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ON A SUBCLASS OF BAZILEVIČ FUNCTIONS

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ABSTRACT. The authors show that certain Bazilevič functions are spiral-like. Then the authors study the growth and Hardy classes of those special functions.

Introduction. I. E. Bazilevič [2] gave an explicit construction for a class of functions analytic and univalent in the unit disc D (see also [10]). His result was as follows.

Theorem 1. *Let g be univalent starlike in D with $g(0)=0$, and let h be analytic and satisfy $\mathrm{Re}(e^{i\lambda}h(z))>0$ in D for some real λ. Then if $\alpha>0$ and β is real, the function*

$$f(z)=\left\{\int_0^z g^{\alpha}(\zeta)h(\zeta)\zeta^{i\beta-1}\,d\zeta\right\}^{1/(\alpha+i\beta)} \tag{1}$$

is analytic and univalent in D.

In this paper we consider the functions f that arise from (1) when $h(z)\equiv 1$. Such a function f must satisfy

$$\mathrm{Re}[1+zf''(z)/f'(z)+(\alpha-1+i\beta)zf'(z)/f(z)]>0, \qquad z\in D. \tag{2}$$

Conversely, if f is analytic in D, with $f(0)=0$, $f(z)f'(z)/z\neq 0$ $(z\in D)$, and if f satisfies (2) for some $\alpha>0$, β real, then f can be written in the form (1), with $h(z)\equiv 1$. Let us denote the class of such functions f by $B(\alpha+i\beta)$. The class obtained when $\beta=0$ has been studied extensively [5], [6], [7], [8], [9]. The class $B(1+i\beta)$ has recently been considered by H. Yoshikawa [12]; he showed that if $f\in B(1+i\beta)$ then f is γ-spiral-like, where $\gamma=$ arc tan β. We generalize this to

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Theorem 2. *If* $f \in B(\alpha + i\beta)$ *then* f *is* γ*-spiral-like, where* γ *satisfies* $\alpha + i\beta = (\alpha^2 + \beta^2)^{1/2} e^{i\gamma}$, $-\pi/2 < \gamma < \pi/2$.

Proof. Define a function w analytic in D by

$$e^{i\gamma} \frac{zf'(z)}{f(z)} = \cos \gamma \left(\frac{1 + w(z)}{1 - w(z)} \right) + i \sin \gamma, \qquad z \in D. \tag{3}$$

One easily checks that $w(0) = 0$, $w(z) \neq \pm 1$ $(z \in D)$. It suffices to show $|w(z)| < 1$ for $z \in D$. Let $w(z) = R(z) e^{i\phi(z)}$ for $z = re^{i\theta}$ and suppose that $z_0 = r_0 e^{i\theta_0}$ is a point of D such that

$$\max_{|z| \le r_0} |w(z)| = |w(z_0)| = 1.$$

Then $(\partial R/\partial\theta)|_{z=z_0} = 0$, and since

$$\frac{zw'(z)}{w(z)} = \frac{\partial\phi}{\partial\theta} - i \frac{1}{R} \frac{\partial R}{\partial\theta}$$

we have at the point z_0,

$$z_0 w'(z_0)/w(z_0) = (\partial\phi/\partial\theta)|_{z=z_0}.$$

We shall now show that

$$\mathrm{Re}\,[1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta) zf'(z)/f(z)]_{z=z_0} = 0 \tag{4}$$

thus contradicting the assertion $f \in B(\alpha + i\beta)$. By (3), (4) can be written as

$$\mathrm{Re}\left[\frac{zP'(z)}{P(z) + i \tan \gamma} + (\alpha^2 + \beta^2)^{1/2} (\cos \gamma P(z) + i \sin \gamma) \right]_{z=z_0} = 0 \tag{5}$$

where $P(z) = (1 + w(z))/(1 - w(z))$. Since $|w(z_0)| = 1$ and since $[z_0 w'(z_0)/w(z_0)]$ is real, it follows that $P(z_0)$ is imaginary and $z_0 P'(z_0)$ is real. Hence (5) holds at z_0. This completes the proof.

