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Determinants involving q-Stirling numbers

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Abstract

Let $S[i, j]$ denote the q-Stirling numbers of the second kind. We show that the determinant of the matrix $(S[s+i+j, s+j])_{0\leqslant i,j\leqslant n}$ is given by the product $q^{\binom{s+n+1}{3}-\binom{s}{3}} \cdot [s]^0 \cdot [s+1]^1 \cdots [s+n]^n$. We give two proofs of this result, one bijective and one based upon factoring the matrix. We also prove an identity due to Cigler that expresses the Hankel determinant of q-exponential polynomials as a product. Lastly, a two variable version of a theorem of Sylvester and an application are presented.

1. Introduction

The Stirling numbers of the second kind, $S(n, k)$, count the number of partitions of an n-element set into k blocks. They have a natural q-analogue called the q-Stirling numbers of the second kind denoted by $S[n, k]$. They were first defined in the work of Carlitz [4]. A lot of combinatorial work has centered around this q-analogue, the earliest by Milne [12,13]; also see [6,9,19,20].

The goal of this article is to evaluate determinants involving q-Stirling numbers and give bijective proofs whenever possible. Our tool is the juggling interpretation of q-Stirling numbers. Juggling patterns were introduced and studied by Buhler et al. [2]. More combinatorial work was done in [3]. Together with Readdy, the author introduced a crossing statistic in the study of juggling patterns to obtain a q-analogue [8]. Notably, Ehrenborg–Readdy give an interpretation of the q-Stirling numbers of the second kind $S[n, k]$ in terms of juggling patterns. This combinatorial interpretation is useful in obtaining identities involving the q-Stirling numbers; see for instance Theorem 3.3. This interpretation of q-Stirling numbers is equivalent to the rook placement interpretation of Garsia and Remmel [9].

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In Section 3 we evaluate the determinant $\det(S[s+i+j, s+j])$ and give two different proofs. The bijective proof is based upon the bijection in [7], whereas the second proof uses the *LU*-decomposition of the matrix. Similarly, in Section 4 we give a bijective proof of a result of Cigler that expresses the Hankel determinant of the q-exponential polynomials $\tilde{e}_n[x]$ as a product [5]. In the last section we extend a result of Sylvester to evaluate the determinant $\det(S(s+i+j, s+j)/(s+i+j)!)$.

2. q-analogues

We summarize the basic q-analogue notations. For n a non-negative integer, let $[n]$ denote the sum $1+q+\cdots+q^{n-1}$. The *q-factorial* $[n]!$ is the product $[1]\cdot[2]\cdots[n]$. We have that

$$\sum_{\sigma} q^{\mathrm{inv}(\sigma)} = [n]!,$$

where σ ranges over all permutations on n elements. The *Gaussian coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\cdot[n-k]!}.$$

It has the following combinatorial interpretation. Define the rank of a set $S=\{s_1, s_2, \ldots, s_k\}$ of positive integers of cardinality k to be the difference $\rho(S) = s_1+s_2+\cdots+s_k-1-2-\cdots-k$. Then the Gaussian coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{S} q^{\rho(S)},$$

where the sum ranges over all subsets S of $\{1,\ldots,n\}$ of cardinality k.

The Stirling number of the second kind $S(n,k)$ is the number of partitions of a set of cardinality n into k blocks. The *q-Stirling numbers of the second kind* are a natural extension of the classical Stirling numbers. The recursive definition of the q-Stirling numbers is

$$S[n,k] = q^{k-1}\cdot S[n-1,k-1] + [k]\cdot S[n-1,k],$$

where $n,k \geqslant 1$. When $n=0$ or $k=0$, define $S[n,k]=\delta_{n,k}$. The q-Stirling numbers are well-studied; see for instance [6,8,9,11–13,19,20]. There are several combinatorial interpretations of the q-Stirling numbers. We now introduce the interpretation of Ehrenborg and Readdy [8].

Let π be a partition of $\{1,\ldots,n\}$ into k blocks, that is, $\pi=\{B_1,\ldots,B_k\}$. To this partition π we associate a juggling pattern consisting of k paths with each path corresponding to a block of the partition. The ith path touches down at the nodes belonging

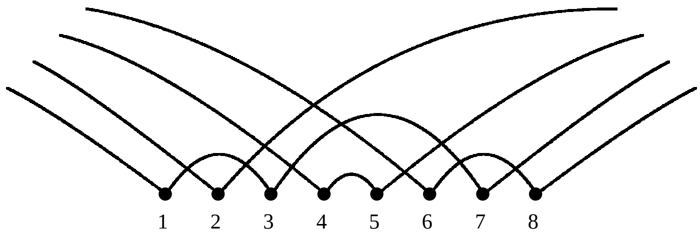

Fig. 1. The juggling pattern associated with the partition $\pi = \{\{1, 3, 7\}, \{2\}, \{4, 5\}, \{6, 8\}\}$. Observe that there are 10 crossings.

to the elements in block B_i. The juggling pattern is drawn so that arcs do not cross each other multiple times and that no more than two arcs intersect at a point. See Fig. 1 for the juggling pattern corresponding the partition $\pi = \{\{1, 3, 7\}, \{2\}, \{4, 5\}, \{6, 8\}\}$. Let cross (π) be the number of crossings in the juggling pattern associated with the partition π. We have the following interpretation of the q-Stirling numbers of the second kind [8].

