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Composition operators with univalent symbol in Schatten classes ☆

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Abstract

We study composition operators, induced by a sub-domain of the unit disc whose boundary intersects the unit circle at 1 and which has, in a neighborhood of 1, a polar equation $1 - r = \gamma(|\theta|)$ (see Fig. 1). We obtain an explicit characterization for the membership in Schatten p-classes, in terms of γ.

Keywords: Hardy space; Dirichlet space; Bergman space; Composition operators; Univalent functions; Harmonic measure

1. Introduction

In this paper, we study composition operators with univalent symbols acting on weighted spaces of analytic functions in the unit disc $\mathbb{D}$. Denote by $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. The Hardy space H^2 consists of those analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

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where $f(z)=\sum_{n\geqslant 0}a_n z^n$. By Fatou's radial limits theorem, every function $f\in H^2$ has non-tangential limits almost everywhere on the unit circle $\partial\mathbb{D}$. The limit function f^* belongs to $L^2(\partial\mathbb{D})$ and

$$\|f\|_{H^2}^2=\frac{1}{2\pi}\int_{\partial\mathbb{D}}\left|f^*\left(e^{i\theta}\right)\right|^2 d\theta=\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\partial\mathbb{D}}\left|f\left(re^{i\theta}\right)\right|^2 d\theta.$$

By the classical Littlewood–Paley identity we have

$$\left\|f-f(0)\right\|_{H^2}^2=2\int_{\mathbb{D}}\left|f'(z)\right|^2\log\frac{1}{|z|}\,dA(z)<\infty,$$

where $dA(z)=dx\,dy/\pi$ is the Lebesgue normalized area measure on the disc $\mathbb{D}$.

Using this formula, it is clear that

$$H^2=\left\{f\in H(\mathbb{D}):\ \int_{\mathbb{D}}\left|f'(z)\right|^2\left(1-|z|^2\right)dA(z)<\infty\right\}.$$

More generally, we define for $\alpha\geqslant 0$, the weighted analytic spaces $\mathcal{H}_\alpha$ by

$$\mathcal{H}_\alpha:=\left\{f\in H(\mathbb{D}):\ \int_{\mathbb{D}}\left|f'(z)\right|^2 dA_\alpha(z)<\infty\right\},$$

where

$$dA_\alpha(z)=(1+\alpha)\left(1-|z|^2\right)^\alpha dA(z).$$

For $\alpha\in[0,1)$, $\mathcal{H}_\alpha:=\mathcal{D}_\alpha$ are the weighted Dirichlet spaces (the classical Dirichlet space corresponds to $\alpha=0$). Note also that for $\alpha>1$, $\mathcal{H}_\alpha$ are weighted Bergman spaces. More precisely, if A^2_β denotes the weighted Bergman space defined by

$$A^2_\beta:=\left\{f\in H(\mathbb{D}):\ \int_{\mathbb{D}}\left|f(z)\right|^2 dA_\beta(z)<\infty\right\}\quad(\beta>-1),$$

then $\mathcal{H}_\alpha=A^2_{\alpha-2}$.

Let φ be a holomorphic self map of $\mathbb{D}$. The composition operator C_φ acting on $\mathcal{H}_\alpha$ with symbol φ is defined by

$$C_\varphi f=f\circ\varphi,\qquad f\in\mathcal{H}_\alpha.$$

The boundedness, compactness and membership in Schatten classes of composition operators was subject of many papers. See, for example, [14,10,19,18,2,16,6]. It is known, from Schwarz lemma, that

$$\sup_{|z|<1}\frac{1-|z|}{1-|\varphi(z)|}<\infty.$$

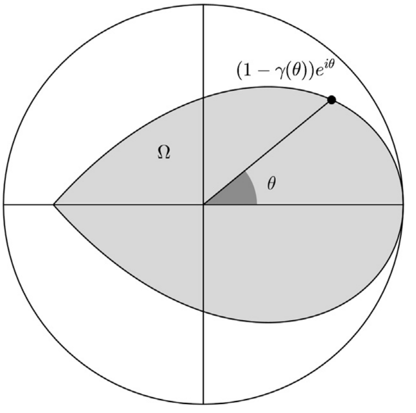

Fig. 1. Domain Ω.

Indeed, if $\varphi(0)=0$ this is a direct consequence of Schwarz lemma. In the general case one can consider $\varphi_a \circ \varphi$, $a=\varphi(0)$, where φ_a is an automorphism of $\mathbb{D}$ and use

$$C_1 \leqslant \frac{1-|\varphi_a(z)|}{1-|z|} \leqslant C_2$$

where C_1, C_2 depend only on a. As consequence and by a change of variables formula it is easy to see that if φ is univalent then C_φ is bounded on $\mathcal{H}_\alpha$. Note also that C_φ is compact in this case, for $\alpha>0$, if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|}{1-|\varphi(z)|}=0.$$

Note that if φ, ψ are two univalent analytic functions such that $\varphi(\mathbb{D})=\psi(\mathbb{D})$ then $C_\varphi = C_h C_\psi$ with h an automorphism of the disc, and therefore C_φ and C_ψ are equal up to multiplicative invertible operator. Then C_φ is compact or in a p-Schatten class if and only if C_ψ satisfies the same property. So, it is natural to ask how to characterize these properties in terms of the geometry of $\varphi(\mathbb{D})$.

Throughout this paper Ω denotes a simply connected sub-domain of $\mathbb{D}$ such that $\partial\Omega \cap \partial\mathbb{D}=\{1\}$. We will also suppose that $\partial\Omega$ has, in a neighborhood of $+1$, a polar equation $1-r=\gamma(|\theta|)$, where $\gamma:[0,\delta]\to[0,1]$ is a continuous, increasing function with $\gamma(0)=0$ (see Fig. 1).