Theorem 3. *If* $\alpha' + i\beta' = t(\alpha + i\beta)$, $t \ge 1$, *then* $B(\alpha + i\beta) \subset B(\alpha' + i\beta')$.

Proof. Since f is γ-spiral-like $(\alpha + i\beta = (\alpha^2 + \beta^2)^{1/2} e^{i\gamma})$,

$$\mathrm{Re}\,[(t-1)(\alpha^2 + \beta^2)^{1/2} e^{i\gamma} zf'(z)/f(z)] \ge 0, \qquad z \in D.$$

Then

$$\begin{aligned} &\mathrm{Re}\,[1 + zf''(z)/f'(z) + (\alpha' - 1 + i\beta') zf'(z)/f(z)] \\ &\qquad = \mathrm{Re}\,[1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta) zf'(z)/f(z)] \\ &\qquad\quad + \mathrm{Re}\,[(t-1)(\alpha^2 + \beta^2)^{1/2} e^{i\gamma} zf'(z)/f(z)] \ge 0, \qquad z \in D. \end{aligned}$$

In the integral representation for functions in $B(\alpha + i\beta)$, namely,

$$f(z) = \left\{\int_0^z g^{\alpha}(\zeta)\zeta^{i\beta-1}\, d\zeta\right\}^{1/(\alpha+i\beta)}, \tag{6}$$

let us denote by $f_{\alpha+i\beta}$ the function obtained by letting g be the Koebe function $z/(1-z)^2$. The following theorem illustrates the dependence of the growth of f on the parameters α and β.

Theorem 4. *Suppose $f \in B(\alpha + i\beta)$.*

(A) *If $0 < \alpha \leq \frac{1}{2}$, then, unless f is a rotation or magnification of $f_{1/2+i\beta}$, f is bounded.*

(B) *If $\alpha > \frac{1}{2}$ and f is not a rotation or magnification of $f_{\alpha+i\beta}$, then there exists $\epsilon = \epsilon(f) > 0$ such that $f \in H^{\lambda+\epsilon}$ and $f' \in H^{(\lambda/(1+\lambda))+\epsilon}$, where $\lambda = (\alpha^2 + \beta^2)/\alpha(2\alpha - 1)$.*

(C) *For $\alpha > \frac{1}{2}$ the function $f_{\alpha+i\beta}$ belongs to H^p, $\forall p < \lambda$, but does not belong to H^{λ}.*

Proof. Following Sheil-Small's construction [11] of f in "analytic stages" from the representation (6), we set

$$F(z) = \left(\frac{g(z)}{z}\right)^{\alpha} = \sum_{n=0}^{\infty} C_n z^n,$$

for a suitable branch of the nonvanishing function $(g(z)/z)^{\alpha}$. Let

$$G(z) = \sum_{n=0}^{\infty} \frac{C_n}{n + \alpha + i\beta} z^n, \tag{7}$$

so that G is analytic in D and satisfies the differential equation

$$(\alpha + i\beta)G(z) + zG'(z) = F(z). \tag{8}$$

Sheil-Small [11] shows that $G(z) \neq 0$ in D. We now define f by the formula

$$f(z) = z[G(z)]^{1/(\alpha+i\beta)}. \tag{9}$$

One can easily verify that apart from a constant factor, this defines an analytic branch of the formula (6). By (9) we may write

$$G(z) = [(f(z)/z)^{1+i\beta/\alpha}]^{\alpha} = [s(z)/z]^{\alpha}, \tag{10}$$

where (10) is the defining equation for s. Since f is γ-spiral-like (Theorem 2), it follows easily that s is starlike in D. From (8) we have