Theorem 2.1 (Ehrenborg–Readdy). *The q-Stirling number of the second kind $S[n, k]$ is given by*

$$S[n, k] = \sum_{\pi} q^{\operatorname{cross}(\pi)},$$

where the sum ranges over all partitions π of the set $\{1, \ldots, n\}$ into k blocks.

One of the major tools in studying juggling patterns are juggling cards. The juggling card C_i is the picture that consists of one node and k paths, where the $(i + 1)$st path from the bottom touches down at the node and then continues as the lowest path. The juggling cards C_0, C_1, C_2, and C_3 are displayed in Fig. 2. Observe that the juggling card C_i has exactly i crossings.

Let π be a partition on the set $\{1, \ldots, n\}$. For S a subset of $\{1, \ldots, n\}$, define the restricted partition $\pi|_S$ to be the partition $\pi|_S = \{B \cap S\colon\ B \in \pi, B \cap S \neq \emptyset\}$. Moreover,

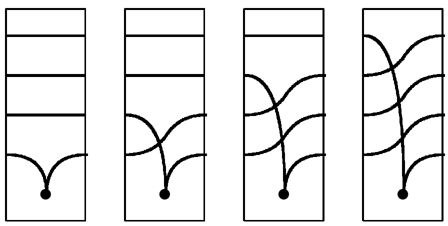

Fig. 2. The four juggling cards C_0, C_1, C_2, and C_3.

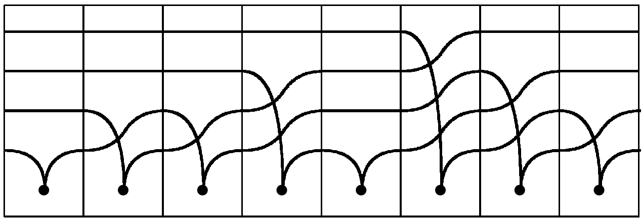

Fig. 3. The partition $\pi = \{\{1,3,7\},\{2\},\{4,5\},\{6,8\}\}$ represented by juggling cards.

for $1 \leqslant i \leqslant n$, we will define c_i so that we can represent our partition as the juggling cards $C_{c_1}, \ldots, C_{c_n}$. To do this, let j be the maximum of the set

$$\{0\} \cup \{h\colon\ h < i, h \text{ and } i \text{ belong to the same block of } \pi\}.$$

Let c_i be the number blocks in the restricted partition $\pi|_{\{j+1,\ldots,i-1\}}$. It is straightforward to verify that the partition π is given by the juggling cards $C_{c_1}, \ldots, C_{c_n}$. For instance, the partition $\pi = \{\{1,3,7\},\{2\},\{4,5\},\{6,8\}\}$ is represented by the juggling cards C_0, C_1, C_1, C_2, C_0, C_3, C_2, and C_1 in Fig. 3. Note that the sum of the indices of the cards is the number of crossings.

Observe that $q^{\binom{k}{2}}$ always divides the q-Stirling number $S[n,k]$. Sometimes it is convenient to work with the modified q-Stirling number $\widetilde{S}[n,k]$ defined by

$$\widetilde{S}[n,k] = q^{-\binom{k}{2}} \cdot S[n,k].$$

The modified q-Stirling number of the second kind $\widetilde{S}[n,k]$ has the natural interpretation when we omit the incoming paths and then count the crossings in the remaining pattern. Let $\widetilde{\text{cross}}(\pi)$ denote the number of crossings in such a pattern, that is, $\widetilde{\text{cross}}(\pi) = \text{cross}(\pi) - \binom{k}{2}$. Thus we have

$$\widetilde{S}[n,k] = \sum_{\pi} q^{\widetilde{\text{cross}}(\pi)}.$$

3. The determinant of q-Stirling numbers

We now consider the determinant of the matrix consisting of q-Stirling numbers. We present two proofs for evaluating this determinant.

