, this follows immediately from ([26, Lemma 2]).

It remains to disprove (iii) for this X. Assume it holds. Then, by Theorem 3(iii), there are $1<p_n\leqslant 2$, $n\in\mathbb{N}$, a set Γ, and a linear bounded mapping T: $(\sum_{n=1}^{\infty}\ell_{p_n}(\Gamma))_{\ell_2}\to X$ with dense range. Putting $q_n=\frac{p_n}{p_n-1}$, and denoting by P_n

the canonical "projection" from $(\sum_{n=1}^{\infty} \ell_{q_n}(\Gamma))_{\ell_2}$ onto $\ell_{q_n}(\Gamma)$, we get the mappings $P_n \circ T^*$: $X^* \to \ell_{q_n}(\Gamma)$ for every $n \in \mathbb{N}$.

Assume for a while that $P_n \circ T^*$ has separable range for every $n \in \mathbb{N}$. Then, as T^* is injective, X^* linearly and continuously injects into the (separable) space $(\sum_{n=1}^{\infty} \ell_{q_n})_{\ell_2}$, and, finally, $(\sum_{n=1}^{\infty} \ell_{p_n})_{\ell_2}$ is mapped linearly and continuously onto a dense set in X. Therefore X is separable, which is a contradiction.

Hence, there is $n \in \mathbb{N}$ so that $P_n \circ T^*$: $X^* \to \ell_{q_n}(\Gamma)$ has a nonseparable range. In what follows we will put $q := q_n$ and $S := P_n \circ T^*$. Thus $S : X^* \to \ell_q(\Gamma)$, where $S(X^*)$ is nonseparable, and we will deduce a contradiction. Let $\alpha_1 < \omega_1$ be the first ordinal with $Se_{\alpha_1} \neq 0$. Consider $1 < \xi < \omega_1$ and assume that we already found $\alpha_\beta < \omega_1$ for $1 \leqslant \beta < \xi$. Since $S(X^*)$ is not separable and Γ is uncountable, there is $\sup_{\beta<\xi} \alpha_\beta < \alpha_\xi < \omega_1$ so that $Se_{\alpha_\xi} \neq 0$ and the support of Se_{α_ξ} is disjoint from the support of Se_{α_β} for every $\beta < \xi$. Thus we can construct α_β for every $\beta < \omega_1$. Denote by Y the closed linear span of $\{e_{\alpha_\beta};\ \beta < \omega_1\}$. Clearly, Y is nonseparable. It remains to check that the restriction of S to Y is injective. Take $y \in Y$ so that $Sy = 0$. Since $\{(e_\gamma, f_\gamma)\}_{\gamma<\omega_1}$ is a basis for X^*, we have $y = \sum_{\beta<\omega_1} a_\beta e_{\alpha_\beta}$ for some $a_\beta \in \mathbb{R}$. Then $0 = Sy = \sum_{\beta<\omega_1} a_\beta Se_{\alpha_\beta}$. As the vectors Se_{α_β}, $\beta < \omega_1$, have pairwise disjoint supports, we conclude that $a_\beta = 0$ for every $\beta < \omega_1$, and hence $y = 0$. Therefore S is injective and this contradicts the property of the space X^* mentioned at the beginning.

Note that if the space X satisfies (iv) but not (iii), Day's norm and the formula (5) are needed for constructing a strongly UG smooth norm, and we have to leave the superreflexive frame. □

(v) $\Rightarrow$ (iv) is false

This can be seen, assuming the Continuum Hypothesis, as follows. We construct a uniform Eberlein compact K in $(c_0(\mathbb{N}^{\mathbb{N}}), w)$. Using the ideas from [1,5], we use K to build a reflexive space Y (with an unconditional basis), whose unit ball is a uniform Eberlein compact. Finally, we show that the dual to Y disproves our implication. This will be done by using the properties of the compact K and Theorem 4.

For $p \in \mathbb{N}$, let K_p denote the set of all elements of the form $\frac{1}{p}\chi_A$ where A is any finite subset of $\mathbb{N}^{\mathbb{N}}$ and $\gamma(p) \geqslant \#A$ for every $\gamma \in A$. We claim that K_p is closed in $\{0, \frac{1}{p}\}^{\mathbb{N}^{\mathbb{N}}}$, and so is a weakly compact set in $c_0(\mathbb{N}^{\mathbb{N}})$. Indeed, let $(\frac{1}{p}\chi_{A_\tau})_{\tau \in T}$ be a net in K_p converging to some $\frac{1}{p}\chi_B \in \{0, \frac{1}{p}\}^{\mathbb{N}^{\mathbb{N}}}$. Assume that B contains an infinite sequence $\gamma_1, \gamma_2, \ldots$ of elements of $\mathbb{N}^{\mathbb{N}}$. Put $i = \gamma_1(p) + 1$. Find $\tau \in T$ so that $\gamma_1, \ldots, \gamma_i \in A_\tau$. Then $\gamma_1(p) \geqslant \#A_\tau \geqslant i$, a contradiction. Hence, B is a finite set. Since $B \subset A_\tau$ for some $\tau \in T$, we get that $\frac{1}{p}\chi_B \in K_p$.

