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Application of generalized Hadamard product on special classes of analytic univalent functions

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Abstract In this paper, the author established certain results concerning the quasi-Hadamard product for generalized subclasses of univalent functions with positive coefficients.

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1. Introduction

Let A denote the class of analytic univalent functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f(z) \in A$ is called starlike of order α if $f(z)$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \tag{1.2}$$

for $0 \leq \alpha < 1$ and $z \in U$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α . Also, a function $f(z) \in A$ is called convex of order α if $f(z)$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \tag{1.3}$$

for $0 \leq \alpha < 1$ and $z \in U$. We denote by $C(\alpha)$ the class of convex functions of order α .

For $\beta > 1$ and $z \in U$, let $\mathfrak{M}(\beta)$ denotes the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \beta, \tag{1.4}$$

and let $\mathfrak{N}(\beta)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the condition

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta, \tag{1.5}$$

it follows from (1.4) and (1.5) that

$$f(z) \in \mathfrak{N}(\beta) \iff zf'(z) \in \mathfrak{M}(\beta). \tag{1.6}$$

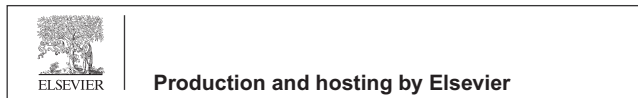
The subclasses $\mathfrak{M}(\beta)$ and $\mathfrak{N}(\beta)$ and some related classes have been studied by several authors (e.g. [1–4]).

Furthermore, let V denote the subclass of analytic univalent functions of the form:

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$$f(z) = a_0z + \sum_{n=2}^{\infty} |a_n|z^n \quad (a_0 > 0). \tag{1.7}$$

Also, let

$$f_i(z) = a_{0,i}z + \sum_{n=2}^{\infty} |a_{n,i}|z^n \quad (a_{0,i} > 0), \tag{1.8}$$

and

$$g_j(z) = b_{0,j}z + \sum_{n=2}^{\infty} |b_{n,j}|z^n \quad (b_{0,j} > 0), \tag{1.9}$$

the quasi-Hadamard product $(f_i * g_j)(z)$ of the functions $f_i(z)$ and $g_j(z)$ by

$$(f_i * g_j)(z) = a_{0,i}b_{0,j}z + \sum_{n=2}^{\infty} |a_{n,i}||b_{n,j}|z^n \quad (i, j \in \mathbb{N} = 1, 2, 3, \dots).$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

Also, let $V(\beta) = \mathfrak{R}(\beta) \cap V$ and $U(\beta) = \Re(\beta) \cap V$, following the results obtained by Uraleghaddi et al. [5], we can obtain the following lemmas.

Lemma 1. Let the function $f(z) \in V$, then $f(z) \in V(\beta)$ ($1 < \beta \leq \frac{4}{3}$) if and only if

$$\sum_{n=2}^{\infty} (n - \beta)|a_n| \leq (\beta - 1)a_0. \tag{1.10}$$

Lemma 2. Let the function $f(z) \in V$, then $f(z) \in U(\beta)$ ($1 < \beta \leq \frac{4}{3}$) if and only if

$$\sum_{n=2}^{\infty} n(n - \beta)|a_n| \leq (\beta - 1)a_0. \tag{1.11}$$

Let $\varphi(z)$ be a fixed function of the form:

$$\varphi(z) = c_0z + \sum_{n=2}^{\infty} c_nz^n \quad (c_0, c_n \geq 0). \tag{1.12}$$

Using the function defined by (1.12), we now define the following new classes.

Definition 1. A function $f(z) \in V_\varphi(c_n, \delta)$ ($c_n \geq c_2 > 0$) if and only if

$$\sum_{n=2}^{\infty} c_n|a_n| \leq \delta a_0 \quad (\delta > 0). \tag{1.13}$$

Definition 2. A function $f(z) \in U_\varphi(c_n, \delta)$ ($c_n \geq c_2 > 0$) if and only if

$$\sum_{n=2}^{\infty} nc_n|a_n| \leq \delta a_0 \quad (\delta > 0). \tag{1.14}$$

Also, we introduce the following class of analytic functions which plays an important role in the discussion that follows.