Theorem 3.1. *Let n and s be non-negative integers. Then we have*

$$\det\bigl(S[s+i+j, s+j]\bigr)_{0 \leqslant i,j \leqslant n} = q^{\binom{s+n+1}{3} - \binom{s}{3}} \cdot [s]^0 \cdot [s+1]^1 \cdots [s+n]^n.$$

Proof. Let T denote the set of all $(n+2)$-tuples $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$ where σ is a permutation on the set $\{0, 1, \ldots, n\}$ and π_i is a partition of the set $\{1, \ldots, s+i+\sigma(i)\}$ into $s+\sigma(i)$ blocks. Expanding the determinant we have

$$\det\bigl(S[s+i+j, s+j]\bigr)_{0 \leqslant i,j \leqslant n} = \sum_{(\sigma,\pi_0,\ldots,\pi_n)\in T} (-1)^{\sigma} \cdot q^{\operatorname{cross}(\pi_0)+\cdots+\operatorname{cross}(\pi_n)}.$$

Let $(\sigma, \pi_0, \ldots, \pi_n)$ be in the set T. Let X_i be the set $\{1, \ldots, s+i\}$ and Y_i the set $\{s+i+1, \ldots, s+i+\sigma(i)\}$. Define the integer $a_i = |\pi_i|_{X_i}| - s$. That is, $a_i + s$ is the number of blocks in π_i that intersect non-trivially the set X_i. From this we conclude that $a_i \leqslant i$. Since the total number of blocks is $s + \sigma(i)$ we also obtain that $a_i \leqslant \sigma(i)$. Finally, observe that the number of blocks that are contained in the set Y_i is $(s+\sigma(i)) - (a_i + s) = \sigma(i) - a_i$. This number must be less than or equal to the cardinality of the set Y_i, which is $\sigma(i)$. Thus we conclude that $a_i \geqslant 0$.

Let T_1 consist of all tuples $(\sigma, \pi_0, \ldots, \pi_n)$ in T such that the a_i's are distinct. Let us now consider those tuples that are in T_1. Observe that the inequalities $a_i \leqslant i$ and $a_i \leqslant \sigma(i)$ imply that $a_i = i = \sigma(i)$ for all indices i. Hence the partition π_i consists of $s+i$ blocks with each block containing one element from the set $\{1, \ldots, s+i\}$. Observe that such a partition is represented by the juggling cards $C_0, C_1, \ldots, C_{s+i-1}, C_{\alpha_1}, \ldots, C_{\alpha_i}$, where $0 \leqslant \alpha_1, \ldots, \alpha_i \leqslant s+i-1$. Thus summing q to the power of the crossing statistic $\operatorname{cross}(\pi_i)$ over all such possible partitions π_i, we have

$$\sum_{\pi_i} q^{\operatorname{cross}(\pi_i)} = q^{\binom{s+i}{2}} \cdot [s+i]^i.$$

Hence we have

$$\sum_{(\sigma,\pi_0,\ldots,\pi_n)\in T_1} (-1)^{\sigma} \cdot q^{\operatorname{cross}(\pi_0)+\cdots+\operatorname{cross}(\pi_n)} = \prod_{i=0}^{n} q^{\binom{s+i}{2}} \cdot [s+i]^i. \tag{3.1}$$

Let T_2 be the complement of T_1, that is, $T_2 = T - T_1$. We define a sign-reversing involution on T_2 as follows. For $(\sigma, \pi_0, \ldots, \pi_n)$ in T_2 we know that there exists a pair of indices (j,k) such that $a_j = a_k$. Let (j,k) be the least such pair in the lexicographic order. Let σ' be the permutation $\sigma'(j) = \sigma(k)$, $\sigma'(k) = \sigma(j)$ and $\sigma'(i) = \sigma(i)$ for $i \neq j,k$. Clearly, $(-1)^{\sigma'} = -(-1)^{\sigma}$. Moreover, let $\pi_i' = \pi_i$ for $i \neq j,k$.

Assume that π_j is constructed by the juggling cards $D(1), \ldots, D(s+j), D(s+j+1), \ldots, D(s+j+\sigma(j))$ and π_k is constructed by the cards $E(1), \ldots, E(s+k), E(s+k+1), \ldots, E(s+k+\sigma(k))$. We now define two new partitions π_j' and π_k'. Let π_j' be constructed by the juggling cards $D(1), \ldots, D(s+j), E(s+k+1), \ldots, E(s+j+\sigma(k))$ and π_k' constructed by the cards $E(1), \ldots, E(s+k), D(s+j+1), \ldots, D(s+j+\sigma(j))$.

Notice that we need to add $\sigma(k) - \sigma(j)$ paths at the top of each of the cards $D(1), \ldots, D(s+j)$ and similarly, remove $\sigma(k) - \sigma(j)$ paths from the top of the cards $E(1), \ldots, E(s+k)$ in order that each card has the same number paths as blocks in the partition.

The map $(\sigma, \pi_0, \ldots, \pi_n) \mapsto (\sigma', \pi'_0, \ldots, \pi'_n)$ on T_2 defines a sign-reversing involution. Moreover, the quantity $\text{cross}(\pi_0) + \cdots + \text{cross}(\pi_n)$ is invariant under the involution. Thus the determinant is given by the product in Eq. (3.1). □

