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When is the Cuntz–Krieger algebra of a higher-rank graph approximately finite-dimensional? ☆

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Abstract

We investigate the question: when is a higher-rank graph C^*-algebra approximately finite-dimensional? We prove that the absence of an appropriate higher-rank analogue of a cycle is necessary. We show that it is not in general sufficient, but that it is sufficient for higher-rank graphs with finitely many vertices. We give a detailed description of the structure of the C^*-algebra of a row-finite locally convex higher-rank graph with finitely many vertices. Our results are also sufficient to establish that if the C^*-algebra of a higher-rank graph is AF, then its every ideal must be gauge-invariant. We prove that for a higher-rank graph C^*-algebra to be AF it is necessary and sufficient for all the corners determined by vertex projections to be AF. We close with a number of examples which illustrate why our question is so much more difficult for higher-rank graphs than for ordinary graphs.

Keywords: Graph C^*-algebra; C^*-algebra; AF algebra; Higher-rank graph; Cuntz–Krieger algebra

1. Introduction

A directed graph E consists of countable sets E^0 and E^1 and maps $r, s : E^1 \to E^0$. We call elements of E^0 *vertices* and elements of E^1 *edges* and think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. When $r^{-1}(v)$ is finite and nonempty for all v, the graph C^*-algebra $C^*(E)$ is

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the universal C^*-algebra generated by a family of mutually orthogonal projections $\{p_v\colon v \in E^0\}$ and a family of partial isometries $\{s_e\colon e \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$ and $p_v = \sum_{r(e)=v} s_e s_e^*$ for all $v \in E^0$ [18,33].

Despite the elementary nature of these relations, the class of graph C^*-algebras is quite rich. It includes, up to strong Morita equivalence, all AF algebras [16,52], all Kirchberg algebras whose K_1-group is free abelian [51] and many other interesting C^*-algebras besides [25,26]. We know this because we can read off a surprising amount of the structure of a graph C^*-algebra (for example its K-theory [35,42], and its whole primitive ideal space [27]) directly from the graph. In particular, a graph C^*-algebra is AF if and only if the graph contains no directed cycles [32, Theorem 2.4]. Moreover, if E contains a directed cycle and $C^*(E)$ is simple, then $C^*(E)$ is purely infinite. So every simple graph C^*-algebra is classifiable either by Elliott's theorem or by the Kirchberg–Phillips theorem.

In 2000, Kumjian and Pask introduced higher-rank graphs, or k-graphs, and their C^*-algebras [31] as a generalisation of graph algebras designed to model Robertson and Steger's higher-rank Cuntz–Krieger algebras [45]. These have proved a very interesting source of examples in recent years [15,36], but remain far less well understood than their 1-dimensional counterparts, largely because their structure theory is much more complicated. In particular, a general structure result for simple k-graph algebras is still lacking; even a satisfactory characterisation of simplicity itself is in full generality fairly recent [47]. The examples of [36] show that there are simple k-graph algebras which are neither AF nor purely infinite, indicating that the question is more complicated than for directed graphs. Some fairly restrictive sufficient conditions have been identified which ensure that a simple k-graph C^*-algebra is AF [31, Lemma 5.4] or is purely infinite [49, Proposition 8.8], but there is a wide gap between the two.

Deciding whether a given C^*-algebra is AF is an interesting and notoriously difficult problem. The guiding principle seems to be that if, from the point of view of its invariants, it looks AF and it smells AF, then it is probably AF. This point of view led to the discovery and analyses of non-AF fixed point subalgebras of group actions on non-standard presentations of AF algebras initiated by [2] and [30] and continued by [6,19] and others. Numerous powerful AF embeddability theorems (the canonical example is [38]; and more recently for example [11,28,50]) have also been uncovered. These results demonstrate that algebraic obstructions — beyond the obvious one of lack of stable finiteness — to approximate finite-dimensionality of C^*-algebras are hard to come by. On the other hand, proving that a given C^*-algebra is AF can be a highly nontrivial task (cf. [6,9] and the series of penetrating analyses of actions of finite subgroups of $SL_2(\mathbb{Z})$ on the irrational rotation algebra initiated by [5,8,54] and culminating in [17]). Moreover, non-standard presentations of AF algebras have found applications in classification theory [38], and also to long-standing questions such as the Powers–Sakai conjecture [29].

In this paper, we consider more closely the question of when a k-graph C^*-algebra is AF. The question is quite vexing, and we have not been able to give a complete answer (see Example 4.2). However, we have been able to weaken the existing necessary condition for the presence of an infinite projection, and also to show that for a k-graph C^*-algebra to be AF, it is necessary that the k-graph itself should contain no directed cycles; indeed, we identify a notion of a higher-dimensional cycle the presence of which precludes approximate finite-dimensionality of the associated C^*-algebra. Our results are sufficiently strong to completely characterise when a unital k-graph C^*-algebra is AF, and to completely describe the structure of unital k-graph C^*-algebras associated to row-finite k-graphs. We also provide some examples confirming some earlier conjectures of the first author. Specifically, we construct a 2-graph Λ which contains no cycles and in which every infinite path is aperiodic, but such that $C^*(\Lambda)$ is finite but not AF, and we construct an example of a 2-graph which does not satisfy [20, Condition (S)] but does satisfy [20, Condition (Γ)] and whose C^*-algebra is AF. We close with an intriguing example of a 2-graph Λ_{II} whose infinite-path space contains a dense set of periodic points, but whose C^*-algebra is simple, unital and AF-embeddable, and shares many invariants with the 2^∞ UHF algebra. If, as seems likely, the C^*-algebra of Λ_{II} is strongly Morita equivalent to the 2^∞ UHF algebra, it will follow that the structure theory of simple k-graph algebras is much more complex than for graph algebras.

We remark that a proof that $C^*(\Lambda_{\mathrm{II}})$ is indeed AF would provide another interesting non-standard presentation of an AF algebra. It would open up the possibility that known constructions for k-graph C^*-algebras might provide new insights into questions about AF algebras.

2. Background

We introduce some background relating to k-graphs and their C^*-algebras. See [31,40,41] for details.

2.1. Higher-rank graphs

Fix an integer $k > 0$. We regard $\mathbb{N}^k$ as a semigroup under pointwise addition with identity element denoted 0. When convenient, we also think of it as a category with one object. We denote the generators of $\mathbb{N}^k$ by $e_1, \dots, e_k$, and for $n \in \mathbb{N}^k$ and $i \leqslant k$ we write n_i for the ith coordinate of n; so $n = (n_1, n_2, \dots, n_k) = \sum_{i=1}^k n_i e_i$. For $m, n \in \mathbb{N}^k$, we write $m \leqslant n$ if $m_i \leqslant n_i$ for all i, and we write $m \vee n$ for the coordinatewise maximum of m and n, and $m \wedge n$ for the coordinatewise minimum of m and n. Observe that $m \wedge n \leqslant m, n \leqslant m \vee n$, and that $m' := m - (m \wedge n)$ and $n' := n - (m \wedge n)$ is the unique pair such that $m - n = m' - n'$ and $m' \wedge n' = 0$. For $n \in \mathbb{N}^k$, we write $|n|$ for the *length* $|n| = \sum_{i=1}^k n_i$ of n.

As introduced in [31], a *graph of rank k* or a k-graph is a countable small category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree functor*, which satisfies the *factorisation property*: for all $m, n \in \mathbb{N}^k$ and all $\lambda \in \Lambda$ with $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

We write Λ^n for $d^{-1}(n)$. If $d(\lambda) = 0$ then $\lambda = \mathrm{id}_o$ for some object o of Λ. Hence $r(\lambda) := \mathrm{id}_{\mathrm{cod}(\lambda)}$ and $s(\lambda) := \mathrm{id}_{\mathrm{dom}(\lambda)}$ determine maps $r, s : \Lambda \to \Lambda^0$ which restrict to the identity map on Λ^0 (see [31]). We think of elements of Λ^0 both as vertices and as paths of degree 0, and we think of each $\lambda \in \Lambda$ as a path from $s(\lambda)$ to $r(\lambda)$. If $v \in \Lambda^0$ and $\lambda \in \Lambda$, then the composition $v\lambda$ makes sense if and only if $v = r(\lambda)$. With this in mind, given a subset E of Λ, and a vertex $v \in \Lambda^0$, we write vE for the set $\{\lambda \in E\colon\ r(\lambda) = v\}$. Similarly, Ev denotes $\{\lambda \in E\colon\ s(\lambda) = v\}$. In particular, for $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $v\Lambda^n = \{\lambda \in \Lambda\colon\ d(\lambda) = n \text{ and } r(\lambda) = v\}$. Moreover, given a subset H of Λ^0, we let EH denote the set $\{\lambda \in E\colon\ s(\lambda) \in H\}$ and set $HE = \{\lambda \in E\colon\ r(\lambda) \in H\}$.

We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that Λ has *no sources* if $v\Lambda^n$ is nonempty for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that Λ is *locally convex* if, whenever $\mu \in \Lambda^{e_i}$ and $r(\mu)\Lambda^{e_j} \neq \emptyset$ with $i \neq j$, we have $s(\mu)\Lambda^{e_j} \neq \emptyset$ also.

For $\lambda \in \Lambda$ and $m \leqslant n \leqslant d(\lambda)$, we denote by $\lambda(m, n)$ the unique element of Λ^{n-m} such that $\lambda = \lambda'\lambda(m, n)\lambda''$ for some $\lambda', \lambda'' \in \Lambda$ with $d(\lambda') = m$ and $d(\lambda'') = d(\lambda) - n$.

For $\mu, \nu \in \Lambda$, a *minimal common extension* of μ and ν is a path λ such that $d(\lambda) = d(\mu) \vee d(\nu)$ and $\lambda = \mu\mu' = \nu\nu'$ for some $\mu', \nu' \in \Lambda$. Equivalently, λ is a minimal common extension of μ and ν if $d(\lambda) = d(\mu) \vee d(\nu)$ and $\lambda(0, d(\mu)) = \mu$ and $\lambda(0, d(\nu)) = \nu$. We write $\mathrm{MCE}(\mu, \nu)$ for the set of all minimal common extensions of μ and ν, and we say that Λ is *finitely aligned* if $\mathrm{MCE}(\mu, \nu)$

is finite (possibly empty) for all $\mu, \nu \in \Lambda$. If Γ is a sub-k-graph of Λ, then for $\mu, \nu \in \Gamma$ we write $\mathrm{MCE}_\Gamma(\mu, \nu)$ and $\mathrm{MCE}_\Lambda(\mu, \nu)$ to emphasise in which k-graph we are computing the set of minimal common extensions. We have $\mathrm{MCE}_\Gamma(\mu, \nu) = \mathrm{MCE}_\Lambda(\mu, \nu) \cap (\Gamma \times \Gamma)$.

For $\lambda \in \Lambda$ and $E \subseteq r(\lambda)\Lambda$, the set of paths $\tau \in s(\lambda)\Lambda$ such that $\lambda\tau \in \mathrm{MCE}(\lambda, \mu)$ for some $\mu \in E$ is denoted $\mathrm{Ext}(\lambda; E)$. That is,

$$\mathrm{Ext}(\lambda; E) = \bigcup_{\mu \in E} \big\{\tau \in s(\lambda)\Lambda \colon \lambda\tau \in \mathrm{MCE}(\lambda, \mu)\big\}.$$

By [22, Proposition 3.12], we have $\mathrm{Ext}(\lambda\mu; E) = \mathrm{Ext}(\mu; \mathrm{Ext}(\lambda; E))$ for all composable λ, μ and all $E \subseteq r(\lambda)\Lambda$.

Fix a vertex $v \in \Lambda^0$. A subset $F \subseteq v\Lambda$ is called *exhaustive* if for every $\lambda \in v\Lambda$ there exists $\mu \in F$ such that $\mathrm{MCE}(\lambda, \mu) \neq \emptyset$. By [41, Lemma C.5], if $E \subset r(\lambda)\Lambda$ is exhaustive, then $\mathrm{Ext}(\lambda; E) \subseteq s(\lambda)\Lambda$ is also exhaustive.

2.2. Higher-rank graph C^*-algebras

Let Λ be a finitely aligned k-graph. A Cuntz–Krieger Λ-family is a subset $\{t_\lambda \colon \lambda \in \Lambda\}$ of a C^*-algebra B such that

(CK1) $\{t_v \colon v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(CK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $t_\mu^* t_\nu = \sum_{\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu,\nu)} t_\alpha t_\beta^*$ for all $\mu, \nu \in \Lambda$; and
(CK4) $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$ for all $v \in \Lambda^0$ and finite exhaustive sets $E \subseteq v\Lambda$.

The C^*-algebra $C^*(\Lambda)$ of Λ is the universal C^*-algebra generated by a Cuntz–Krieger Λ-family; the universal family in $C^*(\Lambda)$ is denoted $\{s_\lambda \colon \lambda \in \Lambda\}$.

The universal property of $C^*(\Lambda)$ ensures that there exists a strongly continuous action γ of $\mathbb{T}^k$ on $C^*(\Lambda)$ satisfying $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for all $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$, where $z^{d(\lambda)}$ is defined by the standard multi-index formula $z^{d(\lambda)} = z_1^{d(\lambda)_1} z_2^{d(\lambda)_2} \dots z_k^{d(\lambda)_k}$.

The Cuntz–Krieger relations can be simplified significantly under additional hypotheses. For details of the following, see [41, Appendix B]. Suppose that Λ is row-finite and locally convex. For $n \in \mathbb{N}^k$, define

$$\Lambda^{\leqslant n} := \bigcup_{m \leqslant n} \big\{\lambda \in \Lambda^m \colon s(\lambda)\Lambda^{e_i} = \emptyset \text{ for all } i \leqslant k \text{ such that } m_i < n_i\big\}.$$

Then (CK3) and (CK4) are equivalent to

(CK3′) $t_\mu^* t_\mu = t_{s(\mu)}$ for all $\mu \in \Lambda$, and
(CK4′) $t_v = \sum_{\lambda \in v\Lambda^{\leqslant n}} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

If Λ is has no sources, then $\Lambda^{\leqslant n} = \Lambda^n$ for all n, so if Λ is row-finite and has no sources then (CK4′) is equivalent to

(CK4″) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Note that (CK3) implies (CK3′) for all k-graphs Λ.

Recall from [37] that a *graph trace* on a row-finite k-graph Λ with no sources is a function $g : \Lambda^0 \to \mathbb{R}^+$ such that $g(v) = \sum_{\lambda \in v\Lambda^n} g(s(\lambda))$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. A graph trace g is called *faithful* if $g(v) \neq 0$ for all $v \in \Lambda^0$. Proposition 3.8 of [37] describes how faithful graph traces on Λ correspond with faithful gauge-invariant semifinite traces on $C^*(\Lambda)$. We call a graph trace g *finite* if $\sum_{v \in \Lambda^0} g(v)$ converges to some $T \in \mathbb{R}^+$, and we say that a finite graph trace g is *normalised* if $\sum_{v \in \Lambda^0} g(v) = 1$.

Lemma 2.1. *Let Λ be a row-finite k-graph with no sources. Each normalised finite faithful graph trace g on Λ determines a faithful bounded gauge-invariant trace τ_g on $C^*(\Lambda)$ which is normalised in the sense that the limit over increasing finite subsets F of Λ^0 of $\tau_g(\sum_{v \in F} s_v)$ is equal to* 1: *specifically, $\tau_g(s_\mu s_\nu^*) = \delta_{\mu,\nu} g(s(\mu))$ for all $\mu, \nu \in \Lambda$. Moreover, $g \mapsto \tau_g$ is a bijection between normalised finite faithful graph traces on Λ and normalised faithful gauge-invariant traces on $C^*(\Lambda)$.*

Proof. By [37, Proposition 3.8], the map $g \mapsto \tau_g$ is a bijection between faithful (not necessarily finite or normalised) graph traces on Λ and faithful semifinite lower-semicontinuous gauge-invariant traces on $C^*(\Lambda)$. So it suffices to show that τ_g is finite if and only if g is finite, and that τ_g is normalised if and only if g is normalised. For this, for each finite $F \subseteq \Lambda^0$ let $P_F := \sum_{v \in F} s_v \in C^*(\Lambda)$. Then the P_F form an approximate identity, and so τ_g is finite if and only if $\lim_F \tau_g(P_F) = \sum_{v \in \Lambda^0} g(v)$ converges. Moreover, each of g and τ_g is normalised if and only if $\sum_{v \in \Lambda^0} g(v)$ converges to 1. $\square$

2.3. *Infinite paths and aperiodicity*

For each $m \in (\mathbb{N} \cup \{\infty\})^k$, we define a k-graph $\Omega_{k,m}$ by

$$\Omega_{k,m} = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k\colon\ p \leqslant q \leqslant m\}, \quad \text{with}$$
$$r(p,q) = (p,p), \qquad s(p,q) = (q,q) \quad \text{and} \quad d(p,q) = q - p.$$

It is standard to identify $\Omega_{k,m}^0$ with $\{p \in \mathbb{N}^k\colon\ p \leqslant m\}$ by $(p,p) \mapsto p$, and we shall silently do so henceforth.

If Λ and Γ are k-graphs, then a k-graph morphism $\phi : \Lambda \to \Gamma$ is a functor from Λ to Γ which preserves degree: $d_\Gamma(\phi(\lambda)) = d_\Lambda(\lambda)$ for all $\lambda \in \Lambda$.