Let φ be a univalent map of $\mathbb{D}$ onto Ω with $\varphi(1)=1$. Tsuji–Warschawski's theorem [15], asserts that φ has an angular derivative at $+1$ if and only if

$$\int_0^\delta \frac{\gamma(t)}{t^2}\,dt < +\infty.$$

This means, by Julia–Caratheodory's theorem and compactness criterion, that for $\alpha>0$ we have

$$C_\varphi \text{ is compact on } \mathcal{H}_\alpha \quad \Longleftrightarrow \quad \int_0^\delta \frac{\gamma(t)}{t^2}\,dt = +\infty. \tag{1}$$

In the sequel we will suppose that

$$\lim_{t\to 0^+}\frac{\gamma(t)}{t}=0 \quad \text{and} \quad \gamma'(t)=O\big(\gamma(t)/t\big) \quad \big(t\to 0^+\big). \tag{2}$$

We will denote

$$\Gamma(t)=\frac{2}{\pi}\int_t^\delta \frac{\gamma(s)}{s^2}\,ds. \tag{3}$$

Our aim in this paper is to study the membership in Schatten p-ideals of such composition operators. Our main result is the following theorem.

Theorem 1.1. *Let Ω, γ and φ as above. Suppose that*

$$\gamma(t)=O\big(t/\log^\beta(1/t)\big) \quad \textit{for some } \beta>1/2. \tag{4}$$

For $\alpha>0$ and $p>0$, we have $C_\varphi\in\mathcal{S}_p(\mathcal{H}_\alpha)$ if and only if

$$\int_0^\delta \frac{e^{-\frac{p\alpha}{2}\Gamma(t)}}{\gamma(t)}\,dt \tag{5}$$

converges.

As applications, we give two corollaries. The first one shows, in particular, that condition (4) is not very restrictive.

Corollary 1.2. *Let Ω, γ and φ as above. Let $\alpha>0$ and $p>0$. If $\gamma(t)\geqslant ct/\log^\beta(1/t)$ for some $\beta\in(0,1)$ and $c>0$, then*

$$C_\varphi\in\bigcap_{p>0}\mathcal{S}_p(\mathcal{H}_\alpha).$$

Proof. Let $\beta\in(1/2,1)$ and $\gamma_\beta(t)=ct/\log^\beta(1/t)$. Let φ_β be a univalent self map of the disc such that $\varphi(\mathbb{D})\subset\varphi_\beta(\mathbb{D})$ and the boundary of $\varphi_\beta(\mathbb{D})$ intersects the unit circle at 1 and has, in neighborhood of 1, the polar equation $1-r=\gamma_\beta(|\theta|)$. By Theorem 1.1, $C_{\varphi_\beta}\in\bigcap_{p>0}\mathcal{S}_p(\mathcal{H}_\alpha)$. Since $\varphi(\mathbb{D})\subset\varphi_\beta(\mathbb{D})$, there exists a univalent self map of $\mathbb{D}$, h, such that $\varphi=\varphi_\beta\circ h$. Then, it follows that $C_\varphi=C_hC_{\varphi_\beta}\in\bigcap_{p>0}\mathcal{S}_p(\mathcal{H}_\alpha)$. □

In the second corollary we construct composition operators in a prescribed Schatten class (see also [1,5,7]).

Corollary 1.3. *Let $\alpha>0$.*

(1) *There exists a compact composition operator C_φ on $\mathcal{H}_\alpha$ which belongs to no Schatten class.*

(2) *Let $p_0 > 0$, there exists a compact composition operator C_φ such that*

$$C_\varphi \in \bigcap_{p>p_0} \mathcal{S}_p(\mathcal{H}_\alpha) \setminus \mathcal{S}_{p_0}(\mathcal{H}_\alpha).$$

(3) *Let $p_0 > 0$, there exists a compact composition operator C_φ such that*

$$C_\varphi \in \mathcal{S}_{p_0}(\mathcal{H}_\alpha) \setminus \bigcup_{p<p_0} \mathcal{S}_p(\mathcal{H}_\alpha).$$

Proof. We apply the above theorem, with the following examples:

1.
$$\gamma(t) = \frac{t}{\log(\frac{1}{t}) \log\log(\frac{1}{t})},$$
$$\int_0^\delta \frac{e^{-\frac{p\alpha}{2}\Gamma(t)}}{\gamma(t)}\,dt \asymp \int_0^\delta \frac{\log(\frac{1}{t})}{t\log^{\frac{\alpha p}{\pi}-1}\log(\frac{1}{t})}\,dt = +\infty, \quad \forall p > 0.$$

2.
$$\gamma(t) = c\frac{t}{\log\frac{1}{t}}$$
with $c = \frac{2\pi}{\alpha p_0}$
$$\int_0^\delta \frac{e^{\frac{-p\alpha}{2}\Gamma(t)}}{\gamma(t)}\,dt \asymp \int_0^\delta \frac{1}{t\log^{\frac{2p}{p_0}-1}(\frac{1}{t})}\,dt,$$
which converge if and only if $p > p_0$.