We now present a second proof of Theorem 3.1. It requires some more notation, but as byproduct we obtain identities for q-Stirling numbers. For non-negative integers n, k and h, let $F^n(k, h)$ be the collection of all sequences $(c_1, \ldots, c_n) \in \{0, \ldots, k-1\}^n$ such that in the juggling pattern $(C_{c_1}, \ldots, C_{c_n})$ each of the h highest paths at time 0, that is, the paths labeled $k-h+1, k-h+2, \ldots, k$, touch down at one of nodes in $\{1, 2, \ldots, n\}$. Let $f^n[k, h]$ denote the q-analogue of the cardinality of the set $F^n(k, h)$, that is,

$$f^n[k,h] = \sum_{(c_1,\ldots,c_n) \in F^n(k,h)} q^{c_1+\cdots+c_n}.$$

Lemma 3.2. *The polynomial $f^n[k, h]$ is given by*

$$f^n[k,h] = \sum_{j=0}^{h} (-1)^j \cdot q^{\binom{j}{2}} \cdot \begin{bmatrix} h \\ j \end{bmatrix} \cdot [k-j]^n.$$

Proof. The expression $[k]^n$ q-enumerates all sequences of n juggling cards with each card having k paths. We will enumerate this set in a second way to obtain a different expression from which the lemma will follow.

Observe that $f^n[k-j, h-j]$ enumerates the collection of patterns where the j highest paths do not touch down, but the $h-j$ next highest are forced to touch down. We generalize this observation as follows. Let $S = \{i_1 < i_2 < \cdots < i_j\}$ be a subset of $\{1, \ldots, h\}$. The collection of patterns where the $i_1, i_2, \ldots, i_j$ highest paths do not touch down but the paths in $\{1, \ldots, h\} - \{i_1, i_2, \ldots, i_j\}$ do touch down is counted by

$$q^{i_1+i_2+\cdots+i_j-1-2-\cdots-j} \cdot f^n[k-j, h-j] = q^{\rho(S)} \cdot f^n[k-j, h-j].$$

Summing over all subsets S of cardinality j, we have

$$[k]^n = \sum_{j=0}^{h} \begin{bmatrix} h \\ j \end{bmatrix} \cdot f^n[k-j, h-j].$$

Applying the q-inversion formula, see [10, Eq. (5)], the lemma follows. □

Theorem 3.3. *The q-Stirling number $S[m+n, k]$ can be expressed by*

$$S[m+n,k] = \sum_{i} S[m,i] \cdot \frac{f^n[k, k-i]}{[k-i]!},$$

where i ranges between $\max(0, k-n)$ *and* $\min(m, k)$.

Proof. Consider a partition π of $\{1, \ldots, m+n\}$ into k blocks. Let $c_1, \ldots, c_{m+n}$ be the corresponding sequence. When restricting this partition to the set $\{1, \ldots, m\}$, that is, to consider the sequence $c_1, \ldots, c_m$, we obtain a partition into i blocks. The remaining part of the sequence $c_{m+1}, \ldots, c_{m+n}$ corresponds to a pattern where the $k-i$ highest paths touch down. However, these $k-i$ paths touch down in order of height. Thus we need to divide the term $f^n[k, k-i]$ with the q-factorial $[k-i]!$ to take the order of the $k-i$ paths in account.

Finally observe that we need $0 \leqslant i \leqslant m$, $i \leqslant k$, and $k-i \leqslant n$ to make the terms in the sum non-zero. □

Second proof of Theorem 3.1. By Theorem 3.3 we have with $m = s+i$, $n = j$, and $k = s+j$,

$$S[s+i+j, s+j] = \sum_{\alpha=s}^{s+\min(i,j)} S[s+i, \alpha] \cdot \frac{f^j[s+j, s+j-\alpha]}{[s+j-\alpha]!}$$
$$= \sum_{\beta=0}^{\min(i,j)} S[s+i, s+\beta] \cdot \frac{f^j[s+j, j-\beta]}{[j-\beta]!}.$$

This shows that the matrix $\mathbf{M} = (S[s+i+j, s+j])_{0 \leqslant i,j \leqslant n}$ factors into a lower triangular matrix $\mathbf{L} = (S[s+i, s+\beta])_{0 \leqslant i,\beta \leqslant n}$ and an upper triangular matrix $\mathbf{U} = (f^j[s+j, j-\beta]/[j-\beta]!)_{0 \leqslant \beta,j \leqslant n}$. Hence the determinant of $\mathbf{M}$ is the product of the elements on the diagonals of $\mathbf{L}$ and $\mathbf{U}$. Hence

$$\det(\mathbf{M}) = \prod_{i=0}^{n} S[s+i, s+i] \cdot f^i[s+i, 0] = \prod_{i=0}^{n} q^{\binom{s+i}{2}} \cdot [s+i]^i. \qquad \square$$

4. The Hankel determinant for q-exponential polynomials

The exponential polynomials $e_n(z)$ are defined by $e_n(z) = \sum_{k=0}^{n} S(n,k) \cdot z^k = \sum_{\pi} z^{|\pi|}$ where π ranges over all partitions of an n-element set. Observe that $e_n(1)$ is the nth Bell number. The Hankel determinant of the Bell numbers and more generally, the exponential polynomials have been considered in several articles [1,7,14–18]. Cigler [5] obtained the q-analogue of this Hankel determinant, namely a similar factorization for the Hankel determinant of the q-exponential polynomials. We present two proofs of his identity. The first proof is an extension of the bijective proof appearing in [7].