$$zG'(z) = (g(z)/z)^{\alpha} - (\alpha + i\beta)(s(z)/z)^{\alpha}. \tag{11}$$

If g is not a rotation of the Koebe function, then there exists $\epsilon = \epsilon(g) > 0$ such that $g \in H^{1/2+\epsilon}$ [4]. Furthermore, it is easy to see from (7) that s cannot be a rotation of the Koebe function. Thus, $G' \in H^{1/(2\alpha)+\epsilon}$, ϵ denoting a positive number, not necessarily the same in each instance. Hence, if $0 < \alpha \le \frac{1}{2}$, G is bounded and so f is bounded, by (9). Whence, (A) is proved. For $\alpha > \frac{1}{2}$, a Hardy-Littlewood theorem [3, p. 88] shows that $G \in H^{1/(2\alpha-1)+\epsilon}$; hence, from (10), $s \in H^{\alpha/(2\alpha-1)+\epsilon}$. From the identity

$$(f(z)/z)^{1+i\beta/\alpha} = s(z)/z, \tag{12}$$

we have

$$\left|\frac{f(z)}{z}\right| = \left|\frac{s(z)}{z}\right|^{\alpha^2(\alpha^2+\beta^2)^{-1}} \cdot \exp\left[\frac{\alpha\beta}{\alpha^2+\beta^2}\arg\left(\frac{s(z)}{z}\right)\right]. \tag{13}$$

Since $s \in H^{\alpha/(2\alpha-1)+\epsilon}$ and the exponential factor is bounded, it follows that

$$f \in H^{\lambda+\epsilon}, \qquad \lambda = \frac{\alpha^2+\beta^2}{\alpha(2\alpha-1)}. \tag{14}$$

To complete the proof of (B), we must show the existence of an $\epsilon > 0$ such that $f' \in H^{\lambda(1+\lambda)^{-1}+\epsilon}$. By Theorem 2, there exists h, $\mathrm{Re}(h(z)) > 0$ in D, such that

$$e^{i\gamma} z f'(z) = f(z)h(z). \tag{15}$$

Fix ϵ in (14), and for small positive δ, let

$$k = k(\delta) = (\lambda + \epsilon)(1 + \lambda + \epsilon + \delta\lambda + \delta\epsilon)^{-1}. \tag{16}$$

Choosing $p = (\lambda + \epsilon)k^{-1}$, $q = (1 + \delta)^{-1}k^{-1}$, and applying Hölder's inequality to (15) with conjugate indices p and q, it follows that

$$\int_{-\pi}^{\pi} |f'(z)|^k \, d\theta \le \left(\int_{-\pi}^{\pi} \left|\frac{f(z)}{z}\right|^{kp} d\theta\right)^{1/p} \left(\int_{-\pi}^{\pi} |h(z)|^{kq}\, d\theta\right)^{1/q}.$$

By (14) and the fact that $kq < 1$, we have that $\int_{-\pi}^{\pi}|f'(z)|^k d\theta$ remains bounded as $r \to 1$. Hence, the proof of (B) is complete if we show the existence of $\delta > 0$ such that $k = k(\delta) > \lambda(1 + \lambda)^{-1}$. But this is easily checked by consideration of (16).

Finally, we leave the verification of (C) to the reader.

Remark 1. If we take $\beta = 0$ in Theorem 4, the result is the same as that obtained in [5].

Remark 2. Note that if we divide (2) by $(\alpha^2 + \beta^2)^{1/2}$ and let $\alpha + i\beta \to \infty$ along the ray $te^{i\gamma}$ the classes $B(\alpha + i\beta)$ "tend" to the full class of γ-spiral-like functions. If we also let $\alpha + i\beta \to \infty$ in Theorem 4, we find that $\lambda = \lambda(\alpha, \beta) \to (2\cos^2\gamma)^{-1}$; and this is precisely the Hardy class result previously known for γ-spiral-like functions [1].

Remark 3. It is interesting to observe that the level curves of $\lambda = (\alpha^2 + \beta^2)[\alpha(2\alpha - 1)]^{-1}$ are the right branches of certain hyperbolas that are symmetric with respect to the real line; and they converge on the vertical line $\alpha = \frac{1}{2}$ (as $\lambda \to \infty$). For instance, if $\lambda = 1$ we obtain the right branch of the hyperbola $4(\alpha - \frac{1}{2})^2 - 4\beta^2 = 1$.

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