For $i, p \in \mathbb{N}$ put $\Gamma_i^p = \{\gamma \in \mathbb{N}^{\mathbb{N}};\ \gamma(p) = i\}$. Clearly, $\bigcup_{i=1}^{\infty} \Gamma_i^p = \mathbb{N}^{\mathbb{N}}$ for every $p \in \mathbb{N}$. We observe that for any $\varepsilon > 0$, any $i \in \mathbb{N}$, and any $x = \frac{1}{p}\chi_A \in K_p$,

$$\#\{\gamma \in \Gamma_i^p;\ x(\gamma) > \varepsilon\} \leqslant \#\Gamma_i^p \cap A \begin{cases} = 0 & \text{if } \#A > i, \\ \leqslant \#A & \text{if } \#A \leqslant i. \end{cases}$$

Define $K = \bigcup_{p=1}^{\infty} K_p$. It is easy to check that K is weak compact in $c_0(\mathbb{N}^{\mathbb{N}})$. We will show that it is a uniform Eberlein compact. So fix $\varepsilon > 0$. Find $p \in \mathbb{N}$ so that $\varepsilon > \frac{1}{p}$. Fix any $i_1, \dots, i_p \in \mathbb{N}$. Take any $x \in K$. We observe that if $x(\gamma) > \varepsilon$ for some $\gamma \in \mathbb{N}^{\mathbb{N}}$, then necessarily $x \in K_j$ for some $j \in \{1, \dots, p-1\}$. Thus we can estimate

$$\#\{\gamma \in \Gamma_{i_1}^1 \cap \cdots \cap \Gamma_{i_p}^p;\ x(\gamma) > \varepsilon\} \leqslant \#\{\gamma \in \Gamma_{i_j}^j;\ x(\gamma) > \varepsilon\} \leqslant i_j.$$

Having this estimate and observing that

$$\bigcup \{\Gamma_{i_1}^1 \cap \cdots \cap \Gamma_{i_p}^p;\ i_1, \dots, i_p \in \mathbb{N},\ p \in \mathbb{N}\} = \mathbb{N}^{\mathbb{N}},$$

the easier part of ([1, Theorem 1.7]) guarantees that K is a uniform Eberlein compact.

Define

$$W = \{y \in c_0(\mathbb{N}^{\mathbb{N}});\ \text{there is } x \in \overline{\text{co}}(K \cup -K)$$
$$\text{so that } |y(\gamma)| \leqslant |x(\gamma)| \text{ for all } \gamma \in \mathbb{N}^{\mathbb{N}}\}.$$

This is a uniform Eberlein compact by ([1, Proposition 1.5, Lemma 3.6]). Then the interpolation theorem [5] produces a reflexive Banach space $(Y, |\,||\cdot||\,|)$ such that Y is a subset of $c_0(\mathbb{N}^{\mathbb{N}})$ and that $W \subset B_{(Y,|\,||\cdot||\,|)} \subset B_{c_0(\mathbb{N}^{\mathbb{N}})}$. ([1, Lemma 3.5]) then says that $B_{(Y,|\,||\cdot||\,|)}$ is even a uniform Eberlein compact. Then, by ([3, Theorem 3.2]), the space $C((B_{(Y,|\,||\cdot||\,|)}, w))$ is Hilbert-generated, and hence it is UG smoothly renormable. Therefore the space $X := (Y, |\,||\cdot||\,|)^*$, is reflexive and admits an equivalent UG smooth norm, see Theorem 1 or ([6, Theorem II.6.8(ii)]). Note that, according to ([5, Lemma 1(x)]), if $\{e_\gamma;\ \gamma \in \mathbb{N}^{\mathbb{N}}\}$ is the canonical basis in $c_0(\mathbb{N}^{\mathbb{N}})$, then it is an unconditional basis for $(Y, |\,||\cdot||\,|)$), and hence X also has an unconditional basis.