Define the q-analogue of $e_n(z)$, the *q-exponential polynomials*, by

$$\tilde{e}_n[z] = \sum_{k=0}^{n} \widetilde{S}[n,k] \cdot z^k = \sum_{\pi} q^{\widetilde{\mathrm{cross}}(\pi)} \cdot z^{|\pi|}.$$

Theorem 4.1 (Cigler). *The Hankel determinant of the q-exponential polynomials is*

$$\det\big(\tilde{e}_{i+j}[z]\big)_{0\leqslant i,j\leqslant n} = q^{\binom{n+1}{3}} \cdot [0]! \cdot [1]! \cdots [n]! \cdot z^{\binom{n+1}{2}}.$$

Proof. Let T denote the set of all $(n+2)$-tuples $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$ where σ is a permutation on the set $\{0, 1, \ldots, n\}$ and π_i is a partition of the set $\{1, \ldots, i+\sigma(i)\}$. Expanding the determinant we have

$$\det\big(\tilde{e}_{i+j}[z]\big)_{0\leqslant i,j\leqslant n} = \sum_{(\sigma,\pi_0,\ldots,\pi_n)\in T} (-1)^{\sigma} \cdot q^{\widetilde{\text{cross}}(\pi_0)+\cdots+\widetilde{\text{cross}}(\pi_n)} \cdot z^{|\pi_0|+\cdots+|\pi_n|}.$$

For $(\sigma, \pi_0, \ldots, \pi_n)$ in T define a_i to be the number of blocks in π_i that intersect non-trivially both $\{1, \ldots, i\}$ and $\{i+1, \ldots, i+\sigma(i)\}$. It is clear from both intersections that $a_i \leqslant i$ and $a_i \leqslant \sigma(i)$.

Let T_1 consist of all tuples $(\sigma, \pi_0, \ldots, \pi_n)$ in T such that the a_i's are distinct. Let us now consider thoses tuples that are in T_1. Observe that the inequalities $a_i \leqslant i$ and $a_i \leqslant \sigma(i)$ imply that $a_i = i = \sigma(i)$ for all indices i. Hence the partition π_i consists of i blocks with each block containing one element from $\{1, \ldots, i\}$ and one from $\{i+1, \ldots, 2\cdot i\}$. There are $i!$ such partitions. They are described by the juggling cards $C_0, \ldots, C_{i-1}, C_{\alpha_0}, \ldots, C_{\alpha_{i-1}}$, where $i \leqslant \alpha_i \leqslant n-1$. Thus summing q to the power of the crossing statistic $\widetilde{\text{cross}}(\pi_i)$ over all such possible partitions π_i, we have $\sum_{\pi_i} q^{\widetilde{\text{cross}}(\pi_i)} \cdot z^{|\pi_i|} = q^{\binom{i}{2}} \cdot [i]! \cdot z^i$. Thus we conclude that

$$\sum_{(\sigma,\pi_0,\ldots,\pi_n)\in T_1} (-1)^{\sigma} \cdot q^{\widetilde{\text{cross}}(\pi_0)+\cdots+\widetilde{\text{cross}}(\pi_n)} \cdot z^{|\pi_0|+\cdots+|\pi_n|} = q^{\binom{n+1}{3}} \cdot [0]! \cdot [1]! \cdots [n]! \cdot z^{\binom{n+1}{2}}.$$

Now we define a sign-reversing involution on the set T_2. For $(\sigma, \pi_0, \ldots, \pi_n)$ in T_2 let (j,k) be the least such pair in the lexicographic order such that $a_j = a_k$. Define σ' and π_i' for $i \neq j,k$ as in the first proof of Theorem 3.1. We need to define the partitions π_j' and π_k'.

Let a denote $a_j = a_k$. For $i = j,k$, let X_i be the set $\{1, \ldots, i\}$ and Y_i be the set $\{i+1, \ldots, i+\sigma(i)\}$. Let κ_i denote the partition π_i restricted to the set X_i and λ_i denote the partition restricted to Y_i. In each of these two partitions mark the a blocks that are restrictions of the blocks having elements in both X_i and Y_i.

Define π_j' to be the join of the partitions κ_j and λ_k on the set $X_j \cup Y_k$ where we join the a marked blocks of κ_j with the a marked blocks of λ_k. Merge the a blocks in the order described by the partition π_j. Define π_k' similarly. It is clear that $|\pi_j| + |\pi_k| = |\pi_j'| + |\pi_k'|$. It remains to show that

$$\widetilde{\text{cross}}(\pi_j) + \widetilde{\text{cross}}(\pi_k) = \widetilde{\text{cross}}\big(\pi_j'\big) + \widetilde{\text{cross}}\big(\pi_k'\big). \tag{4.1}$$

We prove this identity by carefully analyzing the types of crossings in the partition π_i, where $i = j,k$. Let x_i be the number blocks of κ_i that are not marked and similarly define y_i. Let α_i be the number of crossings occurring between the a paths leaving the