Given a k-graph Λ and $m \in \mathbb{N}^k$, each $\lambda \in \Lambda^m$ determines a k-graph morphism $x_\lambda : \Omega_{k,m} \to \Lambda$ by $x_\lambda(p,q) := \lambda(p,q)$ for all $(p,q) \in \Omega_{k,m}$. Moreover, each k-graph morphism $x : \Omega_{k,m} \to \Lambda$ determines an element $x(0,m)$ of Λ^m. Thus we identify the collection of k-graph morphisms from $\Omega_{k,m}$ to Λ with Λ^m when $m \in \mathbb{N}^k$. Extending this idea, given $m \in (\mathbb{N} \cup \{\infty\})^k \setminus \mathbb{N}^k$, we regard k-graph morphisms $x : \Omega_{k,m} \to \Lambda$ as paths of degree m in Λ and write $d(x) := m$ and $r(x)$ for $x(0)$; we denote the set of all such paths by Λ^m. When $m = (\infty, \infty, \dots, \infty)$, we denote $\Omega_{k,m}$ by Ω_k and we call a path x of degree m in Λ an *infinite path*. We denote by W_Λ the collection $\bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^m$ of all paths in Λ; our conventions allow us to regard Λ as a subset of W_Λ.

For each $n \in \mathbb{N}^k$ there is a *shift map* $\sigma^n : \{x \in W_\Lambda\colon\ n \leqslant d(x)\} \to W_\Lambda$ such that $d(\sigma^n(x)) = d(x) - n$ and $\sigma^n(x)(p,q) = x(n+p, n+q)$ for $0 \leqslant p \leqslant q \leqslant d(x) - n$. Given $x \in W_\Lambda$ and $\lambda \in \Lambda r(x)$, there is a unique $\lambda x \in W_\Lambda$ satisfying $d(\lambda x) = d(\lambda) + d(x)$, $(\lambda x)(0, d(\lambda)) = \lambda$ and $\sigma^{d(\lambda)}(\lambda x) = x$. For $x \in W_\Lambda$ and $n \leqslant d(x)$, we then have $x(0,n)\sigma^n(x) = x$.

A *boundary path* in Λ is a path $x : \Omega_{k,m} \to \Lambda$ with the property that for all $p \in \Omega_{k,m}^0$ and all finite exhaustive sets $E \subseteq x(p)\Lambda$, there exists $\mu \in E$ such that $x(p, p+d(\mu)) = \mu$. We denote by $\partial\Lambda$ the collection of all boundary paths in Λ. Lemma 5.15 of [22] implies that for each $v \in \Lambda^0$, the set $v\partial\Lambda := \{x \in \partial\Lambda\colon r(x) = v\}$ is nonempty. Fix $x \in \partial\Lambda$. If $n \leqslant d(x)$, then $\sigma^n(x) \in \partial\Lambda$, and if $\lambda \in \Lambda r(x)$, then $\lambda x \in \partial\Lambda$ [22, Lemma 5.13]. Recall also from [40] that if Λ is row-finite and locally convex, then $\partial\Lambda$ coincides with the set

$$\Lambda^{\leqslant\infty} = \{x \in W_\Lambda\colon\ x(n)\Lambda^{e_i} = \emptyset \text{ whenever } n \leqslant d(x) \text{ and } n_i = d(x)_i\}.$$

Recall from [34] that a finitely-aligned k-graph Λ is said to be *aperiodic* if for all $\mu, \nu \in \Lambda$ such that $s(\mu) = s(\nu)$ there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. By [34, Proposition 3.6 and Theorem 4.1], the following are equivalent:

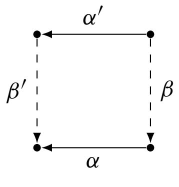
(1) Λ is aperiodic;
(2) for all distinct $m, n \in \mathbb{N}^k$ and $v \in \Lambda^0$ there exists $x \in v\partial\Lambda$ such that either $m \vee n \nleqslant d(x)$ or $\sigma^m(x) \neq \sigma^n(x)$;
(3) for all $v \in \Lambda^0$ there exists $x \in v\partial\Lambda$ such that for distinct $m, n \leqslant d(x)$, $\sigma^m(x) \neq \sigma^n(x)$;
(4) for every nontrivial ideal I of $C^*(\Lambda)$ there exists $v \in \Lambda^0$ such that $s_v \in I$.

Here, and in the rest of the paper, an "ideal" of a C^*-algebra always means a closed 2-sided ideal.

2.4. Skeletons

We will frequently wish to present a k-graph visually. To do this, we draw its *skeleton* and, if necessary, list the associated *factorisation rules.*

Given a k-graph Λ, the *skeleton* of Λ is the coloured directed graph E_Λ with vertices $E_\Lambda^0 := \Lambda^0$, edges $E_\Lambda^1 := \bigcup_{i=1}^k \Lambda^{e_i}$ and with colouring map $c : E_\Lambda^1 \to \{1, \dots, k\}$ given by $c(\alpha) = i$ if and only if $\alpha \in \Lambda^{e_i}$. In pictures in this paper, edges of degree e_1 will be drawn as solid lines and those of degree e_2 as dashed lines. If $\alpha, \beta \in E_\Lambda^1$ have distinct colours, say $c(\alpha) = i$ and $c(\beta) = j$, and if $s(\alpha) = r(\beta)$, then $\alpha\beta \in \Lambda^{e_i+e_j}$ and the factorisation property in Λ implies that there are unique edges $\beta', \alpha' \in E_\Lambda^1$ such that $c(\beta') = c(\beta)$ and $c(\alpha') = c(\alpha)$ and such that



is a commuting diagram in Λ. We call such a diagram a *square* and we denote by $\mathcal{C}$ the collection of all such squares. We write $\alpha\beta \sim_{\mathcal{C}} \beta'\alpha'$, or just $\alpha\beta \sim \beta'\alpha'$. We call the list of all such relations the *factorisation rules* for E_Λ. It turns out that Λ is uniquely determined up to isomorphism by its skeleton and factorisation rules [23,24]. Moreover, given a k-coloured directed graph E and a collection of factorisation rules of the form $\alpha\beta \sim \beta'\alpha'$ where $\alpha\beta$ and $\beta'\alpha'$ are bi-coloured paths of opposite colourings with the same range and source, there exists a k-graph with this skeleton and set of factorisation rules if and only if both of the following conditions are satisfied: (1) the relation $\sim$ is bijective in the sense that for each ij-coloured path $\alpha\beta$, there is exactly one ji-coloured path $\beta'\alpha'$ such that $\alpha\beta \sim \beta'\alpha'$; and (2) if $\alpha\beta \sim \beta^1\alpha^1$, $\alpha^1\gamma \sim \gamma^1\alpha^2$ and $\beta^1\gamma^1 \sim \gamma^2\beta^2$, and if $\beta\gamma \sim \gamma_1\beta_1$, $\alpha\gamma_1 \sim \gamma_2\alpha_1$ and $\alpha_1\beta_1 \sim \beta_2\alpha_2$, then $\alpha^2 = \alpha_2$, $\beta^2 = \beta_2$ and $\gamma^2 = \gamma_2$. Observe that (2) is vacuous unless α, β and γ are of three distinct colours, so if $k = 2$, then condition (1) by itself characterises those lists of factorisation rules which determine 2-graphs.

If E_Λ has the property that given any two vertices v, w and any two colours $i, j \leqslant k$, there is at most one path fg from w to v such that $c(f) = i$ and $c(g) = j$, then there is just one possible complete collection of squares possible for this skeleton. In this situation, we just draw the skeleton to specify Λ, and do not bother to list the squares.

3. Cycles and generalised cycles

In this section we present a necessary condition on an arbitrary finitely aligned k-graph for its C^*-algebra to be AF.

As with graph C^*-algebras, the necessary conditions for k-graph C^*-algebras to be AF which we have developed involve the presence of cycles of an appropriate sort in the k-graph. To formulate a result sufficiently general to deal with the examples which we introduce later, we propose the notion of a *generalised cycle*. We have not been able to construct a non-AF k-graph C^*-algebra which could not be recognised as such by the presence of a generalised cycle in the complement of some hereditary subgraph, but we have no reason to believe that such an example does not exist. For the origins of the following definition, see [20, Lemma 4.3]

Definition 3.1. Let Λ be a finitely aligned k-graph. A *generalised cycle* in Λ is a pair $(\mu, \nu) \in \Lambda \times \Lambda$ such that $\mu \neq \nu$, $s(\mu) = s(\nu)$, $r(\mu) = r(\nu)$, and $\mathrm{MCE}(\mu\tau, \nu) \neq \emptyset$ for all $\tau \in s(\mu)\Lambda$.

Lemma 3.2. *Let Λ be a finitely aligned k-graph. Fix a pair $(\mu, \nu) \in \Lambda \times \Lambda$ such that $\mu \neq \nu$, $s(\mu) = s(\nu)$ and $r(\mu) = r(\nu)$. Then the following are equivalent*:

(1) *the pair (μ, ν) is a generalised cycle*;
(2) *the set* $\mathrm{Ext}(\mu, \{\nu\})$ *is exhaustive*; *and*
(3) $\{\mu x\colon x \in s(\mu)\partial\Lambda\} \subseteq \{\nu y\colon y \in s(\nu)\partial\Lambda\}$.

Proof. Suppose that (μ, ν) is a generalised cycle. Fix $\lambda \in s(\mu)\Lambda$. Then $\mathrm{MCE}(\mu\lambda, \nu) \neq \emptyset$, and hence $\mathrm{Ext}(\mu\lambda, \{\nu\}) \neq \emptyset$. By [22, Proposition 3.12], we have

$$\mathrm{Ext}\big(\mu\lambda, \{\nu\}\big) = \mathrm{Ext}\big(\lambda; \mathrm{Ext}\big(\mu, \{\nu\}\big)\big),$$

and hence there exists $\alpha \in \mathrm{Ext}(\mu, \{\nu\})$ such that $\mathrm{MCE}(\lambda, \alpha) \neq \emptyset$. Hence $\mathrm{Ext}(\mu, \{\nu\})$ is exhaustive. This proves (1) $\Longrightarrow$ (2).

Now suppose that $\mathrm{Ext}(\mu, \{\nu\})$ is exhaustive. Since Λ is finitely aligned, $\mathrm{Ext}(\mu, \{\nu\})$ is also finite, and hence it is a finite exhaustive subset of $s(\mu)\Lambda$. Fix $x \in s(\mu)\partial\Lambda$. By definition of $\partial\Lambda$ there exists $\alpha \in \mathrm{Ext}(\mu, \{\nu\})$ such that $x(0, d(\alpha)) = \alpha$. Hence $(\mu x)(0, d(\mu) \vee d(\nu)) = \mu\alpha \in \mathrm{MCE}(\mu, \nu)$, and it follows that $(\mu x)(0, d(\nu)) = (\mu\alpha)(0, d(\nu)) = \nu$. Thus $y := \sigma^{d(\nu)}(\mu x)$ satisfies $y \in s(\nu)\partial\Lambda$ and $\mu x = \nu y$. This proves (2) $\Longrightarrow$ (3).

Finally suppose that $\{\mu x\colon x \in s(\mu)\partial\Lambda\} \subseteq \{\nu y\colon y \in s(\nu)\partial\Lambda\}$. Fix $\tau \in s(\mu)\Lambda$. Since $s(\tau)\partial\Lambda \neq \emptyset$ [22, Lemma 5.15], we may fix $z \in s(\tau)\partial\Lambda$, and then $x := \tau z \in s(\mu)\partial\Lambda$ also [22, Lemma 5.13]. By hypothesis, we then have $\mu x = \nu y$ for some $y \in s(\nu)\partial\Lambda$. In particular, $(\mu x)(0, d(\mu\tau) \vee d(\nu)) \in \mathrm{MCE}(\mu\tau, \nu)$, and hence the latter is nonempty. This proves (3) $\Longrightarrow$ (1). □

In the language of [22], condition (3) of Lemma 3.2 says that the cylinder sets $Z(\mu)$ and $Z(\nu)$ are nested: $Z(\mu) \subseteq Z(\nu)$.

For the remainder of the paper, the term *cycle*, as distinct from *generalised cycle*, will continue to refer to a path $\lambda \in \Lambda \setminus \Lambda^0$ such that $r(\lambda) = s(\lambda)$. When — as in Section 5 — we wish to emphasise that we mean a cycle in the traditional sense, rather than a generalised cycle, we will also use the term *conventional cycle.*

To see where the definition of a generalised cycle comes from, observe that if λ is a conventional cycle in a k-graph, then $(\lambda, r(\lambda))$ is a generalised cycle. There are plenty of examples of k-graphs containing generalised cycles but no cycles (see Example 6.1), but when $k = 1$, the two notions more or less coincide:

Lemma 3.3. *Let Λ be a* 1*-graph. Suppose that (μ, ν) is a generalised cycle in Λ. Then there is a conventional cycle $\lambda \in \Lambda \setminus \Lambda^0$ such that either $\mu = \nu\lambda$ or $\nu = \mu\lambda$.*

Proof. Since Λ is a 1-graph, either $d(\mu) \leqslant d(\nu)$ or vice versa. We will assume that $d(\mu) \leqslant d(\nu)$ and show that $\nu = \mu\lambda$ for some conventional cycle λ; if instead $d(\nu) \leqslant d(\mu)$ then the same argument gives $\mu = \nu\lambda$. If $d(\mu) = d(\nu)$, then $\operatorname{MCE}(\mu, \nu) \neq \emptyset$ forces $\mu = \nu$ which is impossible for a generalised cycle, so $d(\mu) < d(\nu)$. Then $\tau := s(\mu) \in s(\mu)\Lambda$ satisfies $\operatorname{MCE}(\mu\tau, \nu) \neq \emptyset$. This forces $\nu = \mu\lambda$ for some λ. Now $r(\lambda) = s(\mu)$ and $s(\lambda) = s(\nu) = s(\mu)$, so λ is a conventional cycle. □

The main result in this section is the following.

Theorem 3.4. *Let Λ be a finitely aligned k-graph. If $C^*(\Lambda)$ is AF, then Λ contains no generalised cycles.*

The proof deals separately with two cases. To delineate the cases, we introduce the notion of an entrance to a generalised cycle.

Definition 3.5. Let Λ be a finitely aligned k-graph. An *entrance* to a generalised cycle (μ, ν) is a path $\tau \in s(\nu)\Lambda$ such that $\operatorname{MCE}(\nu\tau, \mu) = \emptyset$.

If λ is a conventional cycle then an *entrance to the conventional cycle* λ means an entrance to the associated generalised cycle $(\lambda, r(\lambda))$; that is a path $\tau \in r(\lambda)\Lambda$ such that $\operatorname{MCE}(\tau, \lambda) = \emptyset$.

Remark 3.6. A generalised cycle (μ, ν) has an entrance if and only if the reversed pair (ν, μ) is not a generalised cycle.

Lemma 3.7. *Suppose that (μ, ν) is a generalised cycle. Then $s_\mu s_\mu^* \leqslant s_\nu s_\nu^*$. Moreover, $s_\mu s_\mu^* = s_\nu s_\nu^*$ if and only if the generalised cycle (μ, ν) has no entrance.*

Proof. Since $\operatorname{Ext}(\mu, \{\nu\}) \subset s(\mu)\Lambda^{(d(\mu)\vee d(\nu))-d(\mu)}$, for distinct $\alpha, \beta \in \operatorname{Ext}(\mu, \{\nu\})$, we have $s_\alpha s_\alpha^* s_\beta s_\beta^* = 0$. In particular, applying (CK4),

$$0 = \prod_{\alpha \in \operatorname{Ext}(\mu, \{\nu\})} \left(s_{s(\mu)} - s_\alpha s_\alpha^*\right) = s_{s(\mu)} - \sum_{\alpha \in \operatorname{Ext}(\mu, \{\nu\})} s_\alpha s_\alpha^*.$$

Hence

$$s_\mu s_\mu^* = s_\mu s_{s(\mu)} s_\mu^* = \sum_{\alpha \in \mathrm{Ext}(\mu,\{\nu\})} s_{\mu\alpha} s_{\mu\alpha}^*.$$

For each $\alpha \in \mathrm{Ext}(\mu, \{\nu\})$, we have $\mu\alpha = \nu\beta$ for some $\beta \in \Lambda$, and hence $s_{\mu\alpha} s_{\mu\alpha}^* = s_\nu (s_\beta s_\beta^*) s_\nu^* \leqslant s_\nu s_\nu^*$, giving $s_\mu s_\mu^* \leqslant s_\nu s_\nu^*$.

Suppose that the generalised cycle (μ, ν) has no entrance. Then (ν, μ) is also a generalised cycle, and the preceding paragraph gives $s_\nu s_\nu^* \leqslant s_\mu s_\mu^*$ also.

Now suppose that the generalised cycle (μ, ν) has an entrance τ; so $\mathrm{MCE}(\nu\tau, \mu) = \emptyset$. Then $s_{\nu\tau} s_{\nu\tau}^* \leqslant s_\nu s_\nu^*$ and

$$s_{\nu\tau} s_{\nu\tau}^* s_\mu s_\mu^* = \sum_{\lambda \in \mathrm{MCE}(\nu\tau,\mu)} s_\lambda s_\lambda^* = 0.$$

Hence $s_\nu s_\nu^* - s_\mu s_\mu^* \geqslant s_{\nu\tau} s_{\nu\tau}^* > 0$. □

Corollary 3.8. *(See [20, Lemma 4.3].) Let Λ be a finitely aligned k-graph which contains a generalised cycle with an entrance. Then $C^*(\Lambda)$ contains an infinite projection. In particular $C^*(\Lambda)$ is not AF.*

Proof. Let (μ, ν) be the generalised cycle with an entrance. By Lemma 3.7, we have

$$s_\nu s_\nu^* > s_\mu s_\mu^* = s_\mu s_\nu^* s_\nu s_\mu^* \sim s_\nu s_\mu^* s_\mu s_\nu^* = s_\nu s_\nu^*.$$

Hence $s_\nu s_\nu^*$ is an infinite projection. The last statement follows immediately. □

We must now show that when Λ contains a generalised cycle with no entrance, $C^*(\Lambda)$ is not AF. The following result is the key step. The argument is essentially that of [7, Proposition 4.4.1], and we thank George Elliott for directing our attention to [7].