3.
$$\gamma(\tau) = \frac{2\pi}{\alpha p_0}\left(\frac{t}{\log\frac{1}{t}} + \frac{t}{\log(\frac{1}{t})\log\log(\frac{1}{t})}\right),$$
$$\int_0^\delta \frac{e^{\frac{-p\alpha}{2}\Gamma(t)}}{\gamma(t)}\,dt \asymp \int_0^\delta \frac{1}{t\log^{\frac{2p}{p_0}-1}(\frac{1}{t})\log^{\frac{2p}{p_0}}\log(\frac{1}{t})}\,dt,$$
which converge if and only if $p \geqslant p_0$. □

This paper is organized as follows. In the second section we recall some characterizations of membership to Schatten classes. We will also translate these characterizations in term of harmonic measure of Ω. In the third section, we give estimates of harmonic measure for small arcs. Section four is devoted to the proof of our main theorem.

The notation $A \lesssim B$ means that there is a constant c independent of the relevant variables such that $A \leqslant cB$. We write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

2. Background and notations

The compactness and membership in Schatten p-classes of composition operators has been studied either from the point of view of the Nevanlinna counting functions or from the pull-back measure. In this section we will recall some results which we will be used in the proof of our main results.

If T is a compact operator acting on a separable Hilbert space $\mathcal{H}$, then $(T^*T)^{1/2}$ is a positive compact operator, where T^* denotes the adjoint of T. It follows that the spectrum of $(T^*T)^{1/2}$ consists of countably many eigenvalues (s_n) on $[0,+\infty[$. If the sequence of the eigenvalues of $(T^*T)^{1/2}$ lies in ℓ^p then T is said to belong to the Schatten p-class, denoted by $\mathcal{S}_p(\mathcal{H})$. For more details see [17].

2.1. *Nevanlinna counting functions*

Let φ be an analytic self map of $\mathbb{D}$. The Nevanlinna counting function, $N_{\varphi,\alpha}$, of φ associated to $\mathcal{H}_\alpha$ is defined by

$$N_{\varphi,\alpha}(w) := \sum_{w=\varphi(z)} \left(\log \frac{1}{|z|}\right)^\alpha \quad \text{if } w \in \varphi(\mathbb{D}); \qquad N_{\varphi,\alpha}(w) = 0 \quad \text{if } w \notin \varphi(\mathbb{D}).$$

In 1987, J.H. Shapiro in [14] characterized the compactness of a composition operators on the Hardy space ($\alpha = 1$) in terms of the Nevanlinna function. He proved that C_φ is compact on H^2 if and only if $\lim_{|w|\to 1} \frac{N_\varphi(w)}{1-|w|} = 0$. This result was extended to other spaces including our $\mathcal{H}_\alpha$ spaces (see [6] and references therein).

Five years later, D. Luecking and K. Zhu obtained in [11] a simple characterization for a composition operator C_φ to be in $\mathcal{S}_p(\mathcal{H}_\alpha)$ in terms of $N_{\varphi,\alpha}$ in the case $\alpha \geqslant 1$. Recently, J. Pau and P.A. Pérez in [13] gave an analogous characterization for the weighted *Dirichlet spaces*. We summarize these results in the following theorem.

Theorem 2.1. *Let φ be an analytic self map of $\mathbb{D}$. Let $\alpha > 0$ and $p > 0$. Then $C_\varphi \in \mathcal{S}_p(\mathcal{H}_\alpha)$ if and only if*

$$w \to \frac{N_{\varphi,\alpha}(w)}{(1-|w|)^\alpha} \in L^{p/2}(\mathbb{D}, d\lambda),$$

where $d\lambda$ is the Möbius invariant measure on $\mathbb{D}$ defined by

$$\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

2.2. *Pull-back measures*

Let φ be an analytic self map of $\mathbb{D}$. Recall that the pull-back measure, m_φ, associated to φ is defined on borelian sets B of $\overline{\mathbb{D}}$ by

$$m_\varphi(B) = m\left(\left\{e^{it}/\varphi^*\left(e^{it}\right) \in B\right\}\right),$$

where m is the normalized Lebesgue measure on the unit circle $\mathbb{T}$ and φ^* is the radial limit of φ.

In [12], B. MacCluer and J.H. Shapiro formulated the compactness of C_φ on H^2, in terms of the pull-back measure. Namely, C_φ is compact if and only if

$$\sup_{\zeta\in\mathbb{T}} m_\varphi\big[W(\zeta,h)\big]=o(h),$$

where

$$W(\zeta,h):=\big\{z\in\mathbb{D};\ |z|\geqslant 1-h;\ \big|\arg(z\bar{\zeta})\big|\leqslant h\big\}$$

are the Carleson windows.

In 1987, D. Luecking [10] characterized a membership to $\mathcal{S}_p(H^2)$ of composition operators in terms of the pull-back measure m_φ. He obtained the following theorem.

Theorem 2.2. *For every $p>0$, the composition operator $C_\varphi: H^2\to H^2$ assuming that $|\varphi^*|<1$ a.e. on $\mathbb{T}$, is on the Schatten p-ideals $\mathcal{S}_p$ if and only if*

$$\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1}\big[2^n m_\varphi(R_{n,j})\big]^{p/2}<\infty, \tag{6}$$

where

$$R_{n,j}=\left\{z\in\mathbb{D};\ 1-2^{-n}\leqslant|z|<1-2^{-n-1}\ \textit{and}\ \frac{2j\pi}{2^n}\leqslant\arg z<\frac{2(j+1)\pi}{2^n}\right\}.$$

As observed in [7], the condition (6) is equivalent to

$$\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1}\big[2^n m_\varphi(W_{n,j})\big]^{p/2}<\infty, \tag{7}$$

where

$$W_{n,j}=\left\{z\in\overline{\mathbb{D}};\ 1-2^{-n}\leqslant|z|\leqslant 1\ \text{and}\ \frac{2j\pi}{2^n}\leqslant\arg z<\frac{2(j+1)\pi}{2^n}\right\}.$$

In this paper, it will be more convenient for us to work with the dyadic Carleson boxes $W_{n,j}$.