Definition 3. A function $f(z) \in V_\varphi^k(c_n, \delta)$ ($c_n \geq c_2 > 0$) if and only if

$$\sum_{n=2}^{\infty} n^k c_n|a_n| \leq \delta a_0 \quad (\delta > 0), \tag{1.15}$$

where k is any fixed nonnegative real number.

For suitable choices of c_n, δ, k and $a_0 = 1$, we obtain:

- (i) $V_\varphi^0((n - \beta), (\beta - 1)) = V(\beta)$ ($1 < \beta \leq \frac{4}{3}$) [5];
- (ii) $V_\varphi^1(n(n - \beta), (\beta - 1)) = U(\beta)$ ($1 < \beta \leq \frac{4}{3}$) [5];
- (iii) $V_\varphi^0((n - 1) + |n - 2\beta + 1|, 2(\beta - 1)) = \mathfrak{R}(\beta)$ ($\beta > 1, a_0 = 1$) ([6,7], with $k = 1$);
- (iv) $V_\varphi^1(n\{n - 1\} + |n - 2\beta + 1\}, 2(\beta - 1)) = \Re(\beta)$ ($\beta > 1, a_0 = 1$) ([6,7], with $k = 1$).

Evidently, $V_\varphi^0(c_n, \delta) = V_\varphi(c_n, \delta)$ and $V_\varphi^1(c_n, \delta) = U_\varphi(c_n, \delta)$. Further $V_\varphi^{\gamma_1}(c_n, \delta) \subset V_\varphi^{\gamma_2}(c_n, \delta)$ if $\gamma_1 > \gamma_2 \geq 0$, the containment being proper. Moreover, for any positive integer k , we have the following inclusion relation

$$V_\varphi^k(c_n, \delta) \subset V_\varphi^{k-1}(c_n, \delta) \subset \dots \subset V_\varphi^2(c_n, \delta) \subset U(c_n, \delta) \subset V(c_n, \delta).$$

We also note that for nonnegative real number k , the class $V_\varphi^k(c_n, \delta)$ is nonempty as the function

$$f(z) = a_0z + \sum_{n=2}^{\infty} n^{-k} \frac{\delta a_0}{c_n} \lambda_n z^n, \tag{1.16}$$

where $a_0 > 0, \lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n \leq 1$, satisfy the inequality (1.15).

The quasi-Hadamard product of two or more univalent functions has been recently defined and studied by Aouf [8], Darwish [9], Frasin [10], Frasin and Aouf [11] and Kumar [12].

The object of this paper is to establish a result concerning the quasi-Hadamard product of functions in the classes $V_\varphi^k(c_n, \delta), U_\varphi(c_n, \delta)$, and $V_\varphi(c_n, \delta)$.

2. The main results

Theorem 1. Let the functions $f_i(z)$ defined by (1.8) belong to the class $U_\varphi(c_n, \delta)$ for every $i = 1, 2, \dots, m$; and let the functions $g_j(z)$ defined by (1.9) belong to the class $V_\varphi(c_n, \delta)$ for every $j = 1, 2, \dots, q$. If $c_n \geq n\delta(n \in \mathbb{N})$, then the quasi-Hadamard product $f_1 * f_2 * \dots * f_m * g_1 * g_2 * \dots * g_q(z)$ belongs to the class $V_\varphi^{2m+q-1}(c_n, \delta)$.

Proof. It is sufficient to show that

$$\sum_{n=2}^{\infty} \left[n^{2m+q-1} c_n \left(\prod_{i=1}^m |a_{n,i}| \cdot \prod_{j=1}^q |b_{n,j}| \right) \right] \leq \delta \left(\prod_{i=1}^m a_{0,i} \cdot \prod_{j=1}^q b_{0,j} \right).$$

Since $f_i(z) \in U_\varphi(c_n, \delta)$, we have

$$\sum_{n=2}^{\infty} nc_n|a_{n,i}| \leq \delta a_{0,i}, \tag{2.1}$$

for every $i = 1, 2, \dots, m$. Therefore,

$$|a_{n,i}| \leq n^{-1} \left(\frac{\delta}{c_n} \right) a_{0,i},$$

and hence

$$|a_{n,i}| \leq n^{-2} a_{0,i}, \tag{2.2}$$

the inequalities (2.1) and (2.2) hold for every $i = 1, 2, \dots, m$. Further, since $g_j(z) \in V_\varphi(c_n, \delta)$, we have

$$\sum_{n=2}^{\infty} c_n |b_{n,j}| \leq \delta b_{0,j}, \quad (2.3)$$

for every $j = 1, 2, \dots, q$. Hence, we obtain

$$|b_{n,j}| \leq n^{-1} b_{0,j}, \quad (2.4)$$

for every $j = 1, 2, \dots, q$.