Proposition 3.9. *Let A be a unital C^*-algebra carrying a normalised trace T, and let $\beta : \mathbb{T} \to \mathrm{Aut}(A)$ be a strongly continuous action. Let U be a unitary in A, and suppose that there exists $n \in \mathbb{Z} \setminus \{0\}$ satisfying $\beta_z(U) = z^n U$ for all $z \in \mathbb{T}$. Then U does not belong to the connected component of the identity in the unitary group $\mathcal{U}(A)$.*

Proof. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ be the action determined by $\alpha_t(a) := \beta_{e^{2\pi i t}}(a)$. Let $\mathcal{D}(\delta) := \{a \in A: \lim_{t\to 0} \frac{1}{t}(\alpha_t(a) - a) \text{ exists}\}$, and let $\delta : \mathcal{D}(\delta) \to A$ be the generator of α; that is $\delta(a) := \lim_{t\to 0} \frac{1}{t}(\alpha_t(a) - a)$ for $a \in \mathcal{D}(\delta)$. Note that $U \in \mathcal{D}(\delta)$ since we have

$$\delta(U) = \lim_{t\to 0} \frac{1}{t}\big(\beta_{e^{2\pi i t}}(U) - U\big) = \lim_{t\to 0} \frac{e^{2n\pi i t} - 1}{t} U = 2n\pi i U. \tag{3.1}$$

Let μ denote the normalised Haar measure on $\mathbb{T}$. Define a map $\tau : A \to \mathbb{C}$ by $\tau(a) := \int_{\mathbb{T}} T(\beta_z(a))\, d\mu(z)$. We claim that τ is a normalised β-invariant (and hence α-invariant) trace on A. Given $a \in A$, for each $z \in \mathbb{T}$ we have $T(\beta_z(a^*a)) = T(\beta_z(a)^*\beta_z(a)) \geqslant 0$ as T is a trace.

Hence $\tau(a^*a) \geqslant 0$ so τ is positive. It is clearly linear, and it satisfies $\tau(1) = 1$ because β fixes 1. For $a, b \in A$ we calculate:

$$\tau(ab) = \int_{\mathbb{T}} T\big(\beta_z(a)\beta_z(b)\big)\, d\mu(z) = \int_{\mathbb{T}} T\big(\beta_z(b)\beta_z(a)\big)\, d\mu(z) = \tau(ba).$$

So τ is a trace. Finally, to see that τ is β-invariant, note that for $a \in A$, we have $\tau(\beta_z(a)) = \int_{\mathbb{T}} T(\beta_w(\beta_z(a)))\, d\mu(w) = \int_{\mathbb{T}} T(\beta_{zw}(a))\, d\mu(w) = \int_{\mathbb{T}} \beta_{w'}(a)\, d\mu(z^{-1}w') = \tau(a)$ by left-invariance of μ.

It now follows from [39, p. 281, lines 7–16] that for a unitary $V \in \mathcal{D}(\delta)$ which is also in the connected component $\mathcal{U}_0(A)$ of the identity, we have $\tau(V^*\delta(V)) = 0$. However, using (3.1), we have $\tau(U^*\delta(U)) = \tau(U^* 2n\pi i U) = \tau(2n\pi i 1_A) = 2n\pi i$, and it follows that $U \notin \mathcal{U}_0(A)$. □

Proposition 3.10. *Let Λ be a finitely aligned k-graph, and let $\phi : \mathbb{Z}^k \to \mathbb{Z}$ be a homomorphism. Suppose that there exist $N \in \mathbb{Z} \setminus \{0\}$ and a partial isometry $V \in \overline{\operatorname{span}}\{s_\mu s_\nu^*\colon \mu, \nu \in \Lambda,\ \phi(d(\mu) - d(\nu)) = N\}$ such that $VV^* = V^*V$. Then $C^*(\Lambda)$ is not AF.*

Proof. Let $P = V^*V$. Then V is a unitary in $PC^*(\Lambda)P$.

For each $i \leqslant k$, let $\phi_i = \phi(e_i)$ so that $\phi(n) = \sum_{i=1}^k \phi_i n_i$ for all $n \in \mathbb{Z}^k$. Define a homomorphism $\iota_\phi : \mathbb{T} \to \mathbb{T}^k$ by $\iota(z)_i = z^{\phi_i}$ for $1 \leqslant i \leqslant k$, and define $\beta : \mathbb{T} \to \operatorname{Aut}(C^*(\Lambda))$ by $\beta_z := \gamma_{\iota_\phi(z)}$ for all $z \in \mathbb{T}$.

For $\mu, \nu \in \Lambda$ we have

$$\beta_z\big(s_\mu s_\nu^*\big) = \gamma_{\iota_\phi(z)}\big(s_\mu s_\nu^*\big) = \iota_\phi(z)^{d(\mu)-d(\nu)} s_\mu s_\nu^* = z^{\phi(d(\mu)-d(\nu))} s_\mu s_\nu^*.$$

In particular, since $V \in \overline{\operatorname{span}}\{s_\mu s_\nu^*\colon \phi(d(\mu) - d(\nu)) = N\}$, we have $\beta_z(V) = z^N V$ for all $z \in \mathbb{T}$ so that β fixes P and hence restricts to an action on $PC^*(\Lambda)P$. Now suppose that $C^*(\Lambda)$ is an AF algebra; we seek a contradiction. Since corners of AF algebras are AF [14, Exercise III.2], $PC^*(\Lambda)P$ is a unital AF algebra, and hence carries a normalised trace. We may therefore apply Proposition 3.9 to see that V does not belong to the connected component of the unitary group of $PC^*(\Lambda)P$. This is a contradiction since the unitary group of any unital AF algebra is connected. □

Proof of Theorem 3.4. We prove the contrapositive statement. Let (μ, ν) be a generalised cycle in Λ. If (μ, ν) has an entrance, then Corollary 3.8 implies that $C^*(\Lambda)$ is not AF. So suppose that (μ, ν) has no entrance.

Since $d(\mu) \neq d(\nu)$ there exists i such that $d(\mu)_i \neq d(\nu)_i$. Define $\phi : \mathbb{Z}^k \to \mathbb{Z}$ by $\phi(n) := n_i$, let $N := d(\mu)_i - d(\nu)_i$, and let $V := s_\mu s_\nu^*$. By Lemma 3.7, we have $VV^* = V^*V$, so Proposition 3.10 applied to V, N, ϕ implies that $C^*(\Lambda)$ is not AF. □

Using the characterisation of gauge-invariant ideals in k-graph algebras of [49], and using also that quotients of AF algebras are AF, we can extend the main theorem somewhat, at the expense of a more technical statement. Example 6.3 indicates that the extended result is genuinely stronger.

Corollary 3.11. *Let Λ be a finitely aligned k-graph. Suppose that there exists a saturated hereditary subset H of Λ^0 such that $\Lambda \setminus \Lambda H$ contains a generalised cycle. Then $C^*(\Lambda)$ is not AF.*

Moreover, given a saturated hereditary subset H of Λ^0, a pair $(\mu, \nu) \in (\Lambda \setminus \Lambda H)^2$ is a generalised cycle in $\Lambda \setminus \Lambda H$ if and only if $d(\mu) \neq d(\nu)$, $s(\mu) = s(\nu)$, $r(\mu) = r(\nu)$, and $\mathrm{MCE}_\Lambda(\nu, \mu\tau) \setminus \Lambda H \neq \emptyset$ for every $\tau \in \Lambda \setminus \Lambda H$.

Proof. For the first statement observe that by [49, Lemma 4.1], $\Lambda \setminus \Lambda H$ is a finitely aligned k-graph, and [49, Corollary 5.3] applied with $B = \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$ implies that $C^*(\Lambda \setminus \Lambda H)$ is a quotient of $C^*(\Lambda)$. If $\Lambda \setminus \Lambda H$ contains a generalised cycle, then Theorem 3.4 implies that $C^*(\Lambda \setminus \Lambda H)$ is not AF, and since quotients of AF algebras are AF, it follows that $C^*(\Lambda)$ is not AF either.

For the final statement, observe that

$$\mathrm{MCE}_{\Lambda \setminus \Lambda H}(\alpha, \beta) = \mathrm{MCE}_\Lambda(\alpha, \beta) \setminus \Lambda H.$$

So by Remark 3.6, a generalised cycle in $\Lambda \setminus \Lambda H$ is a pair of distinct paths (μ, ν) in $\Lambda \setminus \Lambda H$ with the same range and source such that for every $\tau \in s(\mu)\Lambda \setminus \Lambda H$, the set $\mathrm{MCE}_\Lambda(\nu, \mu\tau) \setminus \Lambda H$ is nonempty as claimed. □

Theorem 3.4 combined with the results of [34] shows in particular that aperiodicity of every quotient graph is necessary for $C^*(\Lambda)$ to be AF. We use this to show that if $C^*(\Lambda)$ is AF, then its ideals are indexed by the saturated hereditary subsets of Λ^0.

Proposition 3.12. *Let Λ be a finitely aligned k-graph such that $C^*(\Lambda)$ is AF. Then for every saturated hereditary subset H of Λ, and every pair η, ζ of distinct paths in $\Lambda \setminus \Lambda H$, there exists $\tau \in s(\eta)\Lambda \setminus \Lambda H$ such that $\mathrm{MCE}(\eta\tau, \zeta\tau) \subset \Lambda H$. Moreover every ideal of $C^*(\Lambda)$ is gauge-invariant.*

Proof. For the first statement of the proposition, we prove the contrapositive. Suppose that there exist a saturated hereditary $H \subset \Lambda^0$ and distinct paths $\eta, \zeta \in \Lambda \setminus \Lambda H$ such that for every $\tau \in s(\eta)\Lambda \setminus \Lambda H$, we have $\mathrm{MCE}(\eta\tau, \zeta\tau) \cap (\Lambda \setminus \Lambda H) \neq \emptyset$. Let $\Gamma := \Lambda \setminus \Lambda H$. The paths $\eta, \zeta \in \Gamma$ have the property that for every $\tau \in s(\eta)\Gamma$, we have $\mathrm{MCE}_\Gamma(\eta\tau, \zeta\tau) \neq \emptyset$. That is, Γ is not aperiodic in the sense of [34, Definition 3.1].

By [34, Proposition 3.6 and Definition 3.5], there exist $v \in \Gamma^0$ and distinct $m, n \in \mathbb{N}^k$ such that $m \vee n \leqslant d(x)$ and $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\partial\Gamma$. By [34, Lemma 4.3], there then exist $\mu, \nu, \alpha \in \Gamma$ such that $d(\mu) = m$, $d(\nu) = n$, $r(\mu) = r(\nu)$, $s(\mu) = s(\nu) = r(\alpha)$, and $\mu\alpha x = \nu\alpha x$ for all $x \in s(\alpha)\partial\Gamma$. In particular $\{\mu\alpha x\colon x \in s(\mu\alpha)\partial\Lambda\} \subseteq \{\nu\alpha y\colon y \in s(\nu\alpha)\partial\Lambda\}$. So (3) $\Longrightarrow$ (1) of Lemma 3.2 implies that $(\mu\alpha, \nu\alpha)$ is a generalised cycle in Γ, and then Theorem 3.4 implies that $C^*(\Gamma)$ is not AF.

Corollary 5.3 of [49] implies that $C^*(\Gamma)$ is isomorphic to the quotient of $C^*(\Lambda)$ by the ideal generated by $\{s_v\colon v \in H\}$. Since quotients of AF algebras are AF, it follows that $C^*(\Lambda)$ is also not AF.

To prove the second statement, suppose that $C^*(\Lambda)$ is indeed AF. The previous statement combined with [34, Lemma 4.4] implies that for each saturated hereditary $H \subseteq \Lambda^0$, each $v \in \Lambda^0 \setminus H$ and each finite $F \subset \Lambda v$, there exists $\tau \in v\Lambda \setminus \Lambda H$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$ for all distinct $\mu, \nu \in F$. We may now run the proof of [48, Theorem 6.3], leaving out Lemma 6.4 and the first two paragraphs of the proof of Lemma 6.7 and using τ in place of the path $x(0, N)$ in the remainder of the proof of Lemma 6.7, to see that the conclusion of [48, Theorem 6.3] holds for any relative Cuntz–Krieger algebra associated to $\Lambda \setminus \Lambda H$; and then the argument of [49, Theorem 7.2] implies that every ideal of $C^*(\Lambda)$ is gauge-invariant as claimed. □

4. Corners and skew-products

We begin this section with a characterisation of approximate finite-dimensionality of $C^*(\Lambda)$ in terms of the same property for corners of the form $s_v C^*(\Lambda) s_v$. We then describe a recipe for constructing examples of k-graphs whose C^*-algebras are AF.

Proposition 4.1. *Let (Λ, d) be a finitely aligned k-graph. Then $C^*(\Lambda)$ is AF if and only if the corners $s_v C^*(\Lambda) s_v$, $v \in \Lambda^0$ are all AF.*

Proof. It is standard that corners of AF algebras are AF (see, for example, [14, Exercise III.2]), proving the "only if" implication.

For the "if" direction, suppose that each $s_v C^*(\Lambda) s_v$ is AF. For each finite $F \subset \Lambda^0$, let $P_F := \sum_{v \in F} s_v$. We claim that each ideal $I_F := C^*(\Lambda) P_F C^*(\Lambda)$ is AF. We proceed by induction on $|F|$. If $|F| = 1$, say $F = \{v\}$, then $I_F \sim_{\text{Me}} s_v C^*(\Lambda) s_v$ is AF by hypothesis. Now suppose that I_F is AF whenever $|F| \leqslant n$ and fix $F \subset \Lambda^0$ with $|F| = n+1$. Fix $v \in F$, and let $G = F \setminus \{v\}$. Then I_G and $I_{\{v\}}$ are both AF by the inductive hypothesis, and hence $I_G/(I_{\{v\}} \cap I_G)$ is also AF because quotients of AF algebras are AF. Since $I_F = I_G + I_{\{v\}}$, there is an exact sequence

$$0 \to I_{\{v\}} \to I_F \to I_G/(I_{\{v\}} \cap I_G) \to 0.$$

Since extensions of AF algebras by AF algebras are also AF (see, for example [14, Theorem III.6.3]), it follows that I_F is AF as claimed.

Clearly $G \subseteq F$ implies $I_G \subseteq I_F$. Thus $C^*(\Lambda) = \overline{\bigcup_{F \subset \Lambda^0 \text{ finite}} I_F}$ is AF because the class of AF algebras is closed under taking countable direct limits (this follows from an $\varepsilon/2$ argument using [4, Theorem 2.2]). □

We now describe a class of examples of k-graphs whose C^*-algebras are AF. The primary motivation is the example Λ_{I} discussed in Section 6.

Recall from [31, Definition 1.9] that if $f : \mathbb{N}^k \to \mathbb{N}^l$ is a surjective homomorphism, and Λ is an l-graph, then there is a pullback k-graph $f^*(\Lambda)$ equal as a set to $\{(\lambda, n) \in \Lambda \times \mathbb{N}^k \colon d(\lambda) = f(n)\}$ with pointwise operations and degree map $d(\lambda, n) := n$. Also recall that if $c : \Lambda \to \mathbb{Z}^k$ is a functor from a k-graph to $\mathbb{Z}^k$, then we can form the skew-product k-graph $\Lambda \times_c \mathbb{Z}^k$, which is equal as a set to $\Lambda \times \mathbb{Z}^k$ with structure maps $r(\lambda, m) = (r(\lambda), m)$, $s(\lambda, m) = (s(\lambda), m + c(\lambda))$, and $(\lambda, m)(\mu, m + c(\lambda)) = (\lambda\mu, m)$.

Example 4.2. Let E be a row-finite 1-graph, and denote the degree functor on E by $|\cdot| : E \to \mathbb{N}$. Let c_0 be a function from E^1 to $\{0, e_1, \ldots, e_{k-1}\} \subseteq \mathbb{Z}^k$, and for $\lambda = \lambda_1 \ldots \lambda_n \in E^n$, let $c_0(\lambda) := \sum_{i=1}^n c_0(\lambda_i)$. Define $c_0(v) = 0$ for $v \in E^0$. Define $f : \mathbb{N}^k \to \mathbb{N}$ by $f(n) = \sum_{i=1}^k n_i$, and a functor $c : f^*(E) \to \mathbb{Z}^k$ by

$$c(\lambda, n)_j := \begin{cases} c_0(\lambda)_j - \sum_{i \neq j} n_i & \text{if } j < k, \\ |\lambda| & \text{if } j = k. \end{cases}$$

Let Λ be the skew-product k-graph $\Lambda = f^*(E) \times_c \mathbb{Z}^k$. Identifying $(E \times \mathbb{N}^k) \times \mathbb{Z}^k$ with $E \times \mathbb{N}^k \times \mathbb{Z}^k$, we have

$$f^*(E) \times_c \mathbb{Z}^k = \{(\alpha, m, q) \in E \times \mathbb{N}^k \times \mathbb{Z}^k \colon |\alpha| = f(m)\}.$$

Observe that $p(\lambda, n, a) := \lambda$ defines a functor from Λ to E. In particular, each $v \in \Lambda^0$ has the form $v = (p(v), 0, q)$ for some $q \in \mathbb{Z}^k$, and then $\mu = (p(\mu), d(\mu), q)$ for all $\mu \in v\Lambda$.

We will show that $C^*(\Lambda)$ is AF, with the corners of $C^*(\Lambda)$ determined by vertex projections isomorphic to corresponding corners of the AF core of $C^*(E)$.