Note that recently P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza gave in [8], an explicit relationship between the Nevanlinna counting function and the pull-back measure.

2.3. Composition operators with univalent symbol

Let φ be a univalent analytic self map of $\mathbb{D}$. In this section we will express the membership of C_φ in $\mathcal{S}_p(\mathcal{H}_\alpha)$ in terms of the harmonic measure of the image $\varphi(\mathbb{D})$.

Let us recall the notion of harmonic measures. Let Λ be a Jordan domain in the complex plane. The harmonic measure $\omega(.,E,\Lambda)$ is the harmonic extension of χ_E on Λ, where E is a closed subset of the boundary of Λ and χ_E is the characteristic function of E on $\partial\Lambda$. In the case $\Lambda = \mathbb{D}$ the harmonic measure is given by the Poisson kernel. Indeed

$$\omega(z, E, \mathbb{D}) = \int_E \frac{1-|z|^2}{|\zeta - z|^2}\,|d\zeta| \quad (z \in \mathbb{D}).$$

In particular $\omega(0, E, \mathbb{D}) = m(E)$. Recall also that the harmonic measure is invariant under conformal transformations in the following sense

$$\omega(z, E, \Lambda) = \omega\big(\psi(z), \psi(E), \psi(\Lambda)\big),$$

where ψ is a conformal map and E a closed subset of $\partial\Lambda$. If we suppose that $\varphi(0) = 0$ we have for a closed subset B of $\overline{\mathbb{D}}$,

$$\begin{aligned} m_\varphi(B) &= m\big(\varphi^{-1}(B) \cap \mathbb{T}\big) \\ &= \omega\big(0, \varphi^{-1}(B) \cap \mathbb{T}, \mathbb{D}\big) \\ &= \omega\big(0, B \cap \partial\varphi(\mathbb{D}), \varphi(\mathbb{D})\big). \end{aligned}$$

In the proof of the main theorem we will use the following characterization of membership in Schatten classes.

Theorem 2.3. *Let $p, \alpha > 0$ and φ be a univalent analytic self map of $\mathbb{D}$. We have*

$$C_\varphi \in \mathcal{S}_p(\mathcal{H}_\alpha) \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \big[2^n \omega\big(0, W_{n,j} \cap \partial\varphi(\mathbb{D}), \varphi(\mathbb{D})\big)\big]^{p\alpha/2} < \infty. \tag{8}$$

Proof. Since φ is univalent, $N_{\varphi,\alpha} = N_\varphi^\alpha$. So, using Theorem 2.1, we have

$$\begin{aligned} C_\varphi \in \mathcal{S}_p(\mathcal{H}_\alpha) \quad &\Longleftrightarrow \quad \frac{N_{\varphi,\alpha}(w)}{(1-|w|)^\alpha} \in L^{p/2}(\mathbb{D}, d\lambda) \\ &\Longleftrightarrow \quad \frac{N_\varphi(w)}{1-|w|} \in L^{p\alpha/2}(\mathbb{D}, d\lambda) \\ &\Longleftrightarrow \quad C_\varphi \in \mathcal{S}_{p\alpha}\big(H^2\big). \end{aligned}$$

Then, by Theorem 2.2 and (7),

$$C_\varphi \in \mathcal{S}_p(\mathcal{H}_\alpha) \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \big[2^n m_\varphi(W_{n,j})\big]^{p\alpha/2} < \infty.$$

The result then comes from the above discussion. □

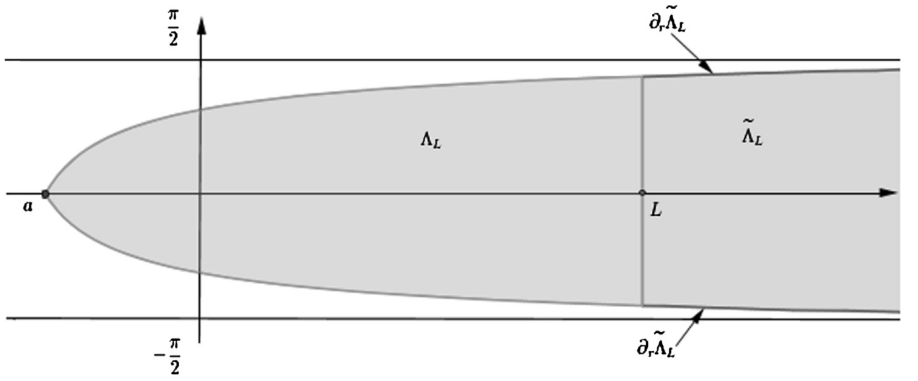

Fig. 2. Sub-domains of Λ.

3. Estimate of harmonic measure

To prove our main theorem we need to estimate $\omega(0, W_{n,j} \cap \partial\varphi(\mathbb{D}), \varphi(\mathbb{D}))$. Due to the invariance of the harmonic measure under conformal maps, it is more natural to consider the strip $S = \{z = x + iy \in \mathbb{C}\colon |y| < \pi/2\}$ instead of the unit disc. Let Λ be the domain defined by

$$\Lambda = \{z = x + iy \in S\colon x > a,\ |y| < \theta(x)/2\},$$

where $a < 0$ and $\theta : (a, +\infty[\to]0, \pi[$ is a function such that

$$\lim_{x\to a} \theta(x) = 0 \quad \text{and} \quad \lim_{x\to +\infty} \theta(x) = \pi.$$

Let $L > 0$, we will use the following notations:

- $\Lambda_L = \{z = x + iy \in \Lambda\colon x < L\}$.
- $\tilde{\Lambda}_L = \{z = x + iy \in \Lambda\colon x > L\}$.
- $L_\Lambda = \{z = x + iy \in \Lambda\colon x = L\}$.
- $\partial_r \tilde{\Lambda}_L = \{z = x + iy \in \partial\Lambda\colon x \geqslant L\}$.

