The Cauchy Mean Value Theorem and L'Hospital's Rule

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1 The Cauchy Mean Value Theorem

The following extension of the Mean Value Theorem, called the *Cauchy Mean Value Theorem*, is very useful.

Theorem 1 *Suppose that a and b are real numbers with $a < b$ and suppose that $f : [a, b] \to R$ and $g : [a, b] \to R$ are both differentiable at each point of (a, b) and continuous at each point of $[a, b]$. Also, suppose that $g(a) \neq g(b)$ and that $f(x)$ and $g(x)$ are not simultaneously zero at any point $x \in (a, b)$. Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Let $F : [a, b] \to R$ be the function defined by

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)).$$

The function F is differentiable at each point of (a, b), continuous at each point of $[a, b]$, and $F(a) = F(b) = 0$. Hence, by Rolle's Theorem, there exists a point $c \in (a, b)$ such that $F'(c) = 0$.

For each $x \in (a, b)$, we see that

$$F'(x) = f'(x) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) g'(x)$$

and since $F'(c) = 0$, we obtain

$$\left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(c) = f'(c).$$

We conclude that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

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Exercise 2 *One of the hypotheses of Theorem 1 is that $g(a) \neq g(b)$. Why is this hypothesis needed in order for the theorem to make sense?*

Exercise 3 *In order to carry out the last step in the proof of Theorem 1, we divided by $g'(c)$. How do we know that $g'(c) \neq 0$?*

Exercise 4 *Suppose that a and b are real numbers with $a < b$ and suppose that $f : [a, b] \to R$ and $g : [a, b] \to R$ are both differentiable at each point of (a, b) and continuous at each point of $[a, b]$. Prove that there exists a point $c \in (a, b)$ such that*

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

HINT: *The proof is similar to the proof of Theorem 1. However, it is obvious that a different function F must be used in the proof because we are not hypothesizing that $g(a) \neq g(b)$.*

2 L'Hospital's Rule

We will use the Cauchy Mean Value Theorem to prove the following famous theorem called L'Hospital's Rule.

Theorem 5 *Suppose that a and b are real numbers with $a < b$. Suppose also that $f : (a, b) \to R$ and $g : (a, b) \to R$ are both differentiable at each point of (a, b) and that*

1. *$g(x) \neq 0$ at any point $x \in (a, b)$.*

2. $g'(x) \neq 0$ *at any point* $x \in (a, b)$.

3. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

4. $\lim_{x \to a} (f'(x) / g'(x))$ *exists (or is* ∞ *or* $-\infty$*).*

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. We define functions $F : (a, b) \to R$ and $G : (a, b) \to R$ as follows:

$$F(x) = \begin{cases} f(x) & \text{if} \quad x \neq a \\ 0 & \text{if} \quad x = a \end{cases}$$

$$G(x) = \begin{cases} g(x) & \text{if} \quad x \neq a \\ 0 & \text{if} \quad x = a \end{cases}.$$

Letting x be an arbitrary point in (a, b), we will show that the hypotheses of Theorem 1 are satisfied for F and G on the interval $[a, x]$ and then apply the theorem on that interval. This will then lead to the proof of the desired result.

For any given $x \in (a, b)$, it is clear that F and G are both differentiable (and thus also continuous) at each point of $(a, x]$. Also, since

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

and

$$\lim_{x \to a} G(x) = \lim_{x \to a} g(x) = 0 = G(a),$$

we see that F and G are both continuous at a. In conclusion, F and G are both differentiable at each point of (a, x) and continuous at each point of $[a, x]$.

Since $G(a) = 0$ and $G(x) = g(x) \neq 0$, we see that $G(a) \neq G(x)$.

Since $G'(t) = g'(t) \neq 0$ at any point $t \in (a, x)$, then obviously $F'(t)$ and $G'(t)$ cannot be simultaneously zero at any point $t \in (a, x)$.

By the Cauchy Mean Value Theorem, there exists a point $c_x \in (a, x)$ such that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(c_x)}{G'(c_x)}.$$

Since $F(a) = 0$, $G(a) = 0$, $F(x) = f(x)$, $G(x) = g(x)$, $F'(c_x) = f'(c_x)$, and $G'(c_x) = g'(c_x)$, we obtain

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}. \tag{1}$$

Now recall that the point x was chosen arbitrarily from the interval (a, b). We have thus shown that for every point $x \in (a, b)$, there exists a point $c_x \in (a, x)$ such that equation (1) is satisfied. Since $\lim_{x\to a} (f'(x)/g'(x))$ exists (or is ∞ or $-\infty$) and since $a < c_x < x$ for all $x \in (a, b)$, we may conclude that

$$\lim_{x\to a} \frac{f'(c_x)}{g'(c_x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

It then follows from equation (1) that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

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The version of L'Hospital's Rule given in Theorem 5 is a limited version (no pun intended). It covers only the situation where the functions f and g are defined on an open interval (a, b) and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. We could obviously prove a similar result if we assume instead that $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$. A more complete statement of L'Hospital's Rule is given in Theorem 6. Proving the more complete version requires considering several cases and using somewhat similar reasoning to that used in the proof of Theorem 5. We will not prove the more complete version.