Lemma 4.3. *Consider the situation of Example* 4.2. *Fix a vertex $v = (p(v), 0, q) \in \Lambda^0$. Fix $\mu, \nu \in v\Lambda$, let $m := d(\mu)$ and $n := d(\nu)$, and express $s(\mu)$ as $(w, 0, q) \in E^0 \times \{0\} \times \mathbb{Z}^k = \Lambda^0$. Then*

$$s_\mu^* s_\nu = \begin{cases} \sum_{\tau \in p(s(\mu))E^{|n|}} s_{(\tau, n, q+c(\mu,m))} s^*_{(\eta\tau, m, q+c(\nu,n))} & \text{if } p(\mu) = p(\nu)\eta, \\ \sum_{\tau \in p(s(\nu))E^{|m|}} s_{(\zeta\tau, n, q+c(\mu,m))} s^*_{(\tau, m, q+c(\nu,n))} & \text{if } p(\nu) = p(\mu)\zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $N := m + n$. We have

$$\begin{aligned} s_\mu^* s_\nu &= \sum_{\lambda \in v\Lambda^{m+n}} s_\mu^* s_\lambda s_\lambda^* s_\nu \\ &= \sum_{\xi \in wE^N} s^*_{(p(\mu), m, q)} s_{(\xi, m+n, q)} s^*_{(\xi, m+n, q)} s_{(p(\nu), n, q)}. \end{aligned}$$

Factorising each $\xi \in wE^N$ as $\xi = \xi_m \xi'_m = \xi_n \xi'_n$ where $|\xi_m| = |m|$ and $|\xi_n| = |n|$ gives

$$s_\mu^* s_\nu = \sum_{\xi \in wE^N} \bigl(s^*_{(p(\mu), m, q)} s_{(\xi_m, m, q)}\bigr) s_{(\xi'_m, n, q+c(\xi_m, m))} s^*_{(\xi'_n, m, q+c(\xi_n, n))} \bigl(s^*_{(\xi_n, n, q)} s_{(p(\nu), n, q)}\bigr). \tag{4.1}$$

The Cuntz–Krieger relations ensure that $s^*_{(p(\mu), m, q)} s_{(\xi_m, m, q)} = 0$ unless $\xi_m = p(\mu)$, and likewise $s^*_{(\xi_n, n, q)} s_{(p(\nu), n, q)} = 0$ unless $\xi_n = p(\nu)$. So $s_\mu^* s_\nu = 0$ unless there exists $\xi \in E^N$ such that $\xi = p(\mu)\xi'_m = p(\nu)\xi'_n$ which occurs if and only if either: (1) $p(\mu) = p(\nu)\eta$, and $\xi'_m = \tau$ and $\xi'_n = \eta\tau$ for some $\tau \in E^{|n|}$; or (2) $p(\nu) = p(\mu)\zeta$, and $\xi'_n = \tau$ and $\xi'_m = \zeta\tau$ for some $\tau \in E^{|m|}$. Using this to simplify (4.1), we obtain the desired formula for $s_\mu^* s_\nu$. □

Lemma 4.4. *Consider the situation of Example* 4.2. *Fix $\alpha, \beta \in E^N$. Define $a, b \in \mathbb{N}^k$ by*

$$a_j = \begin{cases} c_0(\beta)_j & j < k, \\ N - |c_0(\beta)| & j = k \end{cases} \quad \text{and} \quad b_j = \begin{cases} c_0(\alpha)_j & j < k, \\ N - |c_0(\alpha)| & j = k. \end{cases}$$

Then $(\alpha, a), (\beta, b) \in f^*(E)$. If $s(\alpha) = s(\beta)$, then $s(\alpha, a, q) = s(\beta, b, q)$ for all $q \in \mathbb{N}^k$, and if $s(\alpha) \neq s(\beta)$, then $s(\alpha, a, q) \neq s(\beta, b, q)$ for all $q \in \mathbb{N}^k$.

Proof. Clearly $f(a) = f(b) = N = |\alpha| = |\beta|$, so $(\alpha, a), (\beta, b) \in f^*(E)$. For $j \neq k$, we have

$$c(\alpha, a)_j = c_0(\alpha)_j - \sum_{i \neq j} a_i = b_j - (N - a_j) = a_j + b_j - N,$$

and similarly $c(\beta, b)_j = b_j + a_j - N$. Since $c(\alpha, a)_k = N = c(\beta, b)_k$, we have $c(\alpha, a) = c(\beta, b)$ and the result follows. □

Corollary 4.5. *Consider the situation of Example* 4.2. *For $v \in \Lambda^0$ and $N \in \mathbb{N}$, let $B_N(v) := \operatorname{span}\{s_\mu s_\nu^*\colon\ \mu, \nu \in v\Lambda,\ |\mu| = |\nu| = N\}$. Then for each $N \in \mathbb{N}$, the space $B_N(v)$ is a finite-dimensional C^*-algebra with nonzero matrix units*

$$\{w_{\eta,\zeta}\colon\ \eta, \zeta \in p(v)E^N,\ s(\eta) = s(\zeta)\}.$$

Moreover $s_\mu s_\nu^* = \sum_{\xi \in s(\mu)\Lambda^{e_1}} s_{\mu\xi} s_{\nu\xi}^*$ determines an inclusion $B_N(v) \subseteq B_{N+1}(v)$, and $s_v C^*(\Lambda) s_v = \overline{\bigcup_{N=1}^\infty B_N(v)}$. For each $v \in \Lambda^0$, we have $s_v C^*(\Lambda) s_v \cong t_{p(v)} C^*(E)^\gamma t_{p(v)}$. If Λ is cofinal, then $C^*(\Lambda)$ is strongly Morita equivalent to $s_w C^*(E)^\gamma s_w$ for any $w \in E^0$.

Proof. We first claim that if $\mu, \nu, \alpha, \beta \in v\Lambda$ with $|\mu| = |\nu| = |\alpha| = |\beta| = N$, $s(\mu) = s(\nu)$, $s(\alpha) = s(\beta)$, $p(\mu) = p(\alpha)$ and $p(\nu) = p(\beta)$, then $s_\mu s_\nu^* = s_\alpha s_\beta^*$.

To see this, let $m := d(\mu)$, $n := d(\nu)$, $a := d(\alpha)$, and $b := d(\beta)$, and use Lemma 4.3 to calculate

$$\begin{aligned} s_\mu s_\nu^* &= s_\mu s_\nu^* \sum_{\sigma \in p(\nu)E^{2N}} s_{(\sigma, n+b, q)} s_{(\sigma, n+b, q)}^* \\ &= \sum_{\tau \in s(p(\nu))E^N} s_{(p(\mu)\tau, m+b, q)} s_{(p(\nu)\tau, n+b, q)}^*. \end{aligned} \tag{4.2}$$

Likewise,

$$s_\alpha s_\beta^* = \sum_{\tau \in s(p(\beta))E^N} s_{(p(\alpha)\tau, a+n, q)} s_{(p(\beta)\tau, b+n, q)}^*. \tag{4.3}$$

We have $p(\mu) = p(\alpha)$ and $p(\nu) = p(\beta)$ by assumption. So it suffices to show that $a + n = m + b$. That $s(\mu) = s(\nu)$ implies in particular that $q + c(\mu) = q + c(\nu)$ and hence that for $j \neq k$,

$$c_0\big(p(\mu)\big)_j - \sum_{i \neq j} m_i = c_0\big(p(\nu)\big) - \sum_{i \neq j} n_i.$$

Similarly, each

$$c_0\big(p(\alpha)\big)_j - \sum_{i \neq j} a_i = c_0\big(p(\beta)\big) - \sum_{i \neq j} b_i.$$

Since $p(\mu) = p(\alpha)$ and $p(\nu) = p(\beta)$, we may subtract the two equations above to obtain

$$\sum_{i\neq j}(m_i - a_i) = \sum_{i\neq j}(n_i - b_i) \quad \text{for all } j \neq k.$$

Moreover, $\sum_{i=1}^k a_i = N = \sum_{i=1}^k m_i$, and similarly for the n_i and b_i, so we obtain $m_j - a_j = n_j - b_j$ for all $j < k$; and then $m_k - a_k = n_k - b_k$ also because $|m| = |n| = |a| = |b| = N$. So $m - a = n - b$, and rearranging we obtain $a + n = m + b$, proving the claim.

For $\eta, \zeta \in p(v)E^N$, Lemma 4.4 yields $a, b \in \mathbb{N}^k$ with $|a| = |b| = N$ and $c(\eta, a) = c(\zeta, b)$. We then have $s(\eta, a, q) = s(\zeta, b, q)$ if and only if $s(\eta) = s(\zeta)$. For each pair $\eta, \zeta \in p(v)E^n$ such that $s(\eta) = s(\zeta)$, define $w_{\eta,\zeta} := s_{(\eta,a,q)}s^*_{(\zeta,b,q)}$. The above claim shows that the $w_{\eta,\zeta}$ depend only on η and ζ and not on our choice of a and b. The claim also implies that $s_\mu s_\nu^* = w_{(p(\mu),p(\nu))}$ whenever $\mu, \nu \in v\Lambda^N$ with $s(\mu) = s(\nu)$ (this forces $s(p(\mu)) = s(p(\nu))$). Hence

$$B_N = \operatorname{span}\{w_{\eta,\zeta}\colon \eta, \zeta \in p(v)E^N,\ s(\eta) = s(\zeta)\}.$$

The $w_{\eta,\zeta}$ are nonzero because $s_\mu s_\nu^* \neq 0$ in $C^*(\Lambda)$ whenever $s(\mu) = s(\nu)$ [31, Remarks 1.6(iv)]. It remains to check that they are matrix units. We have

$$w^*_{\eta,\zeta} = \left(s_{(\eta,a,q)}s^*_{(\zeta,b,q)}\right)^* = s_{(\zeta,b,q)}s^*_{(\eta,a,q)} = w_{\zeta,\eta}$$

for any choice of a, b for which this makes sense. Lemma 4.3 implies that $w_{\eta,\zeta}w_{\alpha,\beta} = 0$ if $\alpha \neq \zeta$.

Suppose that $\alpha = \zeta$. Let $a, b, m, n \in \mathbb{N}^k$ be the unique elements such that $|a| = |b| = |m| = |n| = N$ and $a_j = c(\beta)_j$, $b_j = c(\alpha)_j$, $m_j = c_0(\zeta)_j$ and $n_j = c(\eta_j)$ for $j < k$. So by Lemmas 4.4 and 4.3, we have $w_{\alpha,\beta} = s_{(\alpha,a,q)}s^*_{(\beta,b,q)}$ and $w_{\eta,\zeta} = s_{(\eta,m,q)}s^*_{(\zeta,n,q)}$. Since $\alpha = \zeta$, Lemma 4.3 and the composition formula in Λ gives

$$\begin{aligned}
w_{\eta,\zeta}w^*_{\alpha,\beta} &= \sum_{\tau\in s(\zeta)E^N} s_{(\eta\tau,m+a,q)}s^*_{(\beta\tau,n+b,q)} \\
&= \sum_{\tau\in s(\zeta)E^N} s_{(\eta,a,q)}s_{(\tau,m,q+c(\eta,a))}s^*_{(\tau,b,q+c(\beta,n))}s^*_{(\beta,n,q)} \\
&= s_{(\eta,a,q)}\left(\sum_{\tau\in s(\zeta)E^N} s_{(\tau,m,q+c(\eta,a))}s^*_{(\tau,b,q+c(\beta,n))}\right)s^*_{(\beta,n,q)}. \qquad (4.4)
\end{aligned}$$

We claim that $c(\eta, a) = c(\beta, n)$. We have $c(\eta, a)_k = N = c(\beta, n)_k$. Fix $j < k$. Then

$$\begin{aligned}
c(\eta, a)_j &= c_0(\eta)_j - N + a_j \\
&= c_0(\eta)_j - N + m_j + (a_j - m_j) \\
&= c_0(\zeta)_j - N + n_j + (a_j - b_j) \quad \text{by definition of } m, n.
\end{aligned}$$

The symmetric calculation gives $c(\beta, n)_j = c_0(\alpha)_j - N + a_j + (n_j - m_j)$. Since $\alpha = \zeta$, we have $b = m$ also, so $c(\eta, a)_j = c(\beta, n)_j$ as claimed.

We now have $s(\eta,a,q)=s(\beta,n,q)$, so that $w_{\eta,\beta}=s_{(\eta,a,q)}s^*_{(\beta,n,q)}$, and (4.4) becomes

$$\begin{aligned} w_{\eta,\zeta}w_{\alpha,\beta} &= s_{(\eta,a,q)}\biggl(\sum_{\tau\in s(\zeta)E^N} s_{(\tau,m,q+c(\eta,a))}s^*_{(\tau,m,q+c(\eta,a))}\biggr)s^*_{(\beta,n,q)} \\ &= s_{(\eta,a,q)}\biggl(\sum_{\lambda\in s(\eta,a,q)\Lambda^m} s_\lambda s_\lambda^*\biggr)s^*_{(\beta,n,q)}, \end{aligned}$$

which is equal to $s_{(\eta,a,q)}s^*_{(\beta,n,q)}$ by (CK4), and hence to $w_{\eta,\beta}$. This proves that $B_N(v)$ is finite-dimensional with matrix units as claimed.

The indicated inclusion $B_N(v)\subseteq B_{N+1}(v)$ is an immediate consequence of the Cuntz–Krieger relations. To see that $s_vC^*(\Lambda)s_v$ is the closure of the union of the B_N, observe that $s_vC^*(\Lambda)s_v$ is spanned by elements of the form $s_\mu s_\nu^*$ where $\mu,\nu\in v\Lambda$ and $s(\mu)=s(\nu)$. Fix such a spanning element. Writing $v=(p(v),0,q)$, we have $\mu=(p(\mu),d(\mu),q)$ and $\nu=(p(\nu),d(\nu),q)$ and then

$$\begin{aligned} s(\mu)=s(\nu) \quad\Longrightarrow\quad c\bigl(p(\mu),d(\mu)\bigr)=c\bigl(p(\nu),d(\nu)\bigr) \quad&\Longrightarrow\quad \bigl|p(\mu)\bigr|=\bigl|p(\nu)\bigr| \\ &\Longrightarrow\quad s_\mu s_\nu^*\in B_{|p(\mu)|}(v). \end{aligned}$$

So $\bigcup_{N=1}^\infty B_N(v)$ contains all the spanning elements of $s_vC^*(\Lambda)s_v$, whence its closure is equal to $s_vC^*(\Lambda)s_v$.

It is routine to check that, for each $N\in\mathbb{N}$, the set $A_N(p(v)):=\{s_\alpha s_\beta^*\colon \alpha,\beta\in p(v)E^N,\ s(\alpha)=s(\beta)\}$ is a set of nonzero matrix units for a finite-dimensional subalgebra of $C^*(E)^\gamma$ and that $s_{p(v)}C^*(E)^\gamma s_{p(v)}$ is the closure of the increasing union of the A_N with inclusions $s_\alpha s_\beta^*\mapsto\sum_{f\in s(\alpha)E^1}s_{\alpha f}s_{\alpha f}^*$. So $w_{\alpha,\beta}\mapsto s_\alpha s_\beta^*$ determines isomorphisms $B_N(v)\to A_N(p(v))$ which respect the inclusion maps. So the inductive limits $s_vC^*(\Lambda)s_v$ and $s_{p(v)}C^*(E)^\gamma s_{p(v)}$ are isomorphic also. For the final statement observe that the proof of [31, Proposition 4.8] shows that if Λ is cofinal then every s_v is full in $C^*(\Lambda)$, and hence $C^*(\Lambda)$ is strongly Morita equivalent to $s_vC^*(\Lambda)s_v$ for any v. □

5. Higher-rank graphs with finitely many vertices

In this section, we completely characterise the higher-rank graphs with finitely many vertices whose C^*-algebras are AF. We then go on to prove that the standard dichotomy for simple graph C^*-algebras persists for row-finite locally convex k-graphs with finitely many vertices, and we describe the structure of non-simple unital finite higher-rank graph C^*-algebras.

Remark 5.1. A standard argument [32, Proposition 1.4] implies that if Λ is a finitely aligned k-graph, then $C^*(\Lambda)$ is unital if and only if Λ^0 is finite. So one may regard the results in this section as results about unital k-graph C^*-algebras.

In the sequel we denote by $\mathcal{K}(\mathcal{H})$ the C^*-algebra of compact operators on a separable Hilbert space $\mathcal{H}$. When $\mathcal{H}$ has finite dimension n we identify $\mathcal{K}(\mathcal{H})$ with $M_n(\mathbb{C})$ in the canonical way. Furthermore, given a countable set S, for each $\alpha,\beta\in S$, $\theta_{\alpha,\beta}$ denotes the canonical matrix unit in $\mathcal{K}(\ell^2(S))$. The following theorem extends [20, Lemma 4.2].

Theorem 5.2. *Let Λ be a finitely aligned k-graph such that Λ^0 is finite. Then*

(1) *$C^*(\Lambda)$ is AF if and only if Λ contains no cycles, and*
(2) *$C^*(\Lambda)$ is finite-dimensional if and only if Λ contains no cycles and is row-finite, in which case there is an isomorphism*

$$\bigoplus_{v\in\Lambda^0, v\Lambda=\{v\}} M_{\Lambda v}(\mathbb{C}) \cong C^*(\Lambda)$$

which takes $\theta_{\alpha,\beta}$ to $s_\alpha s_\beta^*$.

Before proving the theorem, we establish two technical results. Recall from [48] that *satiated* collections $\mathcal{E}$ (see [48, Definition 4.1]) of finite exhaustive subsets of Λ index the relative Cuntz–Krieger algebras $C^*(\Lambda;\mathcal{E})$ which interpolate between the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ and the Cuntz–Krieger algebra $C^*(\Lambda)$ [48, Corollary 5.6]. Moreover, all quotients of $C^*(\Lambda)$ by gauge-invariant ideals can be realised as relative Cuntz–Krieger algebras associated to complements of saturated hereditary subgraphs [49, Theorem 5.5].