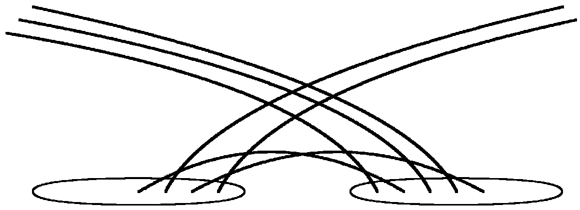

Fig. 4. Sketch of the partition π_j in the proof of Theorem 4.1. Observe that $a = 2$, $x_j = 2$, $y_j = 3$. The crossing displayed are counted by $\alpha_j = 1$, $\beta_j = 2$, $\gamma_j = 4$, and $x_j \cdot y_j = 6$.

set X_i and arriving at the set Y_i. Let β_i be the number of crossings occurring between the x_i paths leaving the set X_i going upwards and the a paths continuing to the set Y_i. Symmetrically, let γ_i be the number crossings occurring between the y_i incoming paths and the a continuing paths. We have now taken into account all the crossings of π_i, that is, $\mathrm{cross}(\pi_i) = \mathrm{cross}(\kappa_i) + \mathrm{cross}(\lambda_i) + x_i \cdot y_i + \alpha_i + \beta_i + \gamma_i$. See Fig. 4 for an example. Since $\binom{a+x_i}{2} + \binom{a+y_i}{2} + x_i \cdot y_i - \binom{a+x_i+y_i}{2} = \binom{a}{2}$, the modified crossing statistic satisfies

$$\widetilde{\mathrm{cross}}(\pi_i) = \widetilde{\mathrm{cross}}(\kappa_i) + \widetilde{\mathrm{cross}}(\lambda_i) + \binom{a}{2} + \alpha_i + \beta_i + \gamma_i. \tag{4.2}$$

Now by the definition of π'_j we have that

$$\widetilde{\mathrm{cross}}(\pi'_j) = \widetilde{\mathrm{cross}}(\kappa_j) + \widetilde{\mathrm{cross}}(\lambda_k) + \binom{a}{2} + \alpha_j + \beta_j + \gamma_k.$$

By adding this equation to the symmetric one for π'_k Eq. (4.1) follows. Hence we obtain a sign-reversing involution that keeps the necessary statistics invariant, thus proving the expansion. □

The next proof is similar to Cigler's proof, namely the objective is to obtain an *LDU*-decomposition of the matrix. However, we are able to obtain this factorization in a purely combinatorial manner. To simplify the notation let us introduce the linear operator D_q by

$$D_q\big(f(z)\big) = \frac{f(z) - f(q \cdot z)}{(1-q) \cdot z}. \tag{4.3}$$

This is the q-analogue of the derivative. For our purposes it is enough to observe that $D_q(z^n) = [n] \cdot z^{n-1}$.

Second proof of Theorem 4.1. Let X be the set $\{1, \ldots, i\}$ and Y the set $\{i+1, \ldots, i+j\}$. We determine the number of ways to choose a partition on $X \cup Y$. First choose a non-negative integer a. Then choose a partition κ on X with $a + x$ blocks, and a partition λ on Y with $a + y$ blocks. Select a blocks of κ and a blocks of λ. This can be done in $\binom{a+x}{a} \cdot \binom{a+y}{a}$ ways. There are $a!$ ways to match these selected blocks. We then obtain a partition π on $X \cup Y$ with $a + x + y$ blocks.

The crossing statistic of the partition π is described by Eq. (4.2) except without any subscripts. However, notice that when summing over all the ways to obtain the partition π from κ and λ, the α-crossings will be counted by $[a]!$. Similarly, the β-crossings will be counted by $\left[{a+x \atop a}\right]$ and the γ-crossings will be counted by $\left[{a+y \atop a}\right]$. Thus we have

$$\begin{aligned}
\tilde{e}_{i+j}[z] &= \sum_{a\geqslant 0}\sum_{x\geqslant 0}\sum_{y\geqslant 0} \widetilde{S}[i,a+x]\cdot \widetilde{S}[j,a+y]\cdot q^{\binom{a}{2}}\cdot [a]!\cdot \begin{bmatrix} a+x \\ a\end{bmatrix}\cdot \begin{bmatrix} a+y \\ a\end{bmatrix}\cdot z^{a+x+y} \\
&= \sum_{a\geqslant 0}\left(\sum_{x\geqslant 0}\widetilde{S}[i,a+x]\cdot \frac{[a+x]!}{[x]!}\cdot z^x\right)\cdot \frac{q^{\binom{a}{2}}\cdot z^a}{[a]!} \\
&\quad\times \left(\sum_{y\geqslant 0}\widetilde{S}[j,a+y]\cdot \frac{[a+y]!}{[y]!}\cdot z^y\right) \\
&= \sum_{a\geqslant 0} D_q^a\big(\tilde{e}_i[z]\big)\cdot \frac{q^{\binom{a}{2}}\cdot z^a}{[a]!}\cdot D_q^a\big(\tilde{e}_j[z]\big).
\end{aligned}$$