For illustration see Fig. 2.

First, we recall some estimates of harmonic measures due to Ahlfors and Warschawski. For more informations see [4, pp. 148–160] and [3, pp. 136–140].

By Ahlfors distortion's theorem, there exists an absolute constant $C_1 > 0$ such that

$$\omega(0, \partial_r \tilde{\Lambda}_L, \Lambda) \leqslant C_1 \exp\left(-\pi \int_0^L \frac{dx}{\theta(x)}\right). \tag{9}$$

Conversely if θ satisfies

$$\int_0^\infty \frac{(\theta'(x))^2}{\theta(x)}\, dx < \infty, \tag{10}$$

then we have, from Warschawski's theorem, a reverse inequality. That means

$$\exp\left(-\pi\int_0^L \frac{dx}{\theta(x)}\right) \leqslant C_2\omega(0,\partial_r\tilde{\Lambda}_L,\Lambda), \tag{11}$$

where C_2 is a positive constant depending on θ.

Note also that if s is an integer satisfying $2C_1C_2e^{-s} \leqslant 1$, then by (9) and (11) we have

$$\omega(0,\partial_r\tilde{\Lambda}_{L+s},\Lambda) \leqslant \frac{1}{2}\omega(0,\partial_r\tilde{\Lambda}_L,\Lambda).$$

It implies that,

$$\omega(0,\partial_r\tilde{\Lambda}_L,\Lambda) \asymp \omega(0,L_\Lambda,\Lambda_L) \asymp \omega(0,\partial_r\tilde{\Lambda}_L \setminus \partial_r\tilde{\Lambda}_{L+s},\Lambda). \tag{12}$$

The characterization to the membership in Schatten classes in Theorem 2.3 needs an estimation of the harmonic measure of small arcs of $\partial\Lambda$. This is the subject of the following lemma.

Lemma 3.1. *Let Λ, θ be as above. Let $\delta \in (0,1]$ and $c > 0$ such that $c\delta < 1$. There exists $C > 0$ such that, for every $b > 2$ satisfying*

$$\sup\{|\theta(x)-\theta(x')|\colon x,x' \in [b-2,b+2]\} < c\delta, \tag{13}$$

we have

$$\omega(0,E,\Lambda) \leqslant C\delta\exp\left(-\pi\int_0^b \frac{dx}{\theta(x)}\right), \tag{14}$$

where $E = \{z = x+iy \in \partial\Lambda\colon |b-x| < \delta\}$.

Before proceeding to the proof of this result, we will need some definitions and results about extremal distance and harmonic measures. For more details see [4, Chap. IV].

A metric on Λ is a non-negative Borel function ρ such that the area

$$A(\Lambda,\rho) = \int_\Lambda \rho(z)^2\,dA(z) \in (0,+\infty).$$

Let $E, F \subset \partial\Lambda$. The extremal distance from E to F is defined by

$$d_\Lambda(E,F) = \sup_\rho \frac{\inf(\int_\gamma \rho(z)\,|dz|)^2}{A(\Lambda,\rho)}.$$

The infimum is taken on all curves in Λ connecting E to F and the supremum is taken over all metrics on Λ. Note that the extremal distance is invariant under conformal transformation. The link between harmonic measure and extremal distance can be expressed as follows

$$\omega(0, E, \Lambda) \leqslant \frac{8}{\pi} e^{-\pi d_\Lambda(\sigma, E)}, \tag{15}$$

where E is an arc of $\partial\Lambda$ and σ is any curve connecting 0 to $\partial\Lambda \setminus F$. A reverse inequality is also true for some such curve σ.

Proof of Lemma 3.1. Let $\sigma = \{iy\colon\ |y| \leqslant \theta(0)\}$. By (15), it suffices to have a lower bound of $d_\Lambda(\sigma, E)$.

Let $L = b - 2$. Since L_Λ separates σ and E, we have

$$d_\Lambda(\sigma, E) \geqslant d_{\Lambda_L}(\sigma, L_\Lambda) + d_{\tilde{\Lambda}_L}(L_\Lambda, E). \tag{16}$$

By [4, p. 148], we have

$$d_\Lambda(\sigma, L_\Lambda) \geqslant \int_0^L \frac{dx}{\theta(x)} = \int_0^b \frac{dx}{\theta(x)} + O(1). \tag{17}$$

To have a lower bound of $d_\Lambda(L_\Lambda, E)$, we consider the following metric

$$\rho(z) = \frac{1}{\sqrt{(x-b)^2 + (y - \theta(b))^2}} \quad (z \in \Lambda_{L+2} \setminus \Lambda_L),$$

and extended by 0 otherwise. First note that if γ is any curve connecting L_Λ to E then

$$\int_\gamma \rho(z)\,|dz| \geqslant \log(2/\delta). \tag{18}$$

Using an elementary geometrical argument, $A(\Lambda_b, \rho)$ is comparable to the ρ-area of the annulus $\Gamma = \{z = b + i\theta(b) + re^{it}\colon\ \delta < r < 2 \text{ and } -\pi < \theta < 0\}$. Since

$$\begin{aligned} A(\Gamma, \rho) &= \int_\Gamma \rho^2(z)\,dA(z) \\ &= \int_\delta^2 \int_0^\pi \frac{1}{r}\,dr\,d\theta \\ &= \pi \log(2/\delta), \end{aligned}$$

we have

$$A(\Lambda_b, \rho) = \pi \log(2/\delta) + O(1). \tag{19}$$

We obtain from (18) and (19) that

$$d_{\tilde{\Lambda}_L}(L_\Lambda, E) \geqslant \frac{1}{\pi} \log\frac{2}{\delta} + O(1). \tag{20}$$

It follows that

$$d_\Lambda(\sigma, E) \geqslant \int_0^\delta \frac{dx}{\theta(x)} + \pi \log(2/\delta) + O(1). \tag{21}$$

By combining Eqs. (15) and (21) we obtain the desired result. □

Remark. For general θ the condition (13) cannot be removed. However, if θ is concave no condition is needed.