Lemma 5.3. *Let Λ be a finitely aligned k-graph. Suppose that Λ^0 is finite, and that Λ contains no cycles. Let $\mathcal{E}$ be a satiated subset of* $\mathrm{FE}(\Lambda)$. *If $v \in \Lambda^0$ satisfies $v\Lambda = \{v\}$, then $C^*(\Lambda;\mathcal{E})s_v C^*(\Lambda;\mathcal{E}) \cong \mathcal{K}(\ell^2(\Lambda v))$.*

Proof. Since $v\Lambda = \{v\}$ for $\mu, \nu \in \Lambda v$, we have

$$\Lambda^{\min}(\mu,\nu) = \begin{cases} \{(v,v)\} & \text{if } \mu=\nu, \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular, the relative Cuntz–Krieger relations imply that if $\mu,\nu,\alpha,\beta \in \Lambda v$, then $s_\mu s_\nu^* s_\alpha s_\beta^* = \delta_{\nu,\alpha} s_\mu s_\beta^*$. Hence $C^*(\Lambda;\mathcal{E})s_v C^*(\Lambda;\mathcal{E}) = \overline{\operatorname{span}}\{s_\mu s_\nu^*\colon \mu,\nu \in \Lambda v\}$, and there is an isomorphism of $\mathcal{K}(\ell^2(\Lambda v))$ with $C^*(\Lambda;\mathcal{E})s_v C^*(\Lambda;\mathcal{E})$ which takes $\theta_{\mu,\nu}$ to $s_\mu s_\nu^*$. □

Proposition 5.4. *Let Λ be a finitely aligned k-graph such that Λ^0 is finite and such that Λ contains no cycles, and let $\mathcal{E}$ be a satiated subset of* $\mathrm{FE}(\Lambda)$. *Then $C^*(\Lambda;\mathcal{E})$ is AF.*

Proof. We proceed by induction on $|\Lambda^0|$. If $|\Lambda^0| = 1$, then since Λ has no cycles, $\Lambda = \{v\}$ where v is the unique element of Λ^0, and hence $C^*(\Lambda) = \mathbb{C}$ is certainly AF.

Now suppose that for any finitely aligned k-graph Γ with no cycles and with fewer vertices than Λ, and for any satiated subset $\mathcal{E}'$ of $\mathrm{FE}(\Gamma)$, the C^*-algebra $C^*(\Gamma;\mathcal{E}')$ is AF. Let $\{s_\lambda\colon \lambda \in \Lambda\}$ denote the universal generating relative Cuntz–Krieger $(\Lambda;\mathcal{E})$-family in $C^*(\Lambda;\mathcal{E})$. Since Λ^0 is finite, and since Λ contains no cycles, there exists $v \in \Lambda^0$ such that $v\Lambda = \{v\}$, and then Lemma 5.3 implies that the ideal $I = C^*(\Lambda;\mathcal{E})s_v C^*(\Lambda;\mathcal{E})$ is AF. Let $H := \{v \in \Lambda^0\colon s_v \notin I\}$ and let

$$\mathcal{E}' := \left\{E \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}\colon \prod_{\lambda\in E}\bigl(s_{r(\lambda)} - s_\lambda s_\lambda^*\bigr) \in I\right\}.$$

If $H = \emptyset$, then $1_{C^*(\Lambda;\mathcal{E})} = \sum_{v \in \Lambda^0} s_v \in I$, so $C^*(\Lambda;\mathcal{E}) = I$ is AF and we are done. So suppose that $H \neq \emptyset$. Let $\Gamma := \Lambda \setminus \Lambda H$. An application of the gauge-invariant uniqueness theorem [48, Theorem 6.1] for relative Cuntz–Krieger algebras shows that $C^*(\Gamma;\mathcal{E}') \cong C^*(\Lambda;\mathcal{E})/I$. Moreover $\Gamma^0 \subset \Lambda^0 \setminus \{v\}$, so Γ has fewer vertices than Λ. The inductive hypothesis therefore implies that $C^*(\Lambda;\mathcal{E})/I$ is AF. Since I is AF and the class of AF algebras is closed under extensions (see, for example, [14, Theorem III.6.3]), it follows that $C^*(\Lambda)$ is itself AF. □

Proof of Theorem 5.2. (1) If Λ contains a cycle, then Theorem 3.4 implies that $C^*(\Lambda)$ is not AF; and if Λ contains no cycle, then $C^*(\Lambda)$ is AF by Proposition 5.4 applied with $\mathcal{E} = \mathrm{FE}(\Lambda)$.

(2) First suppose that Λ is not row-finite. Then there exist $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $v\Lambda^n$ is infinite. Hence $\{s_\lambda s_\lambda^*: \lambda \in v\Lambda^n\}$ is an infinite family of mutually orthogonal nonzero projections in $C^*(\Lambda)$, whence $C^*(\Lambda)$ is not finite-dimensional. Now suppose that Λ is row-finite and contains no cycles. Let Λ^0_{src} denote the collection of vertices $v \in \Lambda^0$ such that $v\Lambda = \{v\}$. Since Λ contains no cycles, $\Lambda^n = \emptyset$ whenever $|n| \geqslant |\Lambda^0|$. Since Λ^0 is finite and Λ is row-finite, Λ itself is finite. In particular, $\Lambda\Lambda^0_{\mathrm{src}}$ is finite. Fix $w \in \Lambda^0$. We claim that $w\Lambda\Lambda^0_{\mathrm{src}}$ is exhaustive. Indeed, fix $\lambda \in w\Lambda$. As above, the set $\{n \in \mathbb{N}^k: s(\lambda)\Lambda^n \neq \emptyset\}$ is bounded; let n be a maximal element of this set, and fix $\tau \in s(\lambda)\Lambda^n$. By definition of n, we have $s(\tau) \in \Lambda^0_{\mathrm{src}}$, so $\lambda\tau \in w\Lambda\Lambda^0_{\mathrm{src}}$ trivially has a common extension with λ. By definition of Λ^0_{src}, as in Lemma 5.3 we have $s_\mu^* s_\nu = \delta_{\mu,\nu} s_{s(\mu)}$ for $\mu, \nu \in \Lambda^0_{\mathrm{src}}$. Hence [41, Proposition 3.5] implies that

$$s_w = \sum_{\lambda \in w\Lambda\Lambda^0_{\mathrm{src}}} s_\lambda s_\lambda^* \prod_{\lambda\lambda' \in w\Lambda\Lambda^0_{\mathrm{src}} \setminus \{\lambda\}} \left(s_\lambda s_\lambda^* - s_{\lambda\lambda'} s_{\lambda\lambda'}^*\right) = \sum_{\lambda \in w\Lambda\Lambda^0_{\mathrm{src}}} s_\lambda s_\lambda^*.$$

Hence

$$C^*(\Lambda) = \bigoplus_{v \in \Lambda^0_{\mathrm{src}}} \operatorname{span}\left\{s_\alpha s_\beta^*: \alpha, \beta \in \Lambda v\right\}.$$

Lemma 5.3 implies that each $\operatorname{span}\{s_\alpha s_\beta^*: \alpha, \beta \in \Lambda v\} \cong M_{\Lambda v}(\mathbb{C})$. □

We show next that for row-finite locally convex k-graphs with finitely many vertices, the standard dichotomy for simple graph C^*-algebras persists: if Λ is a row-finite locally convex k-graph with finitely many vertices and $C^*(\Lambda)$ is simple then $C^*(\Lambda)$ is either finite-dimensional or purely infinite. It seems likely that a similar result holds for arbitrary finitely aligned k-graphs with finitely many vertices (though "finite-dimensional" would be replaced with "isomorphic to $\mathcal{K}(H)$ for some finite- or countably-infinite-dimensional Hilbert space"), but the arguments provided here would require substantial modification. We first need two technical results.

Lemma 5.5. *Let Λ be a row-finite k-graph, and suppose that $\rho \in \Lambda$ is a cycle with no entrance. For each $m \in \mathbb{N}^k$ such that $m \wedge d(\rho) = 0$, define a map $P_\rho : v\Lambda^m \to v\Lambda^m$ by $P_\rho(\mu) := (\rho\mu)(0, m)$. Then P_ρ is bijective.*

Proof. Fix $\mu \in v\Lambda^m$. Since ρ has no entrance, $\Lambda^{\min}(\rho, \mu) \neq \emptyset$. Fix $(\sigma, \tau) \in \Lambda^{\min}(\rho, \mu)$. Then in particular, $\tau \in s(\mu)\Lambda^m$. Now $(\mu\tau)(0, d(\rho)) = \rho$ because ρ does not have an entrance. Hence $\mu = P_\rho((\mu\tau)(d(\rho), d(\rho) + m))$. Since $\mu \in v\Lambda^m$ was arbitrary, it follows that P_ρ is surjective. Since Λ is row-finite, $v\Lambda^m$ is finite, so that P_ρ is surjective implies that it is bijective. □

Lemma 5.6. *Let Λ be a row-finite k-graph, and suppose that $\rho \in \Lambda$ is a cycle with no entrance. For each $\mu \in r(\rho)\Lambda$ such that $d(\mu) \wedge d(\rho) = 0$ and for each $n \in \mathbb{N}$, there is a unique element of $s(\mu)\Lambda^{nd(\rho)}$. Moreover, there exists $p \in \mathbb{N}$ such that the unique element of $s(\lambda)\Lambda^{pd(\rho)}$ is a cycle.*

Proof. Fix $\mu \in r(\rho)\Lambda$ such that $d(\mu) \wedge d(\rho) = 0$, and let $m = d(\mu)$. Observe that for each $n \in \mathbb{N}$,

$$P_\rho^n(\mu) = P_\rho\big(P_\rho^{n-1}(\mu)\big) = \big(\rho P_\rho^{n-1}(\mu)\big)(0, m) = \cdots = \big(\rho^n \mu\big)(0, m). \tag{5.1}$$

Fix $n \in \mathbb{N}$. Let $\tau_n := (\rho^n P_\rho^{-n}(\mu))(m, m + nd(\rho))$. Since $\mu = P_\rho^n(P_\rho^{-n}(\mu)) = (\rho^n P_\rho^{-n}(\mu))(0, m)$, we have $\mu\tau_n = \rho^n P_\rho^n(\mu)$, and in particular, $\tau_n \in s(\mu)\Lambda^{nd(\rho)}$. To see that $s(\mu)\Lambda^{nd(\rho)} = \{\tau_n\}$, let $\lambda \in s(\mu)\Lambda^{nd(\rho)}$. Then $(\mu\lambda)(0, nd(\rho)) = \rho^n$ because ρ has no entrance. Let $\alpha := (\mu\lambda)(nd(\rho), m + nd(\rho))$, so $\rho^n\alpha = \mu\lambda$. Then $P_\rho^n(\alpha) = \mu$ by (5.1), so $\alpha = P_\rho^{-n}(\mu)$, and hence $\mu\lambda = \rho^n\alpha = \rho^n P_\rho^{-n}(\mu)$. Thus $\lambda = \tau_n$.

Since $r(\mu)\Lambda^m$ is finite, there exist $l, n \in \mathbb{N}$ with $l < n$ such that $P_\rho^l(\mu) = P_\rho^n(\mu)$. Let $p := n - l$. Then

$$\mu = P_\rho^{-n}\big(P_\rho^n(\mu)\big) = P_\rho^{-n}\big(P_\rho^l(\mu)\big) = P_\rho^{-(n-l)}(\mu) = P_\rho^{-p}(\mu),$$

and then by definition of the τ_n, we have

$$s(\tau_p) = s\big(\big(\rho^p P_\rho^{-p}(\mu)\big)\big(m, m + pd(\rho)\big)\big) = s\big(P_\rho^{-(n-l)}(\mu)\big) = s(\mu) = r(\tau_p). \qquad \square$$

In the following proof and some later results, given a cycle τ in a k-graph Λ, we write τ^∞ for the unique element of W_Λ such that $d(\tau^\infty)_i$ is equal to ∞ when $d(\tau)_i > 0$ and equal to 0 when $d(\tau)_i = 0$, and such that $(\tau^\infty)(n \cdot d(\tau), (n+1) \cdot d(\tau)) = \tau$ for all $n \in \mathbb{N}$. For $\lambda \in W_\Lambda$ and $m \leqslant d(\lambda)$ we write $\lambda(m)$ instead of $\lambda(m, m)$.

Corollary 5.7. *Let Λ be a row-finite locally convex k-graph such that Λ^0 is finite and $C^*(\Lambda)$ is simple. If Λ contains no cycles, then Λ^0 contains a unique source v, and $C^*(\Lambda) \cong M_{\Lambda v}(\mathbb{C})$. Otherwise, $C^*(\Lambda)$ is purely infinite.*

Proof. Suppose that Λ does not contain a cycle. Then Theorem 5.2 shows that $C^*(\Lambda)$ is equal to the direct sum over all sources w in Λ^0 of $M_{\Lambda w}(\mathbb{C})$. Since $C^*(\Lambda)$ is simple, there can be just one summand, and the result follows.

Now suppose that Λ contains a cycle. Since $C^*(\Lambda)$ is simple, Λ is cofinal and has no local periodicity by [44, Theorem 3.4]. Hence if Λ contains a cycle with an entrance, then [49, Proposition 8.8] implies that $C^*(\Lambda)$ is purely infinite. It therefore suffices to show that Λ contains a cycle with an entrance.

We suppose for contradiction that no cycle in Λ has an entrance. For each cycle $\lambda \in \Lambda$ let $I_\lambda := \{i \leqslant k\colon\ d(\lambda)_i \neq 0\}$, and fix a cycle ρ in Λ such that I_ρ is maximal with respect to set inclusion amongst the sets I_λ. We claim that if $\mu \in r(\rho)\Lambda$ and $m < n \leqslant d(\mu)$, then $\mu(m) \neq \mu(n)$. To see this, suppose for contradiction that $\mu(m) = \mu(n)$. By Lemma 5.6 there exist $p \in \mathbb{N} \setminus \{0\}$ and a cycle τ of degree $pd(\rho)$ with $r(\tau) = \mu(m)$. Since $d(\mu) \wedge d(\rho) = 0$, we have $(n-m) \wedge d(\rho) = 0$, so $\tau\mu(m, n)$ is a cycle with $I_{\tau\mu(m,n)} = I_\tau \sqcup I_{\mu(m,n)} \supsetneq I_\tau = I_\rho$, contradicting our choice of ρ.

Since Λ^0 is finite, it follows that there exists $\mu \in r(\rho)\Lambda$ such that $d(\mu) \wedge d(\rho) = 0$ and such that $s(\mu)\Lambda^{e_i} = \emptyset$ whenever $e_i \wedge d(\rho) = 0$. Another application of Lemma 5.6 implies that there exist $p \in \mathbb{N}$ and a cycle $\tau \in s(\mu)\Lambda^{pd(\rho)}$. Since $r(\tau)\Lambda^{e_i} = \emptyset$, the graph morphism τ^∞ belongs to $\Lambda^{\leqslant\infty}$, and since cycles in Λ have no entrance, $s(\mu)\Lambda^{\leqslant\infty} = \{\tau^\infty\}$. In particular, $\sigma^{d(\tau)}(x) = x$ for all $x \in s(\mu)\Lambda^{\leqslant\infty}$, which contradicts that Λ has no local periodicity. □

We conclude the section with the following description of the C^*-algebras of row-finite locally convex k-graphs with finitely many vertices: each such C^*-algebra either contains an infinite projection or is strongly Morita equivalent (denoted $\sim_{\mathrm{Me}}$) to a direct sum of matrix algebras over the continuous functions on tori of dimension at most k (with the convention that a dimension zero torus is a point). To prove the result, we need some terminology. Let Λ be a row-finite locally convex k-graph such that Λ^0 is finite, and suppose that $C^*(\Lambda)$ does not contain an infinite projection. We will call paths μ such that $r(\mu) = s(\mu)$ and $r(\mu)\Lambda^{e_i} = \emptyset$ whenever $d(\mu)_i = 0$ *initial cycles*, and we will say that a vertex $v \in \Lambda^0$ is a vertex on the initial cycle μ if $v \in (\mu^\infty)^0 := \{\mu^\infty(n)\colon n \leqslant d(\mu^\infty)\}$. We write $\mathrm{IC}(\Lambda)$ for the collection of initial cycles in Λ, and $\mathrm{IC}(\Lambda)^0$ for the collection of vertices of Λ which lie on an initial cycle.

Lemma 5.8. *Let Λ be a row-finite locally convex k-graph such that Λ^0 is finite, and suppose that $C^*(\Lambda)$ does not contain an infinite projection. Let μ be an initial cycle of Λ. Let $G_\mu := \{m - n\colon m, n \leqslant d(\mu^\infty),\ \mu^\infty(m) = \mu^\infty(n)\}$. Then G_μ is a subgroup of $\mathbb{Z}^k$.*

Proof. It is clear that $0 \in G$ and that $-G = G$, so we just need to show that G is closed under addition. Suppose that $\mu^\infty(m) = \mu^\infty(n)$ and that $\mu^\infty(p) = \mu^\infty(q)$, so that $m - n$ and $p - q$ are elements of G; we must show that $(m - n) + (p - q) \in G$. We calculate:

$$\mu^\infty(m + p) = \sigma^{m+p}\big(\mu^\infty\big)(0) = \sigma^m\big(\sigma^p\big(\mu^\infty\big)\big)(0) = \sigma^m\big(\sigma^q\big(\mu^\infty\big)\big)(0) = \sigma^{m+q}\big(\mu^\infty\big)(0).$$

A symmetric argument shows that $\mu^\infty(n+q) = \sigma^{m+q}(\mu^\infty)(0)$ also. Hence $(m-n)+(p-q) = (m + p) - (n + q) \in G$ as required. □

Proposition 5.9. *Let Λ be a row-finite locally convex k-graph such that Λ^0 is finite, and suppose that $C^*(\Lambda)$ does not contain an infinite projection. Then there exist $n \geqslant 1$ and $l_1, \ldots, l_n \in \{0, \ldots, k\}$ such that $C^*(\Lambda) \sim_{\mathrm{Me}} \bigoplus_{i=1}^n C(\mathbb{T}^{l_i})$.*

Proof. Since $C^*(\Lambda)$ contains no infinite projection, Lemma 3.7 implies that no cycle in Λ has an entrance.