Hence the Hankel matrix $(\tilde{e}_{i+j}[z])_{0\leqslant i,j\leqslant n}$ factors into a lower triangular matrix $\mathbf{L} = (D_q^a(\tilde{e}_i[z]))_{0\leqslant i,a\leqslant n}$, a diagonal matrix $\mathbf{D}$ having $q^{\binom{a}{2}}\cdot z^a/[a]!$ as its (a,a) entry and an upper triangular matrix $\mathbf{U} = \mathbf{L}^*$. Thus the determinant of the Hankel matrix is the product of the diagonal elements of these three matrices, that is,

$$\prod_{i=0}^{n} D_q^i\big(\tilde{e}_i[z]\big)^2\cdot \frac{q^{\binom{i}{2}}\cdot z^i}{[i]!} = \prod_{i=0}^{n}[i]!\cdot q^{\binom{i}{2}}\cdot z^i. \qquad \square$$

5. An extension of a theorem of Sylvester

On the space of infinitely differentiable functions of two variables x and y, define the operator T_n by

$$T_n(f) = \det\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{0\leqslant i,j\leqslant n}.$$

The operator T_n satisfies the following identity.

Theorem 5.1. *The operators T_n satisfy the functional equation*

$$T_1\big(T_n(f)\big) = T_{n-1}(f)\cdot T_{n+1}(f).$$

Proof. Let M denote the $(n+2)\times(n+2)$-matrix $(\partial^{i+j} f/\partial x^i \partial y^j)_{0\leqslant i,j\leqslant n+1}$. For S and T subsets of $\{0,1,\ldots,n+1\}$ having the same cardinality let $m_{S,T}$ denote the minor with the rows indexed by the set $n+1-S = \{n+1-s\colon s \in S\}$ removed and the columns indexed by $n+1-T$ removed. Applying the Desnanot–Jacobi adjoint matrix theorem, we have

$$m_{\{0\},\{0\}} \cdot m_{\{1\},\{1\}} - m_{\{0\},\{1\}} \cdot m_{\{1\},\{0\}} = m_{\{0,1\},\{0,1\}} \cdot m_{\emptyset,\emptyset}.$$

It is now straightforward to verify that this identity is the desired result. □

Corollary 5.2 (Sylvester). *Define the operator S_n by $S_n(g) = \det(\partial^{i+j}/\partial x^{i+j} g)_{0 \leqslant i,j \leqslant n}$. Then the operators S_n satisfy the functional equation*

$$S_1\bigl(S_n(g)\bigr) = S_{n-1}(g) \cdot S_{n+1}(g).$$

Proof. Apply Theorem 5.1 to the function $f(x,y) = g(x+y)$ and then set $y = 0$. □

This result was used by Radoux in one of his proofs of the Hankel determinant of the exponential polynomials [16]. Namely, by induction and Corollary 5.2 compute $S_n(g)$, where

$$g(x) = \exp\bigl(z \cdot (\mathrm{e}^x - 1)\bigr) = \sum_{n \geqslant 0} e_n(z) \cdot x^n/n!$$

and then set $x = 0$.

As an application of Theorem 5.1, we evaluate the following determinant.

Theorem 5.3.

$$\det\left(\frac{S(s+i+j, s+j)}{(s+i+j)!}\right)_{0 \leqslant i,j \leqslant n} = \frac{2^{s \cdot (n+1)}}{(2s)!! \cdot (2s+2)!! \cdots (2s+2n)!!},$$

where $k!!$ denotes the double factorial $k \cdot (k-2) \cdots 2$.

Proof. In the expression $\exp(y \cdot (\mathrm{e}^x - 1)) = \sum_{0 \leqslant j \leqslant k} S(k,j) \cdot x^k/k! \cdot y^j$ substitute y/x for y to obtain

$$\begin{aligned}\exp\bigl(y \cdot (\mathrm{e}^x - 1)/x\bigr) &= \sum_{0 \leqslant j \leqslant k} S(k,j)/k! \cdot x^{k-j} \cdot y^j \\ &= \sum_{0 \leqslant i,j} S(i+j,j)/(i+j)! \cdot x^i \cdot y^j.\end{aligned}$$

By Theorem 5.1 and by induction on n it is straightforward to show that

$$T_n\left(\frac{\partial^s}{\partial y^s} f\right) = 0! \cdot 1! \cdots n! \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{e}^x - 1}{x}\right)^{\binom{n+1}{2}} \cdot \left(\frac{\partial^s}{\partial y^s} f\right)^{n+1}, \tag{5.1}$$

where $f(x, y) = \exp(y \cdot (\mathrm{e}^x - 1)/x)$. Now set $x = y = 0$ in Eq. (5.1). We obtain that

$$\det\bigl(S(s+i+j, s+j)/(s+i+j)! \cdot i! \cdot (s+j)!\bigr)_{0 \leqslant i,j \leqslant n} = 0! \cdot 1! \cdots n! \cdot (1/2)^{\binom{n+1}{2}}.$$

