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CLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH CONIC REGIONS*

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Abstract The object of this article is to introduce new classes of meromorphic functions associated with conic regions. Several properties like the coefficient bounds, growth and distortion theorems, radii of starlikeness and convexity, partial sums, are investigated. Some consequences of the main results for the well-known classes of meromorphic functions are also pointed out.

Key words Meromorphic functions; conic regions; varying argument; subordination; Hadamard product; convolution

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1 Introduction

Let $\widetilde{\mathcal{H}}$ denote the class of functions holomorphic in $\mathcal{D} = \mathcal{D}(1)$, where

$$\mathcal{D}(r) = \{z \in \mathbb{C} :\ 0 < |z| < r\},$$

with a simple pole at the point $z = 0$. By $\mathcal{H}$ we denote the class of functions $f \in \widetilde{\mathcal{H}}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (z \in \mathcal{D}). \tag{1}$$

Also, by $\mathcal{T}_\eta^\varepsilon \quad (\eta \in \mathbb{R}, \varepsilon \in \{0, 1\})$ we denote the class of functions $f \in \mathcal{H}$ of the form (1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \varepsilon\pi - (n + 1)\eta \quad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{2}$$

For $\eta = 0$ we obtain the classes $\mathcal{T}_0^0$ and $\mathcal{T}_0^1$ of functions with positive coefficients and negative coefficients, respectively.

Due to Silverman [1] (see also [2]), we define the class

$$\mathcal{T}^\varepsilon := \bigcup_{\eta \in \mathbb{R}} \mathcal{T}_\eta^\varepsilon.$$

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It is called the class of functions with varying arguments of coefficients.

Let $\alpha \in \langle 0,1)$, $r \in (0,1\rangle$. A function $f \in \mathcal{H}$ is said to be meromorphically convex of order α in $\mathcal{D}(r)$ if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in \mathcal{D}(r)).$$

A function $f \in \mathcal{H}$ is said to be meromorphically starlike of order α in $\mathcal{D}(r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathcal{D}(r)). \tag{3}$$

Denote by $\mathcal{MS}^c(\alpha)$ the class of all functions $f \in \mathcal{H}$ meromorphically convex of order α in $\mathcal{D}$, and by $\mathcal{MS}^*(\alpha)$ the class of all functions $f \in \mathcal{H}$ meromorphically starlikeof order α in $\mathcal{D}$. We also set

$$\mathcal{MS}^c = \mathcal{MS}^c(0) \ \text{ and } \ \mathcal{MS}^* = \mathcal{MS}^*(0).$$

It is shown that, for a function $f \in \mathcal{T}_\eta$, condition (3) is equivalent to

$$\left|\frac{zf'(z)}{f(z)}+1\right| < 1-\alpha \quad (z \in \mathcal{D}(r)). \tag{4}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{H}$. We define the radius of starlikeness of order α and the radius of convexity of order α for the class $\mathcal{B}$ by

$$R^*_\alpha(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left(\sup\left\{r \in (0,1] : f \ \text{ is meromorphically starlike of order } \alpha \ \text{ in } \mathcal{D}(r)\right\}\right),$$
$$R^c_\alpha(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left(\sup\left\{r \in (0,1] : f \ \text{ is meromorphically convex of order } \alpha \ \text{ in } \mathcal{D}(r)\right\}\right),$$

respectively.

Let functions f, g be analytic in $\mathcal{U} := \mathcal{D} \cup \{0\}$. We say that the function f is subordinate to the function g, and write $f(z) \prec g(z)$ (or simply $f \prec g$), if there exists a function ω analytic in $\mathcal{U}$, $|\omega(z)| \le |z| \ \ (z \in \mathcal{U})$, such that

$$f(z) = g(\omega(z)) \quad (z \in \mathcal{U}) .$$

In particular, if g is univalent in $\mathcal{U}$, we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \ \text{ and } \ f(\mathcal{U}) \subset g(\mathcal{U}).$$

For functions $f, g \in \widetilde{\mathcal{H}}$ of the form

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n \ \text{ and } \ g(z) = \sum_{n=-1}^{\infty} b_n z^n,$$

by $f * g$ we denote the Hadamard product (or convolution) of f and g, defined by

$$(f * g)(z) = \sum_{n=-1}^{\infty} a_n b_n z^n \quad (z \in \mathcal{D}) .$$

Let γ, k be real parameters with $k \ge 0$, $0 \le \gamma < 1$, and let $\varphi, \phi \in \mathcal{H}$.