For $p \in \mathbb{N}$, let $\mathbf{p} := (p, p, \ldots, p) \in \mathbb{N}^k$. Let $N := |\Lambda^0|$. Fix $\lambda \in \Lambda^{\leqslant \mathbf{N}}$. Since $N = |\Lambda^0|$, there exist $p < q \leqslant N$ such that the vertices $\lambda(\mathbf{p} \wedge d(\lambda))$ and $\lambda(\mathbf{q} \wedge d(\lambda))$ coincide. By [40, Lemmas 3.12 and 3.6], the path $\mu := \lambda(\mathbf{p} \wedge d(\lambda), \mathbf{q} \wedge d(\lambda))$ belongs to $\Lambda^{\leqslant \mathbf{p}-\mathbf{q}}$, so $r(\mu)\Lambda^{e_i} = \emptyset$ whenever $d(\mu)_i = 0$. Since μ has no entrance, $r(\mu)\Lambda^n = \{\mu^\infty(0, n)\}$ for all $n \leqslant d(\mu^\infty)$.

By the preceding paragraph, for every $\lambda \in \Lambda^{\leqslant \mathbf{N}}$, we have $s(\lambda) \in \mathrm{IC}(\Lambda)^0$. By the Cuntz–Krieger relations,

$$\sum_{\lambda \in \Lambda^{\leqslant \mathbf{N}}} s_\lambda s_{s(\lambda)} s_\lambda^* = \sum_{v \in \Lambda^0} \sum_{\lambda \in v\Lambda^{\leqslant \mathbf{N}}} s_\lambda s_\lambda^* = \sum_{v \in \Lambda^0} p_v = 1_{C^*(\Lambda)},$$

so $\sum_{v\in \mathrm{IC}(\Lambda)^0} s_v$ is a full projection in $C^*(\Lambda)$. For each initial cycle μ, we write P_μ for $\sum_{v\in(\mu^\infty)^0} s_v$.

Given two initial cycles μ, ν either $(\mu^\infty)^0 = (\nu^\infty)^0$, or $(\mu^\infty)^0 \cap (\nu^\infty)^0 = \emptyset$. We write $\mu \sim \nu$ if $(\mu^\infty)^0 = (\nu^\infty)^0$. Since cycles in Λ have no entrance, if $\mu, \nu \in \mathrm{IC}(\Lambda)$, with $\mu \nsim \nu$, then $v\Lambda w = \emptyset$ for all $v \in (\mu^\infty)^0$ and $w \in (\nu^\infty)^0$, and hence $P_\mu C^*(\Lambda) P_\mu \perp P_\nu C^*(\Lambda) P_\nu$. In particular,

$$C^*(\Lambda) \sim_{\mathrm{Me}} \Big(\sum_{v\in \mathrm{IC}(\Lambda)^0} s_v\Big) C^*(\Lambda) \Big(\sum_{v\in \mathrm{IC}(\Lambda)^0} s_v\Big) = \sum_{[\mu]\in \mathrm{IC}(\Lambda)/\sim} P_\mu C^*(\Lambda) P_\mu$$
$$= \bigoplus_{[\mu]\in \mathrm{IC}(\Lambda)/\sim} P_\mu C^*(\Lambda) P_\mu.$$

It therefore suffices to show that for $\mu \in \mathrm{IC}(\Lambda)$, we have $P_\mu C^*(\Lambda) P_\mu \sim_{\mathrm{Me}} C(\mathbb{T}^l)$ for some $l \leqslant k$.

For this, fix $\mu \in \mathrm{IC}(\Lambda)$. For $v \in (\mu^\infty)^0$, we have $v = \mu^\infty(m)$ for some m, and then $s_v = s^*_{\mu(0,m)} s_{\mu(0,m)} = s^*_{\mu(0,m)} s_{r(\mu)} s_{\mu(0,m)}$, so $s_{r(\mu)}$ is full in $P_\mu C^*(\Lambda) P_\mu$. It therefore suffices to show that $s_{r(\mu)} C^*(\Lambda) s_{r(\mu)} \cong C(\mathbb{T}^l)$ for some $l \leqslant k$. By [1, Corollary 3.7], $s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}$ is isomorphic to the universal C^*-algebra generated by elements $\{t_{\alpha,\beta}\colon r(\alpha) = r(\beta) = r(\mu),\ s(\alpha) = s(\beta)\}$ such that

(1) $t^*_{\alpha,\beta} = t_{\beta,\alpha}$,
(2) $t_{\alpha,\beta} t_{\eta,\zeta} = \sum_{(\tau,\rho)\in\Lambda^{\min}\beta,\eta} t_{\alpha\tau,\zeta\rho}$, and
(3) for every finite exhaustive subset E of $r(\mu)\Lambda$, $\prod_{\lambda\in E}(t_{v,v} - t_{\lambda,\lambda}) = 0$.

Since μ has no entrance, relation (3) holds if and only if each $t_{\alpha,\alpha} = t_{v,v}$, and then (2) implies that $t_{v,v}$ is a unit for the corner, and that each $t_{\alpha,\beta}$ is a unitary. If $\alpha \in r(\mu)\Lambda$, then $\alpha = \mu^\infty(0, d(\alpha))$. For $m, n \leqslant d(\mu^\infty)$ with $\mu^\infty(m) = (\mu^\infty)(n)$, let $\alpha = \mu^\infty(0, m - m \wedge n)$ and $\beta = \mu^\infty(0, n - m \wedge n)$. Then (2) implies that

$$t_{\alpha,\beta} - t_{\mu^\infty(0,m),\mu^\infty(0,n)} = t_{\alpha,\beta}\big(t_{r(\mu),r(\mu)} - t_{\mu^\infty(0,n),\mu^\infty(0,n)}\big),$$

and since $\{\mu^\infty(0, n)\}$ is exhaustive in $r(\mu)\Lambda$, it follows that $t_{\alpha,\beta} = t_{\mu^\infty(0,m),\mu^\infty(0,n)}$. In particular, if G_μ is the group obtained from Lemma 5.8, then there is a well-defined function $(m - n) \mapsto u_{m-n} := t_{\mu^\infty(0,m),\mu^\infty(0,n)}$ from G_μ to $s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}$.

For $\alpha, \beta, \eta, \zeta, \tau$ and ρ as in (2), we have

$$d(\alpha\tau) - d(\zeta\rho) = \big(d(\alpha) + \big(d(\beta) \vee d(\zeta)\big) - d(\beta)\big) + \big(d(\zeta) + \big(d(\beta) \vee d(\zeta)\big) - d(\zeta)\big)$$
$$= \big(d(\alpha) - d(\beta)\big) + \big(d(\eta) - d(\zeta)\big),$$

and hence for $g, h \in G_\mu$ we have $u_g u_h = u_{g+h}$.

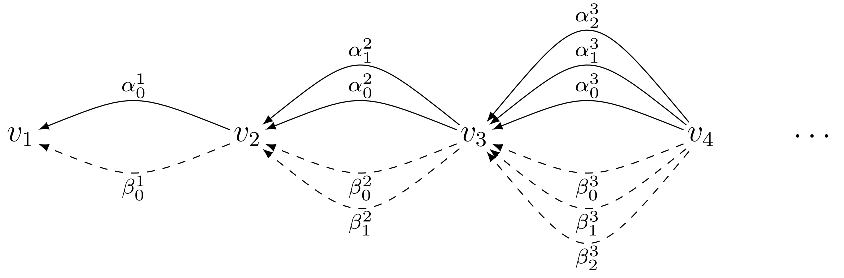
Hence $s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}$ is the universal C^*-algebra generated by a unitary representation of G_μ, namely $C^*(G_\mu)$. Since G_μ is a subgroup of $\mathbb{Z}^k$ it is isomorphic to $\mathbb{Z}^l$ for some $l \leqslant k$, so $C^*(G_\mu) \cong C(\mathbb{T}^l)$ as required. $\square$

6. Examples

In this final section, we present some examples which illustrate our results. We begin with an example that illustrates the need for the fairly technical definition of a generalised cycle.

Before we discuss it, recall that the C^*-algebra of a directed graph is AF if and only if the graph contains no cycle. There are two obvious generalisations of the notion of a cycle to the setting of k-graphs: paths whose range and source coincide, or periodic infinite paths. Examples have appeared previously in the literature to show that there exist k-graphs containing no path whose range and source coincide whose C^*-algebras are not AF (for example the pullback of Ω_1 by the homomorphism $(p,q) \mapsto p+q$; see [31, Example 1.7, Definition 1.9, and Corollary 2.5(iii)]) and that there exist k-graphs in which every infinite path is aperiodic and the C^*-algebra is not AF (see [36, Examples 6.5 and 6.6]). However, to our knowledge, the following is the first known example of a k-graph which contains no (conventional) cycle and in which every infinite path is aperiodic but such that the C^*-algebra is not AF. This confirms the conjecture stated at the opening of [20, Section 4.1].

Example 6.1. Let Λ be the 2-graph with skeleton



and factorisation rules $\alpha^i_j \beta^{i+1}_k \sim \beta^i_{j+1 (\mathrm{mod}\, i)} \alpha^{i+1}_{k+1 (\mathrm{mod}\, i+1)}$. Wright's argument [55] shows that Λ is aperiodic in the sense of [34], meaning that every vertex receives at least one aperiodic infinite path. However, we claim that it has the stronger property that every infinite path in Λ is aperiodic. (For 1-graphs this is equivalent to requiring that the graph contains no cycles; it is also equivalent to the condition that the associated groupoid is principal.)

To see this, fix $x \in \Lambda^{\leqslant\infty}$, say $r(x) = v_{i-1}$, and factorise x as

$$x = \alpha^i_{j_0} \beta^{i+1}_{j_1} \alpha^{i+2}_{j_2} \beta^{i+3}_{j_3} \cdots .$$

Then

$$\begin{aligned}\sigma^{e_1}(x) &= \beta^{i+1}_{j_1} \alpha^{i+2}_{j_2} \beta^{i+3}_{j_3} \alpha^{i+4}_{j_4} \cdots \\ &= \alpha^{i+1}_{j_1 - 1 (\mathrm{mod}\, i+1)} \beta^{i+2}_{j_2 - 1 (\mathrm{mod}\, i+2)} \alpha^{i+3}_{j_3 - 1 (\mathrm{mod}\, i+3)} \beta^{i+4}_{j_4 - 1 (\mathrm{mod}\, i+4)} \cdots\end{aligned}$$

and

$$\begin{aligned}\sigma^{e_2}(x) &= \sigma^{e_2}\big(\beta^i_{j_0+1 (\mathrm{mod}\, i)} \alpha^{i+1}_{j_1+1 (\mathrm{mod}\, i+1)} \beta^{i+2}_{j_2+1 (\mathrm{mod}\, i+2)} \alpha^{i+3}_{j_3+1 (\mathrm{mod}\, i+3)} \beta^{i+4}_{j_4+1 (\mathrm{mod}\, i+4)}\big) \cdots \\ &= \alpha^{i+1}_{j_1+1 (\mathrm{mod}\, i+1)} \beta^{i+2}_{j_2+1 (\mathrm{mod}\, i+2)} \alpha^{i+3}_{j_3+1 (\mathrm{mod}\, i+3)} \beta^{i+4}_{j_4+1 (\mathrm{mod}\, i+4)} \cdots .\end{aligned}$$

Hence for $m, n \in \mathbb{N}$,

$$\sigma^{(m,n)}(x) = \alpha^{i+m+n}_{j_{m+n}+(n-m)(\operatorname{mod} i+m+n)} \beta^{i+m+n+1}_{j_{m+n+1}+(n-m)(\operatorname{mod} i+m+n+1)} \times \alpha^{i+m+n+2}_{j_{m+n+2}+(n-m)(\operatorname{mod} i+m+n+2)} \beta^{i+m+n+3}_{j_{m+n+3}+(n-m)(\operatorname{mod} i+m+n+3)} \cdots.$$

So for $p \in \mathbb{N}^2$, we can recover p from $y := \sigma^p(x)$ as follows:

- If $r(x) = v_l$ and $r(\sigma^p(x)) = v_{l'}$, then $p_1 + p_2 = l' - l$.
- For $i \geqslant 0$, we have $y((i,i),(i+1,i)) = \alpha^{r(y)+2i}_{k(i)}$ for some $k(i) \in \mathbb{Z}/(r(y)+2i)\mathbb{Z}$. Moreover, $p_2 - p_1 \equiv k(i) - j_{r(y)+2i} (\operatorname{mod} r(y) + 2i)$ for all i. Hence the sequence $d_i := k(i) - (j_{r(y)+2i} \in \mathbb{Z})$ is either constant or else increases by 2 at each step. We have $-(r(y)+2i) < p_2 - p_1 < r(y) + 2i$ for $i > 0$, so if (d_i) is constant then $p_2 \geqslant p_1$ and $p_2 - p_1 = d_i$ for $i \geqslant 1$, and if $d_{i+1} = d_i + 2$ for all i, then $p_2 < p_1$ and $p_2 - p_1 = d_i - (r(y) + 2i)$ for $i \geqslant 1$.
- We now know $p_1 + p_2$ and $p_2 - p_1$; we then have $p_2 = \frac{(p_1+p_2)+(p_2-p_1)}{2}$, and then $p_1 = p_1 + p_2 - p_2$.

In particular, if $\sigma^p(x) = \sigma^q(x)$, then $p = q$ by the above, and it follows that x is not periodic. Hence Λ has no periodic boundary paths. It also has no cycles. However, (α^1_0, β^1_0) is a generalised cycle, so $C^*(\Lambda)$ is not AF. Since $g(v_n) := 1/(n-1)!$ defines a finite faithful graph trace on Λ, Lemma 2.1 implies that $C^*(\Lambda)$ carries a faithful trace, and hence is finite.

Remark 6.2. The 2-graph of the preceding example contains a generalised cycle, so we were able to use Theorem 3.4 to see that its C^*-algebra was not AF. We believe that it is possible to construct a similar example which contains no generalised cycle and no periodic paths whose C^*-algebra is simple and finite but not AF, using Proposition 3.10 in place of Theorem 3.4.

Our second example demonstrates a 2-graph which contains no generalised cycle, but so that a quotient graph does contain such a cycle. In particular Corollary 3.11 is a genuinely stronger result than Theorem 3.4.

Example 6.3. Consider the unique 2-graph $\mathcal{S}$ with the skeleton illustrated in Fig. 1. It is straightforward to check that this 2-graph contains no generalised cycles. However, the collection H of vertices to the left of the middle (those contained in the grey rectangle) form a saturated hereditary subset of Λ^0, and the quotient graph does contain a generalised cycle, namely (α, β).

We now present two examples of 2-graphs with the same skeleton, one of them AF, the other not obviously so. The AF example is a 2-graph which satisfies Condition (Γ) of [20, Definition 4.6] but not Condition (S) [20, Definition 4.3], confirming a conjecture of the first author. The other example is intriguing, because it strongly suggests that there are 2-graph C^*-algebras $C^*(\Lambda)$ which are AF but whose canonical diagonal subalgebras (in the sense of Kumjian) as AF algebras are not conjugate to their maximal abelian subalgebras $\overline{\operatorname{span}}\{s_\lambda s_\lambda^*\colon \lambda \in \Lambda\}$. (By contrast, whenever the C^*-algebra of a directed graph E is AF, it has an AF decomposition for which the canonical diagonal subalgebra is precisely $\overline{\operatorname{span}}\{s_\lambda s_\lambda^*\colon \lambda \in E^*\}$, where E^* is the finite path space of E.)

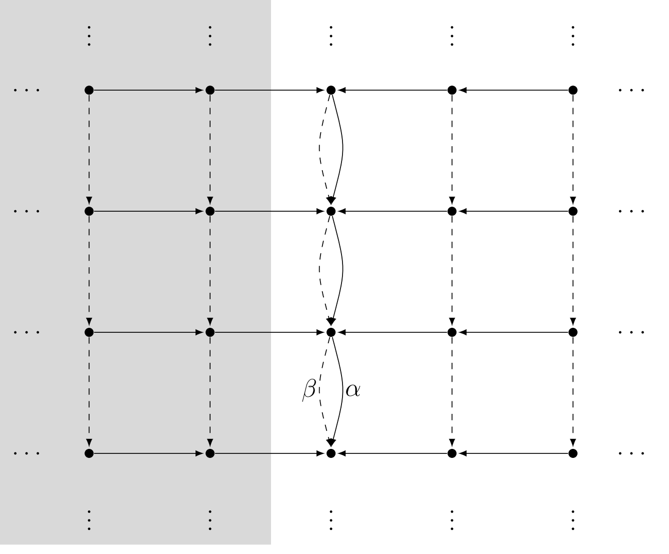


Fig. 1. The skeleton of the 2-graph $\mathcal{S}$.