It remains to show that X *does not admit any equivalent strongly* UG *smooth norm*. Here we assume the Continuum Hypothesis, which allows us to use Theorem 4. If X has a strongly UG smooth norm, then by Theorem 4, there are a set Λ and a linear bounded injective mapping $T: X^* = Y \to c_0(\Lambda)$ such that for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\#\{\lambda \in \Lambda;\ |Ty(\lambda)| > \varepsilon |\,||y||\,|\} < m$ for every $y \in Y$. Note that $K \subset Y$. We claim that *there exists* $p \in \mathbb{N}$ *so that the set*

$$\left\{\gamma \in \mathbb{N}^{\mathbb{N}};\ \gamma(p) = n \text{ and } ||Te_\gamma|| > \frac{1}{p}\right\}$$

is infinite for infinitely many $n \in \mathbb{N}$. Assume this is not true. We will construct, by induction, a sequence $n_1, n_2, \dots$ in $\mathbb{N}$ and finite subsets $M_1, M_2, \dots$ of $\mathbb{N}^{\mathbb{N}}$ as follows. Find $n_1 \in \mathbb{N}$ such that

$$M_1 := \{\gamma \in \mathbb{N}^{\mathbb{N}};\ \gamma(1) = n_1 \text{ and } ||Te_\gamma|| > 1\}$$

is finite. Let $p \in \mathbb{N}$ and assume that we already found $n_p \in \mathbb{N}$ and the finite set M_p. Find then $n_{p+1} \in \mathbb{N}$ so large that $n_{p+1} > \max\{\gamma(p+1);\ \gamma \in M_p\}$ and that the set

$$M_{p+1} = \left\{\gamma \in \mathbb{N}^{\mathbb{N}};\ \gamma(p+1) = n_{p+1} \text{ and } ||Te_\gamma|| > \frac{1}{p+1}\right\}$$

is finite. Performing this for every $p \in \mathbb{N}$, put $\tilde{\gamma} = (n_1, n_2, \ldots)$. Then $\tilde{\gamma} \in M_p$ for no $p \in \mathbb{N}$, and hence $||Te_{\tilde{\gamma}}|| = 0$, a contradiction with the injectivity of the mapping T.

Therefore there exist $p \in \mathbb{N}$, an infinite increasing sequence $n^1, n^2, \ldots$ in $\mathbb{N}$, and a double sequence $\{\gamma^i_j;\ i,j \in \mathbb{N}\}$ of distinct elements of $\mathbb{N}^{\mathbb{N}}$ such that for every $i,j \in \mathbb{N}$,

$$\gamma^i_j(p) = n^i \quad \text{and} \quad ||Te_{\gamma^i_j}|| > \frac{1}{p}.$$

Fix $i \in \mathbb{N}$. Then $e_{\gamma^i_j} \to 0$ weakly as $j \to \infty$, and so $Te_{\gamma^i_j} \to 0$ weakly as $j \to \infty$. Put $a_j = e_{\gamma^i_j}$, $j \in \mathbb{N}$. By induction we find integers $1 = j_1 < j_2 < \cdots < j_{n^i}$ and different $\lambda_1, \lambda_2, \ldots, \lambda_{n^i} \in \Lambda$ so that

$$|Ta_{j_1}(\lambda_1)| > \frac{1}{p}, \ldots, |Ta_{j_{n^i}}(\lambda_{n^i})| > \frac{1}{p}$$

and

$$|Ta_{j_k}(\lambda_l)| < \frac{1}{2pn^i} \quad \text{whenever} \quad 1 \leqslant l < k \leqslant n^i.$$

Put $A = \{\gamma^i_{j_1}, \gamma^i_{j_2}, \ldots, \gamma^i_{j_{n^i}}\}$ and $\bar{y} = \frac{1}{p}\chi_A$. Then $\bar{y} \in K_p$, $T\bar{y} = \frac{1}{p}\sum_{k=1}^{n^i} Ta_{j_k}$, and so

$$T\bar{y}(\lambda_k) > \frac{1}{p}\left(\frac{1}{p} - \frac{n^i - 1}{2pn^i}\right) > \frac{1}{2p^2} \quad \text{for every} \quad k = 1, 2, \ldots, n^i.$$

Thus

$$\#\left\{\lambda \in \Lambda;\ |T\bar{y}(\lambda)| > \frac{1}{2p^2}\right\} \geqslant n^i.$$

Since this can be done for every $i \in \mathbb{N}$, we get

$$\sup\left\{\#\left\{\lambda \in \Lambda;\ |T\bar{y}(\lambda)| \geqslant \frac{1}{2p^2}\right\};\ y \in K_p\right\} \geqslant \sup\{n^i;\ i \in \mathbb{N}\} = +\infty,$$

a contradiction.