By $\mathcal{W}(\phi,\varphi;\gamma,k)$ we denote the class of functions $f\in\mathcal{H}$ such that $(\varphi * f)(z)\neq 0$ $(z\in\mathcal{D})$ and

$$\operatorname{Re}\left\{\frac{(\phi * f)(z)}{(\varphi * f)(z)}-\gamma\right\}>k\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)}-1\right| \quad (z\in\mathcal{D}). \tag{5}$$

Moreover, put

$$\mathcal{TW}^{\varepsilon}(\phi,\varphi;\gamma,k)=\mathcal{T}^{\varepsilon}\cap\mathcal{W}(\phi,\varphi;\gamma,k),$$

$$\mathcal{TW}^{\varepsilon}_{\eta}(\phi,\varphi;\gamma,k)=\mathcal{T}^{\varepsilon}_{\eta}\cap\mathcal{W}(\phi,\varphi;\gamma,k).$$

It is shown that

$$f(z)\in\mathcal{TW}(z\phi'(z),z\varphi'(z);\gamma,k)\Longleftrightarrow -zf'(z)\in\mathcal{TW}(\phi,\varphi;\gamma,k),$$

$$f(z)\in\mathcal{TW}^{0}_{\eta}(z\phi'(z),z\varphi'(z);\gamma,k)\Longleftrightarrow -zf'(z)\in\mathcal{TW}^{1}_{\eta}(\phi,\varphi;\gamma,k).$$

Moreover, a function f belongs to the class $\mathcal{W}(\phi,\varphi;\gamma,k)$ if and only if $\frac{\phi*f}{\varphi*f}$ maps $\mathcal{U}$ onto the conic domain

$$\Omega_{k,\gamma}=\left\{z=u+\mathrm{i}v:\ (u-\gamma)^2>k^2(u-1)^2+k^2v^2\right\} \quad (k>0)$$

or the halfplane

$$\Pi_{\gamma}=\{w\in\mathbb{C}:\operatorname{Re}w>\gamma\} \quad (k=0).$$

Therefore, the classes are related to the class of k-uniformaly convex functions introduced by Kanas and Wisniowska [3] (see also [4]).

For the further investigations, we assume that the functions φ,ϕ are of the forms

$$\phi(z)=\frac{1}{z}-(-1)^{\varepsilon}\sum_{n=1}^{\infty}\beta_n z^n, \quad \varphi(z)=\frac{1}{z}+(-1)^{\varepsilon}\sum_{n=1}^{\infty}\alpha_n z^n \quad (z\in\mathcal{D}), \tag{6}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are nonnegative real, with

$$\alpha_n+\beta_n>0 \quad (n\in\mathbb{N}).$$

Moreover, put

$$d_n:=(k+1)\beta_n+(k+\gamma)\alpha_n \quad (n\in\mathbb{N}). \tag{7}$$

The object of this article is to investigate the coefficient estimates, distortion properties, the radii of starlikeness and convexity, and partial sums for the classes of meromorphic functions with varying arguments of coefficients. Further, we obtain a subordination theorem and integral mean inequalities.

2 Coefficients Estimates

First, we mention a sufficient condition for functions belonging to the class $\mathcal{W}(\phi,\varphi;\gamma,k)$.

Theorem 1 Let $\{d_n\}$ be defined by (7), $0\le\gamma<1$. If a function f of form (1) satisfies the condition

$$\sum_{n=1}^{\infty}d_n|a_n|\le 1-\gamma, \tag{8}$$

then f belongs to the class $\mathcal{W}(\phi, \varphi; \gamma, k)$.

Proof By definition of the class $\mathcal{W}(\phi, \varphi; \gamma, k)$, it suffices to show that

$$k\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| - \operatorname{Re}\left\{\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right\} \leq 1 - \gamma \quad (z \in \mathcal{U}). \tag{9}$$

Simply calculations give

$$\begin{aligned} k\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| - \operatorname{Re}\left\{\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right\} &\leq (k+1)\left|\frac{(\phi * f)(z)}{(\varphi * f)(z)} - 1\right| \\ &\leq (k+1)\frac{\sum\limits_{n=1}^{\infty}(\alpha_n + \beta_n)|a_n||z|^{n+1}}{1 - \sum\limits_{n=1}^{\infty}\alpha_n|a_n||z|^{n+1}}. \end{aligned}$$

Now, the last expression is bounded above by $(1-\gamma)$ if (8) holds. Thus, $f \in \mathcal{W}(\phi, \varphi; \gamma, k)$.