Divide each side by $0! \cdot 1! \cdots n! \cdot s! \cdot (s+1)! \cdots (s+n)!$ and the result follows. □

6. Concluding remarks

Cigler also obtained expressions for the two shifted Hankel determinants:

$$\det\bigl(\tilde{e}_{i+j+1}[z]\bigr)_{0 \leqslant i,j \leqslant n} = q^{\binom{n+2}{2}} \cdot [0]! \cdot [1]! \cdots [n]! \cdot z^{\binom{n+2}{2}},$$

$$\det\bigl(\tilde{e}_{i+j+2}[z]\bigr)_{0 \leqslant i,j \leqslant n} = q^{\binom{n+2}{3}} \cdot [0]! \cdot [1]! \cdots [n]! \cdot z^{\binom{n+2}{2}} \cdot \left(\sum_{k=0}^{n+1} q^{\binom{k}{2}} \cdot z^k \cdot \frac{[n+1]!}{[k]!}\right);$$

see [5, Satz 1]. Can bijective proofs be found for these identities? Moreover, considering the other q-analogue of the exponential polynomials, namely

$$e_n[z] = \sum_{k=0}^{n} S[n,k] \cdot z^k = \sum_{\pi} q^{\mathrm{cross}(\pi)} \cdot z^{|\pi|},$$

he also has expressions for the Hankel determinant and the two shifted Hankel determinants of $e_n[z]$; see [5, Satz 2]. Again it is natural to ask for bijective proofs. However, this might be more challenging since in these cases the determinant is equal to a product whose factors contain terms with negative signs.

One generalization of the q-Stirling numbers is the p, q-Stirling numbers [6,20]. Can any of the results appearing in this paper be extended to them?

We ask if there is a q-analogue of Theorem 5.3. More interestingly, is there a natural q-analogue of the two variable Sylvester's Theorem 5.1. One suggestion is to use the q-analogue of the derivative given in Eq. (4.3).

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References

[1] M. Aigner, A characterization of the Bell numbers, Discrete Math. 205 (1999) 207–210.

[2] J. Buhler, D. Eisenbud, R. Graham, C. Wright, Juggling drops and descents, Amer. Math. Monthly 101 (1994) 507–519.

[3] J. Buhler, R. Graham, A note on the binomial drop polynomial of a poset, J. Combin. Theory Ser. A 66 (1994) 321–326.
[4] L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987–1000.
[5] J. Cigler, Eine Charakterisierung der q-Exponentialpolynome, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 208 (1999) 143–157.
[6] A. de Médicis, P. Leroux, A unified combinatorial approach for q- (and p,q-) Stirling numbers, J. Statist. Plann. Inference 34 (1993) 89–105.
[7] R. Ehrenborg, The Hankel determinant of exponential polynomials, Amer. Math. Monthly 107 (2000) 557–560.
[8] R. Ehrenborg, M. Readdy, Juggling and applications to q-analogues, Discrete Math. 157 (1996) 107–125.
[9] A.M. Garsia, J.B. Remmel, q-Counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41 (1986) 246–275.
[10] J. Goldman, G.-C. Rota, On the foundations of combinatorial theory. IV. Finite vector spaces and Eulerian generating functions, Stud. Appl. Math. 49 (1970) 239–258.
[11] G. Labelle, P. Leroux, E. Pergola, R. Pinzani, Stirling numbers interpolation using permutations with forbidden subsequences, Discrete Math. 246 (2002) 177–195.
[12] S.C. Milne, A q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials, Trans. Amer. Math. Soc. 245 (1978) 89–118.
[13] S.C. Milne, Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers, Adv. Math. 43 (1982) 173–196.
[14] C. Radoux, Calcul effectif de certains déterminants de Hankel, Bull. Soc. Math. Belg. Sér. B 31 (1979) 49–55.
[15] C. Radoux, Déterminant de Hankel construit sur des polynomes liés aux nombres de dérangements, European J. Combin. 12 (1991) 327–329.
[16] C. Radoux, Déterminants de Hankel et théorème de Sylvester, in: Séminaire Lotharingien de Combinatoire, Saint-Nabor, 1992, in: Publ. Inst. Rech. Math. Av., Vol. 498, Univ. Louis Pasteur, Strasbourg, 1992, pp. 115–122.
[17] C. Radoux, Addition formulas for polynomials built on classical combinatorial sequences, J. Comput. Appl. Math. 115 (2000) 471–477.
[18] C. Radoux, The Hankel determinant of exponential polynomials: a very short proof and a new result concerning Euler numbers, Amer. Math. Monthly 109 (2002) 277–278.
[19] B.E. Sagan, A maj statistic for partitions, European J. Combin. 12 (1991) 69–79.
[20] M. Wachs, D. White, p,q-Stirling numbers and set partition statistics, J. Combin. Theory Ser. A 56 (1991) 27–46.