In the sequel h will denote the conformal transform of the unit disc $\mathbb{D}$ onto the strip S given by

$$\zeta \to h(\zeta) = \log \frac{1+\zeta}{1-\zeta}.$$

Let us denote $h(\varphi(\mathbb{D})) = \Lambda$. Clearly Λ has the following form

$$\Lambda = \{z = x + iy \in S\colon\ x > a,\ |y| < \theta(x)/2\}.$$

The following lemma will allow us to estimate the harmonic measure in terms of γ.

Lemma 3.2. *Under the same hypothesis and notations as below we have the following estimation*:

$$\int_0^{\log 2/t} \left(\frac{\pi}{\theta(x)} - 1\right) dx = \frac{2}{\pi} \int_t^2 \frac{\gamma(s)}{s^2}\, ds + \left(\frac{4}{\pi^2} + o(1)\right) \int_t^2 \frac{\gamma^2(s)}{s^3}\, ds. \tag{22}$$

Proof. Let $\zeta = (1-\gamma(t))e^{it} \in \partial\varphi(\mathbb{D})$, $w = \frac{1+\zeta}{1-\zeta}$ and $z = x + iy = \log w$. By a direct calculation we have

$$\operatorname{Re} w = \frac{(2-\gamma(t))\gamma(t)}{\gamma^2(t) + 2(1-\gamma(t))(1-\cos t)}, \tag{23}$$

and

$$\operatorname{Im} w = \frac{2(1-\gamma(t))\sin t}{\gamma^2(t) + 2(1-\gamma(t))(1-\cos t)}. \tag{24}$$

It follows from (23) and (24) that

$$\frac{\theta(x)}{2} = \arg(w) = \frac{\pi}{2} - \arctan\left(\frac{(2-\gamma(t))\gamma(t)}{2(1-\gamma(t))\sin t}\right). \tag{25}$$

Let $u(t) = \frac{(2-\gamma(t))\gamma(t)}{2(1-\gamma(t))\sin t}$. We have

$$\frac{\pi}{\theta(x)} = 1 + \frac{2}{\pi} u + O(u^2).$$

By expanding u and after substitution, we obtain

$$\frac{\pi}{\theta(x)} - 1 = \frac{2}{\pi}\frac{\gamma(t)}{t} + O\left(\frac{\gamma^2(t)}{t^2}\right). \tag{26}$$

To establish (22), we need to express x according to t. We have

$$2x = \log|w|^2 = \log\left(\frac{\gamma^2(t) + 2(1+\cos t)(1-\gamma(t))}{\gamma^2(t) + 2(1-\cos t)(1-\gamma(t))}\right). \tag{27}$$

It gives that

$$\begin{aligned} 2x - \log\frac{4}{t^2} &= \log\frac{\gamma^2(t) + 2(1+\cos(t))(1-\gamma(t))}{4\frac{\gamma^2(t)}{t^2} + 8\frac{1-\cos(t)}{t^2}(1-\gamma(t))} \\ &\asymp \log\left(\frac{1}{1+\frac{\gamma^2(t)}{t^2}}\right) \\ &\asymp -\frac{\gamma^2(t)}{t^2}, \end{aligned}$$

which implies that

$$x = \log(2/t) + O\left(\frac{\gamma^2(t)}{t^2}\right). \tag{28}$$

Since γ satisfies (2), we obtain from (27) the following equality:

$$dx = -\left(1 + O\left(\frac{\gamma(t)}{t}\right)\right)\frac{dt}{t}. \tag{29}$$

Combining (26), (28) and (29), we obtain

$$\int_0^{\log 2/t} \left(\frac{\pi}{\theta(x)} - 1\right) dx = \frac{2}{\pi}\int_t^2 \frac{\gamma(s)}{s^2}\, ds + O\left(\int_t^2 \frac{\gamma^2(s)}{s^3}\, ds\right),$$

which completes the proof. □

4. Proof of the main theorem

The main theorem will derive from the following two lemmas. The first lemma can be seen as a discrete version of Theorem 1.1. To state it we will to precise some notations:

Let $h(\zeta) = \log\frac{1+\zeta}{1-\zeta}$, $\Lambda = h(\Omega)$ with $\Omega = \varphi(\mathbb{D})$ and θ is defined by

$$\Lambda = \{z = x + iy \in S\colon\ x > a,\ |y| < \theta(x)/2\}.$$

Note that under conditions (2) and (4) the condition (10) is satisfied. Indeed, by (25), (27), (28) and (29) we have

$$\int_0^\infty \frac{(\theta'(x))^2}{\theta(x)}\,dx \asymp \int_0^2 \frac{\gamma^2(t)}{t^3}\,dt < \infty.$$

Let us denote

$$h(W_{n,j} \cap \partial\Omega) = E_{j,n} \quad \text{and} \quad h\big(2j\pi/2^n\big) = \zeta_{j,n}.$$

Lemma 4.1. *$C_\varphi \in S_p(\mathcal{H}_\alpha)$ if and only if*

$$\sum_{n=0}^{\infty}\sum_{j=1}^{j_n} \exp\left(\frac{-p\alpha}{2}\int_0^{\mathrm{Re}(\zeta_j)}\left(\frac{\pi}{\theta(x)}-1\right)dx\right) < \infty,$$

where j_n is the integer part of $\frac{2^n}{2\pi}\gamma^{-1}(2\pi/2^n)$.