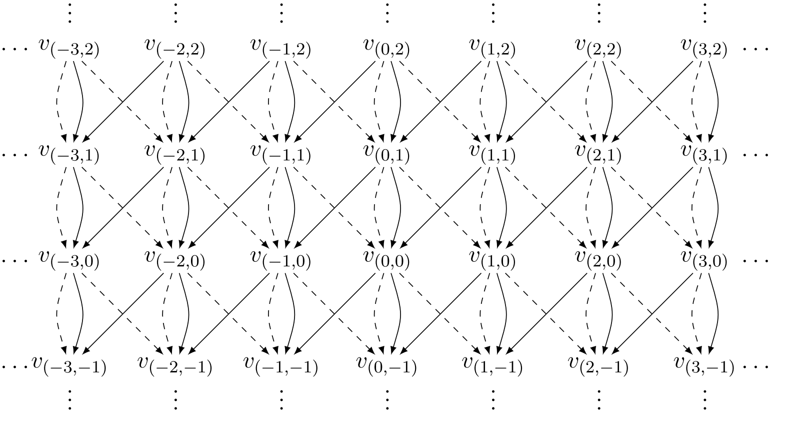
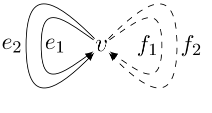


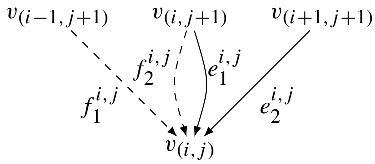
Fig. 2. The common skeleton of the 2-graphs Λ_{I} and Λ_{II}.

Example 6.4. Consider the skeleton



Let $\mathcal{P}_{\mathrm{I}}$ be the 2-graph with this skeleton and factorisation rules $e_i f_j = f_i e_j$. By [31, Corollary 3.5(iii)], $C^*(\mathcal{P}_{\mathrm{I}}) \cong \mathcal{O}_2 \otimes C(\mathbb{T})$.

Let $c(e_1) = c(f_2) := (0, 1)$, $c(e_2) := (1, 1)$ and $c(f_1) := (-1, 1)$. It is straightforward to check that c extends to a functor on $\mathcal{P}_{\mathrm{I}}$. The skew-product graph $\Lambda_{\mathrm{I}} := \mathcal{P}_{\mathrm{I}} \times_c \mathbb{Z}^2$ has the skeleton illustrated in Fig. 2. To keep notation compact, we write $e_l^{i,j}$ for $(e_l, (i, j))$ and $f_l^{i,j}$ for $(f_l, (i, j))$ for all $l \in \{0, 1\}$ and $i, j \in \mathbb{Z}$. So locally, the labelling looks like



The factorisation rules are

$$e_1^{i,j} f_2^{i,j+1} = f_1^{i,j} e_2^{i-1,j+1}, \qquad e_2^{i,j} f_1^{i+1,j+1} = f_2^{i,j} e_1^{i,j+1},$$
$$e_1^{i,j} f_1^{i,j+1} = f_1^{i,j} e_1^{i-1,j+1}, \qquad e_2^{i,j} f_2^{i+1,j+1} = f_2^{i,j} e_2^{i,j+1}.$$

We claim that $C^*(\Lambda_{\mathrm{I}})$ is strongly Morita equivalent to the UHF algebra of type 2^∞. To see this, we invoke Corollary 4.5. Let B_2 be the 1-graph with $B_2^0 = \{v\}$ and $B_2^1 = \{a, b\}$ whose C^*-algebra is canonically isomorphic to $\mathcal{O}_2$. Then $e_1 \to (a, (1, 0))$, $e_2 \to (b, (1, 0))$, $f_1 \to (a, (0, 1))$ and $f_2 \to (b, (0, 1))$ determines an isomorphism of $\mathcal{P}_{\mathrm{I}}$ with the pullback $f^*(B_2)$ under the homomorphism $f : (m, n) \to m + n$ from $\mathbb{N}^2$ to $\mathbb{N}$. The 2-graph Λ_{I} is isomorphic to the one obtained from Example 4.2 by setting $c_0(a) = e_1$ and $c_0(b) = 0$ in $\mathbb{N}$. Since Λ_{I} is cofinal, it follows from Corollary 4.5 that $C^*(\Lambda_{\mathrm{I}})$ is strongly Morita equivalent to $s_v C^*(B_2)^\gamma s_v$. The fixed-point algebra $C^*(B_2)^\gamma$ is precisely the classical core of $\mathcal{O}_2$, which is the 2^∞ UHF algebra [13, 1.5].

Example 6.5. Consider the 2-graph $\mathcal{P}_{\mathrm{II}}$ with the same skeleton as $\mathcal{P}_{\mathrm{I}}$ but with factorisation rules $e_i f_j = f_j e_i$. This is isomorphic to $B_2 \times B_2$, so $C^*(\mathcal{P}_{\mathrm{II}}) \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ as in [31, Corollary 3.5(iv)]. The formula given for c in Example 6.4 also extends to a functor on $\mathcal{P}_{\mathrm{II}}$, and we write Λ_{II} for the corresponding skew-product graph. Then Λ_{II} has the same skeleton as Λ_{I}, but factorisation rules

$$e_1^{i,j} f_2^{i,j+1} = f_2^{i,j} e_1^{i,j+1}, \qquad e_2^{i,j} f_1^{i+1,j+1} = f_1^{i,j} e_2^{i-1,j+1},$$
$$e_1^{i,j} f_1^{i,j+1} = f_1^{i,j} e_1^{i-1,j+1}, \qquad e_2^{i,j} f_2^{i+1,j+1} = f_2^{i,j} e_2^{i,j+1}.$$

For each i, j, let $x^{i,j}$ denote the unique infinite path $x^{i,j} : \Omega_2 \to \Lambda_{\mathrm{II}}$ such that

$$x^{i,j}\big(n, n + (1, 0)\big) = e_1^{i,(j+|n|)} \quad \text{and} \quad x^{i,j}\big(n, n + (0, 1)\big) = f_2^{i,(j+|n|)}$$

for all $n \in \mathbb{N}^2$. Then $\sigma^{(1,0)}(x^{i,j}) = x^{i,(j+1)} = \sigma^{(0,1)}(x^{i,j})$ for all i, j, and in particular every vertex of Λ_{II} receives a periodic infinite path. On the other hand, for each i, j there is a unique infinite path $y^{i,j} : \Omega_2 \to \Lambda_{\mathrm{II}}$ defined by

$$y^{i,j}\big(n, n + (1, 0)\big) := e_2^{(i+n_1-n_2),(j+|n|)} \quad \text{and} \quad y^{i,j}\big(n, n + (0, 1)\big) := f_1^{(i+n_1-n_2),(j+|n|)}$$

for all $n \in \mathbb{N}^2$. Since each $y^{i,j}$ is injective from $\Omega_2^0 \to \Lambda_{\mathrm{II}}^0$ it is aperiodic. So Λ_{II} satisfies the aperiodicity condition. It is also cofinal, so $C^*(\Lambda_{\mathrm{II}})$ is simple by [31, Proposition 4.8].

6.1. The C^*-algebra of Λ_{II}

We will spend some time analysing $C^*(\Lambda_{\mathrm{II}})$. We believe that it is isomorphic to $C^*(\Lambda_{\mathrm{I}})$, but via an isomorphism which cannot easily be described in terms of the presentation of each as a k-graph C^*-algebra. As supporting evidence for this conjecture, setting $p := s_{v_{(0,0)}}$, we prove: that $pC^*(\Lambda_{\mathrm{II}})p$ has a unique tracial state; that $C^*(\Lambda_{\mathrm{II}})$ (and thus $pC^*(\Lambda_{\mathrm{II}})p$) is AF-embeddable; that the K-theory of both $pC^*(\Lambda_{\mathrm{II}})p$ and $C^*(\Lambda_{\mathrm{II}})$ is $(\mathbb{Z}[\frac{1}{2}], \{0\})$ (as groups); that Murray–von Neumann equivalence in $pC^*(\Lambda_{\mathrm{II}})p$ of the canonical representatives of the generators of its K_0-group is equivalent to K_0-equivalence characterised by equality under the trace; and that the order on its K_0-group is the standard unperforated order. So all the evidence suggests that $C^*(\Lambda_{\mathrm{II}})$ is strongly Morita equivalent to the 2^∞ UHF algebra, and hence also to $C^*(\Lambda_{\mathrm{I}})$. To indicate why this might be surprising, we close by showing that if $pC^*(\Lambda_{\mathrm{II}})p$ is indeed the 2^∞ UHF algebra, then its diagonal subalgebra as an AF algebra is not conjugate to the canonical maximal abelian subalgebra $\overline{\operatorname{span}}\{s_\lambda s_\lambda^*\colon \lambda \in v_{(0,0)}\Lambda_{\mathrm{II}}\}$, even though the two subalgebras are canonically isomorphic under an isomorphism which preserves K_0-classes in the enveloping algebras.

Recall that a *normalised trace* on $C^*(\Lambda_{\mathrm{II}})$ is a trace such that $\sum_{v\in F}\tau(s_v)$ converges to 1 as F increases over finite subsets of Λ_{II}^0 and that for a hereditary subset H of Λ_{II}^0, we may identify $C^*(H\Lambda_{\mathrm{II}})$ with the subalgebra of $C^*(\Lambda_{\mathrm{II}})$ generated by $\{s_\lambda\colon \lambda \in H\Lambda_{\mathrm{II}}\}$.

Lemma 6.6. *Let $H := \{v_{(i,j)}\colon\ j \geqslant 0,\ |i| \leqslant j\}$ be the hereditary subset of Λ_{II}^0 generated by v. Let $T := \sum_{j=1}^\infty (2j-1)2^{1-j}$. There is a normalised trace τ on $C^*(H\Lambda_{\mathrm{II}})$ given by $\tau(s_{v_{(i,j)}}) = \frac{1}{T}2^{i-j}$, and $\tau(s_\mu s_\nu^*) = \delta_{\mu,\nu}\tau(s_{s(\mu)})$. Moreover, this is the unique normalised trace on $C^*(H\Lambda_{\mathrm{II}})$.*

Proof. The function $g : v_{(i,j)} \to \frac{1}{T}2^{1-j}$ determines a normalised finite faithful graph trace on each of $H\Lambda_{\mathrm{II}}$ and $H\Lambda_{\mathrm{I}}$. Lemma 2.1 implies that there are faithful normalised traces $\tau_g^{\mathrm{II}} : C^*(H\Lambda_{\mathrm{II}}) \to \mathbb{C}$ and $\tau_g^{\mathrm{I}} : C^*(H\Lambda_{\mathrm{I}}) \to \mathbb{C}$ satisfying $\tau_g(s_\mu s_\nu^*) = \delta_{\mu,\nu} g(s(\mu))$ for all μ, ν. Since $C^*(H\Lambda_{\mathrm{I}})$ is strongly Morita equivalent to M_{2^∞}, τ_g^{I} is the unique such trace on $C^*(H\Lambda_{\mathrm{I}})$, and hence g is the unique normalised finite graph trace on $H\Lambda_{\mathrm{I}}$. It is then also the unique normalised finite graph trace on $H\Lambda_{\mathrm{II}}$, so another application of Lemma 2.1 implies that any trace on $C^*(H\Lambda_{\mathrm{II}})$, which is nonzero on each s_v and zero on each $s_\mu s_\nu^*$ such that $d(\mu) \neq d(\nu)$, must agree with τ_g^{II}.

We claim that τ_g^{II} is the unique trace on $C^*(H\Lambda_{\mathrm{II}})$. To see this, fix a trace τ on $C^*(H\Lambda_{\mathrm{II}})$. By the above, it suffices to show that $\tau(s_v) \neq 0$ for all $v \in H$ and that $\tau(s_\mu s_\nu^*) = 0$ whenever $d(\mu) \neq d(\nu)$. To see that $\tau(s_v) \neq 0$ for all v, fix $v \in H$. Since τ is normalised, we have $\tau(s_w) \neq 0$ for some w. Since Λ is cofinal, [34, Remark A.3] implies that there exists $n \in \mathbb{N}^2$ such that $v\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in w\Lambda^n$. Since $s_w = \sum_{\alpha\in w\Lambda^n} s_\alpha s_\alpha^*$, there exists $\alpha \in w\Lambda^n$ such that $\tau(s_\alpha s_\alpha^*) \neq 0$. Fix $\xi \in v\Lambda s(\alpha)$. Then

$$\tau(s_v) \geqslant \tau\left(s_\xi s_\xi^*\right) = \tau\left(s_\xi s_\alpha^* s_\alpha s_\xi^*\right) = \tau\left(s_\alpha s_\xi^* s_\xi s_\alpha^*\right) = \tau\left(s_\alpha s_\alpha^*\right) \neq 0.$$

It remains to show that $\tau(s_\mu s_\nu^*) = 0$ when $d(\mu) \neq d(\nu)$. If $s(\mu) \neq s(\nu)$, this is trivial, and if $r(\mu) \neq r(\nu)$, then the trace property gives $\tau(s_\mu s_\nu^*) = \tau(s_\nu^* s_\mu) = \tau(s_\nu^* s_{r(\nu)} s_{r(\mu)} s_\mu) = 0$. So we may suppose that $s(\mu) = s(\nu)$ and $r(\mu) = r(\nu)$. Factorise $\mu = \eta\alpha_0$ and $\nu = \zeta\beta_0$ where $d(\eta) = d(\zeta) = d(\mu) \wedge d(\nu)$. Then $d(\alpha_0) \wedge d(\beta_0) = 0$ and

$$\tau\left(s_\mu s_\nu^*\right) = \tau\left(s_\eta s_{\alpha_0} s_{\beta_0}^* s_\zeta^*\right) = \tau\left(s_\zeta^* s_\eta s_{\alpha_0} s_{\beta_0}^*\right) = \delta_{\eta,\zeta}\tau\left(s_{\alpha_0} s_{\beta_0}^*\right).$$

In particular, it suffices to show that $\tau(s_{\alpha_0}s^*_{\beta_0}) = 0$. If $r(\alpha_0) \neq r(\beta_0)$ then by the above argument, we are done. If not then let $K := |\alpha_0|$. Since $r(\alpha_0) = r(\beta_0)$ and $s(\alpha_0) = s(\beta_0)$, we have $\alpha_0 = x^{i,j}(0, Ke_h)$ and $\beta_0 = x^{i,j}(0, Ke_l)$ for some $i, j \in \mathbb{Z}$ and h, l such that $\{h, l\} = \{1, 2\}$. By the Cuntz–Krieger relations and the trace property,

$$\tau\left(s_{\alpha_0}s^*_{\beta_0}\right) = \tau\left(s^*_{\beta_0}s_{\alpha_0}\right) = \sum_{\alpha_0\alpha' = \beta_0\beta' \in \Lambda^{(K,K)}} \tau\left(s_{\alpha'}s^*_{\beta'}\right).$$

Let $\alpha_1 = x^{i,(j+K)}(0, d(\beta_0))$ and $\beta_1 = x^{i,(j+K)}(0, d(\alpha_0))$, then $\mathrm{MCE}(\alpha_0, \beta_0) = \{\alpha_0\alpha_1\} = \{\beta_0\beta_1\}$ so that

$$\tau\left(s_{\alpha_0}s^*_{\beta_0}\right) = \tau\left(s_{\alpha_1}s^*_{\beta_1}\right).$$

Repeating this, we obtain pairs α_n, β_n such that $r(\alpha_n) = r(\beta_n) = s(\alpha_{n-1}) = s(\beta_{n-1})$ and $d(\alpha_n) = d(\beta_{n-1})$ and vice versa for all n, and such that $\tau(s_{\alpha_n}s^*_{\beta_n}) = \tau(s_{\alpha_m}s^*_{\beta_m})$ for all m, n. Now suppose that $K \neq 0$ so that $\alpha_0 \neq \beta_0$ and let $z := \tau(s_{\alpha_0}s^*_{\beta_0})$; we must show that $z = 0$. Let $v_n = r(\alpha_n) = r(\beta_n)$ for all n. Since $K \neq 0$, we have $v_m \neq v_n$ for distinct m, n. It follows that

$$\overline{\mathrm{span}}\left\{s_{\alpha_n}s^*_{\beta_n}\right\} = \bigoplus_{n=0}^{\infty} s_{v_n}\left(\overline{\mathrm{span}}\left\{s_{\alpha_n}s^*_{\beta_n}\right\}\right)s_{v_n} \subseteq \bigoplus_{n=1}^{\infty} s_{v_n}C^*(\Lambda_{\mathrm{II}})s_{v_n}.$$

Since the C^*-norm on a direct sum is the supremum norm, it follows that the series

$$\sum_{n=0}^{\infty} \frac{1}{n} s_{\alpha_n}s^*_{\beta_n}$$

converges to some $S \in C^*(H\Lambda_{\mathrm{II}})$. By continuity of τ, we have

$$\tau(S) = \sum_{n=0}^{\infty} \tau\left(\frac{1}{n} s_{\alpha_n}s^*_{\beta_n}\right) = \sum_{n=0}^{\infty} \frac{z}{n},$$

and this forces $z = 0$ since the harmonic series does not converge. □

Corollary 6.7. *Let τ be the trace on $C^*(\Lambda_{\mathrm{II}})$ constructed in Lemma* 6.6. *There is a unique tracial state τ_0 on $pC^*(\Lambda_{\mathrm{II}})p$ given by $\tau_0 = \frac{1}{\tau(p)}\tau(a)$. In particular $pC^*(\Lambda_{\mathrm{II}})p$ and $C^*(\Lambda_{\mathrm{II}})$ are stably finite.*

Recall that $C^*(\Lambda_{\mathrm{II}})$ is the skew-product of $B_2 \times B_2$ by the $\mathbb{Z}^2$-valued functor c satisfying $c(a, v) = c(v, b) = (0, 1)$, $c(b, v) = (1, 1)$ and $c(v, a) = (-1, 1)$. We write l for the length functor $l(\alpha, \beta) := |\alpha| + |\beta|$ from $B_2 \times B_2$ to $\mathbb{Z}$, and we write γ^l for the corresponding induced action satisfying $\gamma^l_z(s_\lambda) = z^{l(\lambda)}s_\lambda$.