The next theorem shows that condition (8) is necessary as well for functions of form (1), with (2) belonging to the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; \gamma, k)$.

Theorem 2 Let f be a function of the form (1) with (2). Then, f belongs to the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; \gamma, k)$, if and only if the condition (8) holds true.

Proof In view of Theorem 1, we need only show that each function f from the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; \gamma, k)$ satisfies the coefficient inequality (8). Let f be a function of form (1), with (2) belonging the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; \gamma, k)$. Then by (5), we have

$$k\left|\frac{\frac{1}{z} - (-1)^\varepsilon \sum\limits_{n=1}^{\infty}\beta_n a_n z^n}{\frac{1}{z} + (-1)^\varepsilon \sum\limits_{n=1}^{\infty}\alpha_n a_n z^n} - 1\right| < \operatorname{Re}\left\{\frac{\frac{1}{z} - (-1)^\varepsilon \sum\limits_{n=1}^{\infty}\beta_n a_n z^n}{\frac{1}{z} + (-1)^\varepsilon \sum\limits_{n=1}^{\infty}\alpha_n a_n z^n} - \gamma\right\},$$

or

$$k\left|\frac{\sum\limits_{n=1}^{\infty}(\beta_n + \alpha_n) a_n z^{n+1}}{1 + (-1)^\varepsilon \sum\limits_{n=1}^{\infty}\alpha_n a_n z^{n+1}}\right| < \operatorname{Re}\left\{\frac{(1-\gamma) - (-1)^\varepsilon \sum\limits_{n=1}^{\infty}(\beta_n + \gamma\alpha_n) a_n z^{n+1}}{1 + (-1)^\varepsilon \sum\limits_{n=1}^{\infty}\alpha_n a_n z^{n+1}}\right\}.$$

In view of (2), we set $z = re^{\mathrm{i}\eta}$ $(0 \leq r < 1)$ in the above inequality to obtain

$$\frac{\sum\limits_{n=1}^{\infty}k(\beta_n + \alpha_n)|a_n| r^{n+1}}{1 + \sum\limits_{n=1}^{\infty}\alpha_n |a_n| r^{n+1}} < \frac{(1-\gamma) - \sum\limits_{n=1}^{\infty}(\beta_n + \gamma\alpha_n)|a_n| r^{n+1}}{1 + \sum\limits_{n=1}^{\infty}\alpha_n |a_n| r^{n+1}}.$$

Thus, we have

$$\sum_{n=1}^{\infty}\left[(k+1)\beta_n + (k+\gamma)\alpha_n\right]|a_n| r^{n+1} < 1 - \gamma,$$

which, when letting $r \to 1^-$, readily yields assertion (8).

As condition (8) is independent of η, Theorem 2 yields the following theorem.

Theorem 3 Let f be a function of the form (1) with (2). Then, f belongs to the class $\mathcal{TW}^\varepsilon(\phi, \varphi; \gamma, k)$, if and only if condition (8) holds true.

From Theorems 2 and 3, we obtain coefficients estimates for the classes $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; \gamma, k)$ and $\mathcal{TW}^\varepsilon(\phi, \varphi; \gamma, k)$, respectively.

Corollary 1 If a function f of the form (1) belongs to the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi, \varphi; \gamma, k)$, then,

$$|a_n| \leq \frac{1-\gamma}{d_n} \quad (n \in \mathbb{N}), \tag{10}$$

where $\{d_n\}$ is defined by (7). The result is sharp. The functions $f_{n,\eta}$ of the form

$$f_{n,\eta}(z) = \frac{1}{z} + \frac{1-\gamma}{d_n} e^{\mathrm{i}\{\varepsilon\pi-(n+1)\eta\}} z^n \quad (z \in \mathcal{D};\ n \in \mathbb{N}) \tag{11}$$

are the extremal functions.

Corollary 2 If a function f of the form (1) belongs to the class $\mathcal{TW}^{\varepsilon}(\phi, \varphi; \gamma, k)$, then the coefficients estimate (10) holds true. The result is sharp. The functions $f_{n,\eta}$ of the form (11) $(\eta \in \mathbb{R})$ are the extremal functions.