Using (2.2) for $i = 1, 2, \dots, m$, (2.4) for $j = 1, 2, \dots, q - 1$ and (2.3) for $j = q$, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[n^{2m+q-1} c_n \left(\prod_{i=1}^m |a_{n,i}| \cdot \prod_{j=1}^q |b_{n,j}| \right) \right] \\ & \leq \sum_{n=2}^{\infty} \left[n^{2m+q-1} c_n \left(n^{-2m} n^{-(q-1)} \prod_{i=1}^m a_{0,i} \cdot \prod_{j=1}^{q-1} b_{0,j} \right) |b_{n,q}| \right] \\ & = \left(\prod_{i=1}^m a_{0,i} \cdot \prod_{j=1}^{q-1} b_{0,j} \right) \sum_{n=2}^{\infty} c_n |b_{n,q}| \leq \delta \left(\prod_{i=1}^m a_{0,i} \cdot \prod_{j=1}^q b_{0,j} \right). \end{aligned}$$

Hence, $f_1 * f_2 * \dots * f_m * g_1 * g_2 * \dots * g_q \in V_{\varphi}^{2m+q-1}(c_n, \delta)$.

We note that the required estimate can also be obtained by using (2.2) for $i = 1, 2, \dots, m - 1$, (2.4) for $j = 1, 2, \dots, q$, and (2.1) for $i = m$. \square

Taking into account the quasi-Hadamard product functions $f_1(z), f_2(z), \dots, f_m(z)$ only, in the proof of Theorem 1 and using (2.2) for $i = 1, 2, \dots, m - 1$, and (2.1) for $i = m$, we obtain

Corollary 1. *Let the functions $f_i(z)$ defined by (1.8) belong to the class $U_{\varphi}(c_n, \delta)$ for every $i = 1, 2, \dots, m$. If $c_n \geq n\delta$, ($n \in \mathbb{N}$), then the quasi-Hadamard product $f_1 * f_2 * \dots * f_m(z)$ belongs to the class $V_{\varphi}^{2m-1}(c_n, \delta)$.*

Also taking into account the quasi-Hadamard product functions $g_1(z), g_2(z), \dots, g_q(z)$ only, in the proof of Theorem 1 and using (2.4) for $j = 1, 2, \dots, q - 1$, and (2.3) for $j = q$, we obtain

Corollary 2. *Let the functions $g_i(z)$ defined by (1.9) belong to the class $V_{\varphi}(c_n, \delta)$ for every $i = 1, 2, \dots, q$. If $c_n \geq n\delta$, ($n \in \mathbb{N}$). Then the quasi-Hadamard product $g_1 * g_2 * \dots * g_q$ belongs to the class $V_{\varphi}^{q-1}(c_n, \delta)$.*

Remarks

- (i) Putting $c_n = (n - \beta)$ ($n \geq 2$) and $\delta = (\beta - 1)(1 < \beta \leq \frac{4}{3})$ in the above results, we obtain results corresponding to the class $V(\beta)$.

- (ii) Putting $c_n = n(n - \beta)$ ($n \geq 2$) and $\delta = (\beta - 1)(1 < \beta \leq \frac{4}{3})$ in the above results, we obtain results corresponding to the class $U(\beta)$.
- (iii) Putting $c_n = (n - 1) + |n - 2\beta + 1|$ ($n \geq 2$) and $\delta = 2(\beta - 1)$ ($\beta > 1$) in the above results, we obtain results corresponding to the class $\mathfrak{R}(\beta)$.
- (iv) Putting $c_n = n\{(n - 1) + |n - 2\beta + 1|\}$ ($n \geq 2$) and $\delta = 2(\beta - 1)$ ($\beta > 1$) in the above results, we obtain results corresponding to the class $\mathfrak{R}(\beta)$.

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