Lemma 6.8. *The C^*-algebra $C^*(B_2 \times B_2)^{\gamma^l}$ is isomorphic to $\bigotimes_{\mathbb{Z}} M_2 \rtimes_\sigma \mathbb{Z}$, where σ is the* (*Bernoulli*) *shift automorphism that translates each tensor factor one position to the right.*

Proof. Let S_1 and S_2 be the canonical generators of the Cuntz algebra $\mathcal{O}_2$, and let $\mathcal{F}_2$ be the AF core of $\mathcal{O}_2$. We will prove that $C^*(B_2 \times B_2)^{\gamma^l}$ is isomorphic to $C^*(\mathcal{F}_2 \otimes \mathcal{F}_2 \cup \{U\}) \subset \mathcal{O}_2$, where $U := S_1^* \otimes S_1 + S_2^* \otimes S_2$ is unitary. The lemma will follow since $C^*(\mathcal{F}_2 \otimes \mathcal{F}_2 \cup \{U\}) \cong \bigotimes_{\mathbb{Z}} M_2 \rtimes_\sigma \mathbb{Z}$ by [12, Proposition 3.3].

First, note that $u := s_{(v,a)}s^*_{(a,v)} + s_{(v,b)}s^*_{(b,v)}$ is a unitary in $C^*(B_2 \times B_2)$. We will show that

$$C^*(B_2 \times B_2)^{\gamma^l} = C^*\big(C^*(\Lambda)^\gamma \cup \{u\}\big).$$

For this, observe that

$$C^*(B_2 \times B_2)^{\gamma^l} = \overline{\operatorname{span}}\big\{s_\alpha s_\beta^* \colon \alpha, \beta \in B_2 \times B_2,\ l(\alpha) = l(\beta)\big\}.$$

Since $d(\alpha) = d(\beta) \Longrightarrow l(\alpha) = l(\beta)$, and since $\gamma_z^l(u) = u$ for all $z \in \mathbb{T}^2$, we have $C^*(C^*(\Lambda)^\gamma \cup \{u\}) \subseteq C^*(B_2 \times B_2)^{\gamma^l}$.

For the reverse inclusion, since u is unitary, it suffices to show that for each spanning element $s_\alpha s_\beta^*$ of $C^*(B_2 \times B_2)^{\gamma^l}$, there exists $n \in \mathbb{Z}$ such that $u^n s_\alpha s_\beta^* \in C^*(B_2 \times B_2)^\gamma$. For this, fix $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in B_2 \times B_2$ such that $l(\alpha) = l(\beta)$. For each $z \in \mathbb{T}^2$ we have $\gamma_z(u^n s_\alpha s_\beta^*) = (z^{(-1,1)}u)^n (z^{(n,-n)} s_\alpha s_\beta^*) = u^n s_\alpha s_\beta^* \in C^*(B_2 \times B_2)^\gamma$, as required, where $n = |\alpha_1| - |\beta_1| = |\beta_2| - |\alpha_2|$.

By [31, Corollary 3.5(iv)], there is an isomorphism $\psi : C^*(B_2 \times B_2) \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ satisfying $\psi(s_{(\alpha,\beta)}) = S_\alpha \otimes S_\beta$ (with the obvious identification of paths and multi-indices). Hence the restriction of ψ to $C^*(B_2 \times B_2)^{\gamma^l}$ is the required isomorphism, since $\psi(u) = U$ and $\psi(C^*(B_2 \times B_2)^\gamma) = \mathcal{F}_2 \otimes \mathcal{F}_2$. □

Proposition 6.9. *The C^*-algebra $C^*(\Lambda_{\mathrm{II}})$ is AF-embeddable.*

Proof. Since every automorphism of a UHF algebra is approximately inner, $\bigotimes_{\mathbb{Z}} M_2 \rtimes_\sigma \mathbb{Z}$ is AF-embeddable by [53, 3.6 Theorem]. Thus Lemma 6.8 implies that $C^*(B_2 \times B_2)^{\gamma^l}$ is AF-embeddable. For all $\lambda, \mu \in \Lambda_{\mathrm{II}}, c(\lambda) = c(\mu) \Longrightarrow l(\lambda) = l(\mu)$, from which it follows that $C^*(B_2 \times B_2)^{\gamma^c} \subseteq C^*(B_2 \times B_2)^{\gamma^l}$; thus $C^*(B_2 \times B_2)^{\gamma^c}$ is also AF-embeddable. By [46, Proposition], $C^*(\Lambda_{\mathrm{II}})$ is strongly Morita equivalent, and thus stably isomorphic, to $C^*(B_2 \times B_2)^{\gamma^c}$. Hence, $C^*(\Lambda_{\mathrm{II}})$ is itself AF-embeddable. □

Remark 6.10. It is known that $\bigotimes_{\mathbb{Z}} M_2 \rtimes_\sigma \mathbb{Z}$ is a simple, unital (non-AF) A$\mathbb{T}$-algebra of real rank zero and has a unique tracial state [10]. Thus the same is true for $C^*(B_2 \times B_2)^{\gamma^l}$, which is strongly Morita equivalent to $C^*((B_2 \times B_2) \times_l \mathbb{Z})$ by [46, Proposition]. Hence $C^*((B_2 \times B_2) \times_l \mathbb{Z})$ is another example of a simple 2-graph C^*-algebra that is neither AF nor purely infinite.

To prove our K-theory results we will need the fact that the K_0-group of $C^*(\Lambda_{\mathrm{II}})$ is generated by the classes of its vertex projections. The following lemma proves this fact holds in general.

Lemma 6.11. *Let Λ be a row-finite 2-graph with no sources. Suppose that the degree of each cycle in Λ has zero first coordinate or the degree of each cycle has zero second coordinate. Then $K_0(C^*(\Lambda))$ is generated by $\{[s_v] \colon v \in \Lambda^0\}$.*

Proof. As in [21, Definition 3.6], for $i = 1, 2$ we let M_i denote the $\Lambda^0 \times \Lambda^0$ integer matrix $M_i(v, w) = |v\Lambda^{e_i}w|$. By our hypothesis, and without loss of generality, the coordinate graph of Λ corresponding to e_1 (see [31, Examples 1.10(i)]), which we denote by Λ_1, contains no cycles and so its C^*-algebra is AF by [31, Examples 1.7(i)] and [32, Theorem 2.4]. It is well known that the K_1-group of an AF algebra is trivial so that $K_1(C^*(\Lambda_1))$ is the trivial group. From [21, Remarks 3.19] (or the well-known formulae for the K-groups of directed graph C^*-algebras) we get $\ker(1 - M_1^t) \cong K_1(C^*(\Lambda_1)) \cong \{0\}$. Therefore the block-column matrix

$$\begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}$$

of [21, Proposition 3.16] has trivial kernel, and the lemma now follows from [21, Proposition 3.16]. $\square$

Proposition 6.12. *The K_0-group $K_0(pC^*(\Lambda_{\mathrm{II}})p)$ is generated by the classes $\{[s_\lambda s_\lambda^*]\colon \lambda \in v_{(0,0)}\Lambda\}$, and two such classes are equal if and only if the associated projections are Murray–von Neumann equivalent in $pC^*(\Lambda_{\mathrm{II}})p$. Let τ_0 be the unique tracial state on $pC^*(\Lambda_{\mathrm{II}})p$ defined in Corollary 6.7. Then τ_0 induces an order-isomorphism $K_0(\tau_0)$ between the ordered K_0-group of $pC^*(\Lambda_{\mathrm{II}})p$ and $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}] \cap \mathbb{R}^+)$, and hence an order-isomorphism between the ordered K_0-group of $C^*(\Lambda_{\mathrm{II}})$ and $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}] \cap \mathbb{R}^+)$. In particular $K_0(\tau_0)$ carries the class of the identity to 1, and carries $\{[s_\lambda s_\lambda^*]\colon \lambda \in v_{0,0}\Lambda_{\mathrm{II}}\}$ to $[0, 1] \cap \mathbb{Z}[\frac{1}{2}]$. Moreover, $K_1(C^*(\Lambda_{\mathrm{II}}))$, and hence also $K_1(pC^*(\Lambda_{\mathrm{II}})p)$, is trivial.*

Proof. Let $A_0 := pC^*(\Lambda_{\mathrm{II}})p$ and $A := C^*(\Lambda_{\mathrm{II}})$. Then A_0 is unital and A is stably unital [31, Remarks 1.6(v)]. Moreover, both are stably finite by Corollary 6.7 so their K_0-groups are equipped with the canonical partial ordering [3, §6.3]. Furthermore, the trace τ_0 induces a state $K_0(\tau_0)$ on $K_0(A_0)$, whose range is $\mathbb{Z}[\frac{1}{2}]$. Since A_0 is a full corner in A the inclusion mapping $i : A_0 \hookrightarrow A$ induces an order-isomorphism $K_0(i) : K_0(A_0) \to K_0(A)$.

The partial isometry $(s_{e_1^{i,j}} + s_{e_2^{i,j}})(s_{f_1^{i+1,j}} + s_{f_2^{i+1,j}})^*$ implements a Murray von-Neumann equivalence in A between $s_{v_{(i,j)}}$ and $s_{v_{(i+1,j)}}$ for all $i, j \in \mathbb{Z}$. It follows, by induction, that $s_{v_{(i,j)}}$ is Murray von-Neumann equivalent to $s_{v_{(k,j)}}$ in A for all $i, j, k \in \mathbb{Z}$, and hence their K_0-classes are equal. Moreover, for each $i, j, k \in \mathbb{Z}$ such that $k \geqslant j$ we have

$$[s_{v_{(i,j)}}] = \sum_{\lambda \in v_{(i,j)}\Lambda_{\mathrm{II}}^{(k-j)e_1}} [s_\lambda s_\lambda^*] = \left|v_{(i,j)}\Lambda_{\mathrm{II}}^{(k-j)e_1}\right| [s_{v_{(i,k)}}] = 2^{k-j}[s_{v_{(i,k)}}],$$

since, for each $\lambda \in v_{(i,j)}\Lambda_{\mathrm{II}}^{(k-j)e_1}$, $s(\lambda) = v_{(l,k)}$ for some $l \in \mathbb{Z}$, and $s_\lambda s_\lambda^*$ is Murray von-Neumann equivalent to $s_{s(\lambda)}$ in A.

Let $i, j \in \mathbb{Z}$. If $j < 0$, then $[s_{v_{(i,j)}}] = 2^{0-j}[s_{v_{(i,0)}}] = 2^{-j}[s_{v_{(0,0)}}]$. If $j \geqslant 0$ then $[s_{v_{(i,j)}}] = [s_{v(0,j)}] = [s_\lambda s_\lambda^*]$ for some $\lambda \in v_{(0,0)}\Lambda_{\mathrm{II}}^{je_1}$. By Lemma 6.11 $K_0(A)$ is generated by $\{[s_v]\colon v \in \Lambda_{\mathrm{II}}^0\}$. Therefore it is also generated by $\{[s_\lambda s_\lambda^*]\colon \lambda \in s_{v_{(0,0)}}\Lambda_{\mathrm{II}}\}$. Thus $K_0(A_0)$ is generated by the pre-image under $K_0(i)$, namely $\{[s_\lambda s_\lambda^*]\colon \lambda \in s_{v_{(0,0)}}\Lambda_{\mathrm{II}}\}$.

Let $\lambda, \mu \in v_{(0,0)}\Lambda_{\mathrm{II}}$ such that $s(\lambda) = v_{(i,j)}, s(\mu) = v_{(k,l)}$ for some $i, j, k, l \in \mathbb{Z}$ with $j \leqslant k$. Then the following equations are satisfied in $K_0(A)$: $[s_\lambda s_\lambda^*] = [s_{v_{(i,j)}}] = 2^{k-j}[s_{v_{(i,k)}}] = 2^{k-j}[s_\mu s_\mu^*]$. It follows that $[s_\lambda s_\lambda^*] = 2^{k-j}[s_\mu s_\mu^*]$ in $K_0(A_0)$ also.

The above implies that we can write each $x \in K_0(A_0)$ as $x = \sum_{j=0}^{n} x_j[s_{\lambda_j} s_{\lambda_j}^*]$ where $\lambda_j \in v_{(0,0)}\Lambda_{\mathrm{II}} v_{(0,j)}$ for all $j = 1, 2, \dots, n$. Furthermore, $x = (\sum_{j=0}^{n} 2^{n-j} x_j)[s_{\lambda_n} s_{\lambda_n}^*]$ so that each element in $K_0(A_0)$ can be written as an integer multiple of $[s_\lambda s_\lambda^*]$ for some $\lambda \in v_{(0,0)}\Lambda_{\mathrm{II}}$.

We will show that $K_0(\tau_0)$ is injective. Suppose that $K_0(\tau_0)(x) = 0$ for some $x \in K_0(A_0)$. Now $x = m[s_\lambda s_\lambda^*]$ for some $m \in \mathbb{Z}$ and $\lambda \in v_{(0,0)}\Lambda_{\mathrm{II}}$. So we have $0 = mK_0(\tau_0)([s_\lambda s_\lambda^*]) = m\tau_0(s_\lambda s_\lambda^*)$. Thus $m = 0$ since τ_0 is faithful, so that $x = 0$, as required. Since we already showed that projections with the same trace are Murray–von Neumann equivalent, it follows that K_0-equivalence is the same as Murray–von Neumann equivalence.

We have established that $K_0(\tau_0)$ is a positive isomorphism, so to show that it is an isomorphism of ordered groups, it suffices to prove that $\mathbb{Z}[\frac{1}{2}]^+ \subseteq K_0(\tau)(K_0(A_0)^+)$. Let $y \in \mathbb{Z}[\frac{1}{2}]^+$ then $y = m2^{-n}$ for some $m, n \in \mathbb{N}$. But $m2^{-n} = K_0(\tau)(m[s_\lambda s_\lambda^*])$ for some $\lambda \in v_{(0,0)}\Lambda_{\mathrm{II}}$, thus $y = K_0(\tau)(x)$ for some $x \in K_0(A_0)^+$, as required.

As A_0 is strongly Morita equivalent to A, it remains to prove that $K_1(A)$ is isomorphic to the trivial group. This follows immediately from [21, Proposition 3.16] since Λ_{II} and Λ_{I} share the same skeleton. □

Remark 6.13. It follows from the above that the unique normalised traces on $C^*(H\Lambda_{\mathrm{II}})$ and on $C^*(H\Lambda_{\mathrm{I}})$ determine an order-isomorphism $K_0(C^*(\Lambda_{\mathrm{II}})) \cong K_0(C^*(\Lambda_{\mathrm{I}}))$ which carries the class of each vertex projection in $C^*(\Lambda_{\mathrm{II}})$ to the class of the corresponding vertex projection in $C^*(\Lambda_{\mathrm{I}})$.

All of the above is strong evidence suggesting that $C^*(\Lambda_{\mathrm{II}})$ is isomorphic to $C^*(\Lambda_{\mathrm{I}})$. However, since each vertex $v_{i,j}$ receives both a periodic infinite path $x^{i,j}$ and an aperiodic infinite path $y^{i,j}$, every open set in the unit space $\mathcal{G}_\Lambda^{(0)}$ of the k-graph groupoid of [31] contains both a point with trivial isotropy and a point with nontrivial isotropy. So $\mathcal{G}_\Lambda$ is topologically free but not principal: the set of units with trivial isotropy is dense in $G^{(0)}$, but not the whole of $G^{(0)}$. Let $D := \overline{\operatorname{span}}\{s_\lambda s_\lambda^*\colon \lambda \in \Lambda_{\mathrm{II}}\}$. It follows from [43, Proposition 5.11] that the pair $(C^*(\Lambda_{\mathrm{II}}), D)$ is a Cartan pair but not a C^*-diagonal, and in particular that D does not have unique extension of pure states to $C^*(\Lambda_{\mathrm{II}})$. Since the canonical maximal abelian subalgebra in an AF algebra is always a diagonal in the sense of Kumjian and in particular always has the extension property, it follows that if $C^*(\Lambda_{\mathrm{II}})$ is AF, then its canonical diagonal subalgebra (as an AF algebra) is not conjugate to the Cartan subalgebra D. It must, however, be in some sense locally conjugate: the range of the trace on $C^*(\Lambda_{\mathrm{II}})$ is exhausted on D, and since trace equivalence coincides with Murray–von Neumann equivalence for projections in AF algebras, it would follow that each projection in the AF diagonal was Murray–von Neumann equivalent to a projection in D.

We have been unable to determine whether $C^*(\Lambda_{\mathrm{II}})$ is indeed an AF algebra, and leave this interesting question open.

Remark 6.14. Our results in this paper constitute only a start on the problem of when a k-graph algebra is AF. There are numerous examples upon which our results shed little light, and it seems likely that substantially different techniques are required to understand them. One such is the unique 2-graph $\mathcal{X}$ with skeleton as illustrated in Fig. 3 discussed on [20, pp. 52 and 53]. We pass up, for now, analysis of this example: the questions raised by the example Λ_{II} above seem to go more directly to the heart of the difficulty of the question of when a k-graph C^*-algebra is AF.

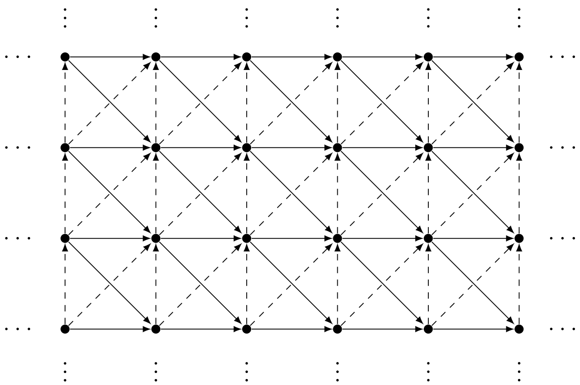


Fig. 3. The skeleton of the 2-graph $\mathcal{X}$.

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