Proof. By Theorem 2.3, $C_\varphi \in \mathcal{S}_p(\mathcal{H}_\alpha)$ if and only if

$$\sum_{n=1}^{\infty}\sum_{j=0}^{2^n-1} \big(2^n\omega(0, W_{n,j} \cap \partial\Omega, \Omega)\big)^{p\alpha/2} < \infty,$$

where

$$W_{n,j} = \left\{z \in \overline{\mathbb{D}};\ 1-2^{-n} \leqslant |z| \leqslant 1 \text{ and } \frac{2j\pi}{2^n} \leqslant \arg z < \frac{2(j+1)\pi}{2^n}\right\}.$$

Thanks to the symmetry of Ω, we can consider only the j's less than 2^{n-1}.

Remark that the set

$$W_{n,j} \cap \partial\Omega = \big\{\big(1-\gamma(t)\big)e^{it}\colon\ \gamma(t) \leqslant 1/2^n \text{ and } 2\pi j/2^n < t < 2\pi(j+1)/2^n\big\}$$

is non-empty if and only if $j \leqslant j_n = \frac{2^n}{2\pi}\gamma^{-1}(1/2^n)$.

By Eq. (28), we have

$$\mathrm{Re}(\zeta_{j,n}) = \log\big(2^n/2j\pi\big) + O\left(\frac{\gamma^2(2j\pi/2^n)}{(2j\pi/2^n)^2}\right).$$

Note that for $j \leqslant j_n$ we have $\frac{\gamma(2j\pi/2^n)}{(2j\pi/2^n)} \leqslant 1/2j\pi$.

It gives

$$\delta := \mathrm{Re}(\zeta_{j,n}) - \mathrm{Re}(\zeta_{j+1,n}) = \frac{1}{2j\pi} + O\left(\frac{1}{j^2}\right),$$

and

$$\mathrm{Re}(\zeta_{j,n}) - \mathrm{Re}(\zeta_{2j,n}) = \log 2 + o(1/j).$$

From (25), we have

$$\sup\{\pi - \theta(\mathrm{Re}(\zeta));\ \zeta \in E_{n,j}\} \asymp \frac{1}{2j\pi}.$$

Then all conditions are met to apply Lemma 3.1. We obtain

$$\omega(0, W_{n,j} \cap \partial\Omega, \Omega) = \omega(0, E_{n,j}, \Lambda) \leqslant \frac{C}{j} \exp\left(-\pi \int_0^{\mathrm{Re}(\zeta_j)} \frac{dx}{\theta(x)}\right).$$

The reverse inequality holds for many $E_{n,j}$. Indeed, let s be an integer satisfying $2C_1C_2e^{-s} \leqslant 1$ with C_1, C_2 are the constants appearing in (9) and (11). Let k be an integer such that $2^k \geqslant 1/s$. By (12), we have

$$\sum_{2^k}^{2^{k+1}-1} \omega(0, E_{n,j}, \Lambda) \asymp \exp\left(-\pi \int_0^{\mathrm{Re}(\zeta_j)} \frac{dx}{\theta(x)}\right).$$

Taking into account that

$$\omega(0, E_{n,j}, \Lambda) \leqslant \frac{C}{2^k} \exp\left(-\pi \int_0^{\mathrm{Re}(\zeta_j)} \frac{dx}{\theta(x)}\right) \quad \text{for all } j \in \{2^k, 2^k+1, \ldots, 2^{k+1}-1\},$$

there exists $\eta > 0$ such that

$$\mathrm{Card}\left\{ j \in \{2^k, \ldots, 2^{k+1}-1\}\colon\ \omega(0, E_{n,j}, \Lambda) \geqslant \frac{\eta}{j} \exp\left(-\pi \int_0^{\mathrm{Re}(\zeta_j)} \frac{dx}{\theta(x)}\right)\right\} \asymp 2^k.$$

We obtain

$$\begin{aligned}\sum_{j=0}^{j_n} \left(2^n \omega(0, W_{n,j} \cap \partial\Omega, \Omega)\right)^{p\alpha/2} &\asymp \sum_{j=0}^{j_n} \frac{2^{np\alpha/2}}{(j+1)^{p\alpha/2}} \exp\left(\frac{-p\alpha\pi}{2} \int_0^{\mathrm{Re}(\zeta_j)} \frac{dx}{\theta(x)}\right) \\ &\asymp \sum_{j=0}^{j_n} \exp\left(\frac{-p\alpha}{2} \int_0^{\mathrm{Re}(\zeta_j)} \left(\frac{\pi}{\theta(x)} - 1\right) dx\right)\end{aligned}$$

which proves the lemma. □

Lemma 4.2. *Let $c > 0$. The two quantities*

$$\sum_{n=0}^{\infty}\sum_{j=1}^{j_n} e^{-c\Gamma(\frac{2\pi j}{2^n})}, \qquad \int_0 \frac{e^{-c\Gamma(t)}}{\gamma(t)}\,dt,$$

have the same nature.