We observe that if X satisfies (v) but not (iv), then B_{X^*} embeds continuously and linearly into a uniform Eberlein weakly compact subset K of $c_0(\Gamma)$, but the Argyros–Farmaki decompositions of K depend upon ε (see [1]). Hence formula (7) of Remark 1 is needed for constructing a UG smooth norm. $\square$

(vi) does not imply (v)

This can be shown on Rosenthal's example R [25]. This is a subspace of an L_1 over a probability space. Thus L_1 is Hilbert-generated (L_2 is dense in L_1) and hence UG smooth. Then R is also UG smooth. Yet R is not weakly compactly generated [25]. Since any operator from an Asplund space into L_1 is weakly compact, this space R also shows that (vi) (or equivalently, the existence of a UG-smooth equivalent norm) does not imply that the space is Asplund generated.

Note that if X satisfies (vi) but not (v), then B_{X^*} is homeomorphic to a weakly compact subset K of $\ell_2(\Gamma)$, but B_{X^*} does not embed continuously and linearly into $c_0(\Gamma)$ equipped with its weak topology. □

Appendix

The purpose of this appendix is to give a simple proof of Theorem 1 in [17]. Recall that a norm $\|\cdot\|$ of a Banach space X is weakly uniformly rotund if $x_n - y_n \to 0$ weakly in X whenever $x_n, y_n \in S_X$ and $\|x_n + y_n\| \to 2$. Note that the norm is weakly uniformly rotund if and only if the dual norm is UG, by the result of Šmulyan (cf. e.g. [6, Theorem II.6.7]) or ([12, Chapter 8]).

Theorem (Hájek [17]). *Assume that the norm of a Banach space X is weakly uniformly rotund. Then X is an Asplund space.*

Proof. We assume that X is separable and prove that X^* is then separable. For $n \in \mathbb{N}$, put

$$V_n = \{f \in B_{X^*}; |f(x-y)| \leqslant 1/3 \text{ if } x, y \in B_X \text{ are such that } \|x+y\| \geqslant 2 - 1/n\}.$$

As X is weakly uniformly rotund, we have $B_{X^*} = \bigcup V_n$. For $n \in \mathbb{N}$, let S_n be a countable and weak* dense subset of V_n. We claim that

$$\overline{\operatorname{span}}^{\|\cdot\|}\left(\bigcup_{n=1}^{\infty} S_n\right) = X^*.$$

If not, take $F \in S_{X^{**}}$ with $F(f) = 0$ for all $f \in \cup S_n$, and choose $f_0 \in S_{X^*}$ with $F(f_0) > 8/9$. There is $n_0 \in \mathbb{N}$ such that $f_0 \in V_{n_0}$. Let $\{x_\alpha\}$ be a net in B_X which weak* converges to F. We have

$$\|x_\alpha + x_\beta\| \geqslant 2 - 1/n_0$$

if α and β are large enough. By definition of V_{n_0}, it follows that there is α_0 such that

$$|f(x_{\alpha_0}) - f(x_\beta)| \leqslant 1/3$$

for all β large enough and all $f \in V_{n_0}$. Since $\{x_\alpha\}$ weak* converges to F it follows that

$$|f(x_{\alpha_0}) - F(f)| \leqslant 1/3$$

for all $f \in V_{n_0}$. Hence for $f \in S_{n_0}$,

$$|(f - f_0)(x_{\alpha_0})| \geqslant |F(f) - F(f_0)| - |F(f_0) - f_0(x_{\alpha_0})| - |f(x_{\alpha_0}) - F(f)| \geqslant 2/9.$$

Thus f_0 does not belong to the weak* closure of S_{n_0}. This contradiction concludes the proof. $\square$

Every weakly uniformly rotund space X admits an equivalent Fréchet differentiable norm. Indeed, then X^* is a subspace of a weakly compactly generated space ([11]) and thus X admits a Fréchet differentiable norm (cf. e.g. [6, Chapter VI]).

An example of a weakly uniformly rotund space Z such that Z^* is not weakly compactly generated is the space Z constructed in ([25, p. 90]).

There are weakly uniformly rotund spaces that are not subspaces of weakly compactly generated spaces. Such is the space U constructed in ([19, p. 222]). There U^* is isomorphic to $\ell_1 \oplus \ell_2(\Gamma)$ and thus U^* admits an equivalent UG norm. By Šmulyan's duality (cf. e.g. [6, Chapter II] or [12, Chapter 8]), U^{**} admits an equivalent weak star uniformly rotund norm and thus U admits an equivalent weakly uniformly rotund norm. It is proved in ([19, p. 222]) that U is not weakly compactly generated and a similar argument gives that U is not a subspace of any weakly compactly generated space (cf. e.g. [12, Chapter 12]).

However, the following problem seems to be open.

Problem. Assume that the norm of a Banach space X is weakly uniformly rotund. Does there exist a bounded linear one-to-one operator from X into $c_0(\Gamma)$ for some Γ?

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