Theorem 6 *Let I be an open interval and let a be a member or an endpoint of I (where we allow the possibility that $a = -\infty$ or $a = \infty$). Let $f : I \to R$ and $g : I \to R$ be differentiable at each point of I and suppose that*

1. *$g(x) \neq 0$ at any point $x \in I - \{a\}$.*

2. $g'(x) \neq 0$ *at any point* $x \in I - \{a\}$.

3. ***Either*** $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ ***or*** $\lim_{x\to a} g(x) = -\infty$ ***or*** $\lim_{x\to a} g(x) = \infty$.

4. $\lim_{x\to a} (f'(x)/g'(x)) = L$ *(where we allow the possibility that* $L = -\infty$ *or* $L = \infty$*).*

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Historical Note Although L'Hospital's Rule is named after the Marquis de L'Hospital because it first appeared in the calculus book that he wrote (around 1690), the theorem was actually discovered by Johann Bernoulli (1667-1748) who taught calculus to L'Hospital. Bernoulli was very upset that L'Hospital did not give him proper credit for the discovery of the theorem. It was not surely established by historians until 1922 (more than 200 years after the time of Bernoulli and L'Hospital) that Bernoulli was the true discoverer of L'Hospital's Rule. We still call it L'Hospital's Rule out of tradition.

We now illustrate L'Hospital's Rule with some examples.

Example 7 *In this example, we will use L'Hospital's Rule to show that*

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

First, we verify that all of the necessary hypotheses are in place: Let $I = (-\infty, \infty)$ *and let* $f : I \to R$ *and* $g : I \to R$ *be the functions defined respectively by* $f(x) = e^x - 1$ *and* $g(x) = x$. *Clearly,* $0 \in I$. *Also,* f *and* g *are both differentiable at each point of* I *and*

1. $g(x) = x \neq 0$ *at any point* $x \in I - \{0\}$.

2. $g'(x) = 1 \neq 0$ *at any point* $x \in I - \{0\}$.

3. $\lim_{x\to 0} f(x) = \lim_{x\to 0} (e^x - 1) = 0$ *and* $\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0$.

4. $\lim_{x\to 0} (f'(x)/g'(x)) = \lim_{x\to 0} (e^x/1) = 1$.

We conclude that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{e^x-1}{x}=1.$$

Example 8 *Let us use L'Hospital's Rule to compute*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

This requires a little trick beforehand. We observe that $\lim_{x\to\infty}(1+1/x)=1$ *and* $\lim_{x\to\infty}x=\infty$*. Thus, the limit we want to compute is of the indeterminate form* 1^∞*. To put the problem into a form for which L'Hospital's Rule may be applied, we make the substitution*

$$y=\left(1+\frac{1}{x}\right)^x$$

which gives us

$$\ln y=x\ln\left(1+\frac{1}{x}\right)=\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

The limit we want to compute is $\lim_{x\to\infty}y$*. We will do this by computing* $\lim_{x\to\infty}\ln y$ *and by then appealing to the continuity of the exponential function. Hence, we now consider the problem*

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

To apply L'Hospital's Rule to this problem, we let $I=(1,\infty)$ *and let* $f:I\to R$ *and* $g:I\to R$ *be the functions defined respectively by* $f(x)=\ln(1+1/x)$ *and* $g(x)=1/x$*. It is clear that* ∞ *is an "endpoint" of* I*. Also* f *and* g *are both differentiable at each point of* I *and:*

1. $g(x)=1/x\neq 0$ *at any point* $x\in I$.
2. $g'(x)=-1/x^2\neq 0$ *at any point* $x\in I$.
3. $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\ln(1+1/x)=0$ *and* $\lim_{x\to\infty}g(x)=\lim_{x\to\infty}(1/x)=0$.

4.

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}=1.$$

We conclude that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\ln y=1.$$

We now obtain (by continuity of the exponential function) that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty}y=\lim_{x\to\infty}e^{\ln y}=e^1=e.$$

Example 9 *In this example, we show an incorrect application of L'Hospital's Rule that leads to a wrong answer (thus illustrating the importance of first checking that all of the hypotheses are satisfied).*

Consider the problem

$$\lim_{x\to 2}\frac{2x-2}{2x}.$$

Hastily attempting to use L'Hospital's Rule, we obtain

$$\lim_{x\to 2}\frac{2x-2}{2x}=\lim_{x\to 2}\frac{2}{2}=1.$$

However, upon closer inspection, we see that $\lim_{x\to 2}(2x-2)=2$ *and* $\lim_{x\to 2}2x=4$ *so (by basic limit properties), the correct result is*

$$\lim_{x\to 2}\frac{2x-2}{2x}=\frac{2}{4}=\frac{1}{2}.$$

Why did L'Hospital's Rule not apply to this problem?

Exercise 10 *In the textbook, Section 4.4, do problems 1(a,d,i,p,q), 10, 11, 14, 24.*