Proof. Since $\lim_{s\to 0^+}\frac{\gamma(s)}{s} = 0$, we have

$$e^{-c\Gamma(\frac{2j\pi}{2^n})} \asymp 2^n \int_{\frac{2j\pi}{2^n}}^{\frac{2(j+1)\pi}{2^n}} e^{-c\Gamma(s)}\,ds.$$

By elementary calculations we have

$$\begin{aligned}\sum_0^{\infty}\sum_{j=1}^{2^n\gamma^{-1}(2^{-n})} e^{-c\Gamma(\frac{j}{2^n})} &\asymp \sum_0^{\infty} 2^n \int_0^{\gamma^{-1}(2^{-n})} e^{-c\Gamma(s)}\,ds \\ &\asymp \sum_0^{\infty}\sum_{k=n}^{+\infty} 2^n \int_{\gamma^{-1}(2^{-k-1})}^{\gamma^{-1}(2^{-k})} e^{-c\Gamma(s)}\,ds \\ &\asymp \sum_{k=0}^{\infty} 2^k \int_{\gamma^{-1}(2^{-k-1})}^{\gamma^{-1}(2^{-k})} e^{-c\Gamma(s)}\,ds \\ &\asymp \sum_{k=0}^{\infty} \int_{\gamma^{-1}(2^{-k-1})}^{\gamma^{-1}(2^{-k})} \frac{e^{-c\Gamma(s)}}{\gamma(s)}\,ds \\ &\asymp \int_0^{\delta} \frac{e^{-c\Gamma(s)}}{\gamma(s)}\,ds. \qquad \square\end{aligned}$$

Proof of Theorem 1.1. Since $\int_0^{\delta}\frac{\gamma(s)^2}{s^3}\,ds < \infty$, by Lemma 3.2

$$\exp\left(\frac{-p\alpha}{2}\int_0^{\mathrm{Re}(\zeta_j)}\left(\frac{\pi}{\theta(x)} - 1\right)dx\right) \asymp \sum_{n=0}^{\infty}\sum_{j=1}^{j_n} e^{-\frac{p\alpha}{2}\Gamma(\frac{2\pi j}{2^n})}.$$

By Lemma 4.1 and Lemma 4.2 we have $C_\varphi \in \mathcal{S}_p(\mathcal{H}_\alpha)$ if and only if

$$\int_0^\delta \frac{e^{-\frac{p\alpha}{2}\Gamma(s)}}{\gamma(s)}\,ds < \infty,$$

which completes the proof. □

5. Final remarks

- The case of the classical Dirichlet space, which corresponds in our notation to $\alpha = 0$, is simpler (see [9]). Indeed the characterization of composition operator to be compact or in Schatten classes depends only on the area of $W(1,\delta) \cap \Omega$. For our domains these characterizations can be expressed as follows

$$C_\varphi \in \mathcal{S}_\infty \quad \Longleftrightarrow \quad \frac{\gamma(\delta)}{\delta} \to \infty \quad (\delta \to 0),$$

  and

$$C_\varphi \in \mathcal{S}_p \quad \Longleftrightarrow \quad \int_0 \frac{\gamma'(t)}{\gamma^{p/2+1}(t)} t^{p/2}\,dt < \infty.$$

- For necessity in Theorem 1.1, one can relax regularity conditions. Indeed, if $\lim_{t\to 0^+} \gamma(t)/t = 0$ and (5) converges then $C_\varphi \in S_p(\mathcal{H}_\alpha)$.

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References

[1] T. Carroll, C.C. Cowen, Compact composition operators not in the Schatten classes, J. Operator Theory 26 (1991) 109–120.
[2] O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi, Level sets and composition operators on the Dirichlet space, J. Funct. Anal. 260 (2011) 1721–1733.
[3] M.A. Evgrafov, Analytic Functions, Dover Publication, Inc., New York, 1996.
[4] J.B. Garnett, D.E. Marshall, Harmonic Measure, Cambridge University Press, New York, 2005.
[5] M. Jones, Compact composition operators not in the Schatten classes, Proc. Amer. Math. Soc. 134 (2006) 1947–1953.
[6] K. Kellay, P. Lefèvre, Compact composition operators on weighted Hilbert spaces of analytic functions, J. Math. Anal. Appl. 386 (2) (2012) 718–727.
[7] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Some examples of compact composition operators on H^2, J. Funct. Anal. 255 (11) (2008) 3098–3124.
[8] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Nevanlinna counting function and Carleson function of analytic maps, Math. Ann. 351 (2) (2011) 305–326.
[9] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Compact composition operators on the Dirichlet space and capacity of sets of contact points, J. Funct. Anal. 264 (4) (2013) 895–919.

[10] D. Luecking, Trace ideal criteria for Toeplitz operators, J. Funct. Anal. 73 (1987) 345–368.
[11] D. Luecking, K. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114 (1992).
[12] B. MacCluer, J.H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38 (1986) 878–906.
[13] J. Pau, P.A. Pérez, Composition operators acting on weighted Dirichlet spaces, J. Math. Anal. Appl. 401 (2) (2013) 682–694.
[14] J.H. Shapiro, The essential norm of a composition operator, Ann. Math. 12 (1987) 375–404.
[15] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
[16] K.J. Wirths, J. Xiao, Global integral criteria for composition operators, J. Math. Anal. Appl. 269 (2002) 702–715.
[17] K. Zhu, Operator Theory in Function Spaces, Monogr. Textbooks Pure Appl. Math., vol. 139, Marcel Dekker, Inc., 1990.
[18] K. Zhu, Schatten class composition operators on weighted Bergman spaces of the disk, J. Operator Theory 46 (2001) 173–181.
[19] N. Zorboska, Composition operators on weighted Dirichlet spaces, Proc. Amer. Math. Soc. 126 (7) (1998) 2013–2023.