METRIC SPACES and COMPLEX ANALYSIS – SHEET 2

Closure. Contraction Mapping Theorem. Completeness. Connectedness.

1. Let M be a metric space and $A, B \subseteq M$. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ but that in general $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

2. Let $f : M \to N$ be a map between metric spaces. Prove that f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of M.

3. Let (X, d_X) and (Y, d_Y) be metric spaces with (Y, d_Y) bounded. Let $\mathcal{C}(X, Y)$ denote the set of all continuous functions from X to Y. Define δ on $\mathcal{C}(X, Y)^2$ by

$$\delta(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

(i) Show that δ is a metric.

(ii) Show further that if (Y, d_Y) is complete then $(\mathcal{C}(X, Y), \delta)$ is complete.

(iii) Consider now the map $R : \mathcal{C}([0,1], \mathbb{R}) \to \mathcal{C}((0,1), \mathbb{R})$ which takes a map on $[0,1]$ to its restriction on $(0,1)$. Is the image of R complete?

4. Let M denote the space of sequences $(x_n)_{n=0}^{\infty}$ where $x_n = 0$ or 1 for each n. Define d on M^2 by

$$d((x_n), (y_n)) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

(i) Show that d is a metric.

(ii) Let U_0 denote the set of sequences that begin with 0. Show that U_0 is open. Deduce that M is disconnected.

(iii) Show that M is complete.

(iv) Let $f : M \to [0,1]$ denote the map $f((x_n)) = \sum_0^{\infty} \frac{x_n}{2^n}$. Is f continuous?

5. Let M denote the space of $n \times n$ real matrices. Define $\| \quad \|$ on M by

$$\|A\| = \max |a_{ij}| \qquad \text{where } A = (a_{ij}).$$

(i) Show that $\| \quad \|$ is a norm on M.

(ii) Show that if $\|A\| < 1/n$ then $\mathbf{x} \mapsto A\mathbf{x}$ is a contraction on M. Deduce that $I - A$ is invertible.

6. (i) Show that a metric space M is connected if and only if every continuous integer-valued function on M is constant.

(ii) Show that $H = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is connected. By considering the function $f(x, y)/x$, or otherwise, show that there are precisely two continuous functions $f : H \to \mathbb{R}$ such that

$$f(x, y)^2 = x^2 \text{ for all } (x, y) \in H.$$

(iii) How many continuous functions $g : \mathbb{R}^2 \to \mathbb{R}$ are there such that $g(x, y)^2 = x^2$ for all $(x, y) \in \mathbb{R}^2$?

7. (i) Prove that if U is an open subset of $\mathbb{R}$ and $c \in U$ then $U \backslash \{c\}$ is disconnected. Give an example to show that this result may fail if U is not open.

(ii) Show that any set, obtained by removing a single point from $\mathbb{R}^2$, is still connected.

(iii) By considering the restriction of f to $(0,1)$, or otherwise, show that there is no invertible continuous function $f : [0,1) \to (0,1)$. [There are bijections which map $[0,1)$ onto $(0,1)$. Can you construct one?]

(iv) Show that there is no continuous 1–1 mapping from $\mathbb{R}^2$ to $\mathbb{R}$.

8. (Optional) Let A be a connected subset of a metric space X.

(i) If C is a closed and open subset of X, prove that either $A \subset C$ or $A \cap C = \emptyset$. Hence or otherwise prove that $\overline{A}$ is a connected subset of X.

(b) Define a relation $\sim$ on X by saying that $x \sim y$ if and only if there is a connected subset of X containing both x and y. Prove that $\sim$ is an equivalence relation and that the equivalence classes, known as *connected components* are closed connected sets.