3 Distortion Theorems

From Theorem 2, we have the following lemma:

Lemma 1 Let a function f of form (1) belong to the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi, \varphi; \gamma, k)$. If the sequence $\{d_n\}$ defined by (7) satisfies the inequality

$$d_1 \leq d_n \quad (n \in \mathbb{N}), \tag{12}$$

then

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\gamma}{d_1}.$$

Moreover, if

$$nd_1 \leq d_n \quad (n \in \mathbb{N}), \tag{13}$$

then

$$\sum_{n=1}^{\infty} na_n \leq \frac{1-\gamma}{d_1}.$$

Theorem 4 Let a function f belong to the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi, \varphi; \gamma, k)$. If the sequence $\{d_n\}$ defined by (7) satisfies (12), then,

$$\frac{1}{r} - \frac{1-\gamma}{d_1} r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\gamma}{d_1} r \quad (|z| = r < 1). \tag{14}$$

Moreover, if (13) holds, then,

$$\frac{1}{r^2} - \frac{1-\gamma}{d_1} \leq |f(z)| \leq \frac{1}{r^2} + \frac{1-\gamma}{d_1} \quad (|z| = r < 1). \tag{15}$$

The result is sharp, with the extremal function $f_{1,\eta}$ of form (11).

Proof Let a function f of form (1) belong to the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi, \varphi; \gamma, k)$, $|z| = r < 1$. As

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \leq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n| r^n \\ &= \frac{1}{r} + r \sum_{n=1}^{\infty} |a_n| r^{n-1} \leq \frac{1}{r} + r \sum_{n=1}^{\infty} |a_n| \end{aligned}$$

and

$$\begin{aligned}|f(z)| &= \left|z+\sum_{n=1}^{\infty}a_nz^n\right| \geq \frac{1}{r}-\sum_{n=1}^{\infty}|a_n|\,r^n\\ &= \frac{1}{r}-r\sum_{n=1}^{\infty}|a_n|\,r^{n-1} \geq \frac{1}{r}-r\sum_{n=1}^{\infty}|a_n|\,,\end{aligned}$$

then by Lemma 1, we have (14). Analogously, we may prove (15).

Theorem 4 implies the following corollary.

Corollary 3 Let a function f belong to the class $\mathcal{TW}^{\varepsilon}(\phi,\varphi;\gamma,k)$. If the sequence $\{d_n\}$ defined by (7) satisfies (12), then assertion (14) holds true. Moreover, if we assume (13), then the assertion (14) holds true. The result is sharp, with the extremal functions $f_{1,\eta}\,(\eta\in\mathbb{R})$ of form (11).

4 The Radii of Convexity and Starlikeness

Theorem 5 The radius of starlikeness of order α for the class $\mathcal{TW}_{\eta}^{0}(\phi,\varphi;\gamma,k)$ is given by

$$R_{\alpha}^{*}\left(\mathcal{TW}_{\eta}^{0}(\phi,\varphi;\gamma,k)\right)=\inf_{n\in\mathbb{N}}\left(\frac{(1-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n+1}}, \tag{16}$$

where d_n is defined by (7).

Proof The function $f\in\mathcal{T}_{\eta}^{0}$ of form (1) is meromorphically starlike of order α in the disk $\mathcal{D}(r)$, $0<r\leq 1$, if and only if it satisfies condition (4). As

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum\limits_{n=1}^{\infty}(n+1\,)a_nz^{n+1}}{1+\sum\limits_{n=1}^{\infty}a_nz^{n+1}}\right| \leq \frac{\sum\limits_{n=1}^{\infty}(n+1\,)\,|a_n|\,|z|^{n+1}}{1-\sum\limits_{n=1}^{\infty}|a_n|\,|z|^{n+1}},$$

by putting $|z|=r$, condition (4) is true if

$$\sum_{n=1}^{\infty}\frac{n-\alpha}{1-\alpha}\,|a_n|\,r^{n+1}\ \leq 1. \tag{17}$$

By Theorem 2, we have

$$\sum_{n=1}^{\infty}\frac{d_n}{1-\gamma}\,|a_n|\leq 1,$$

Thus, condition (17) is true if

$$\frac{n-\alpha}{1-\alpha}r^{n+1}\ \leq\frac{d_n}{1-\gamma}\quad(n\in\mathbb{N})\,,$$

that is, if

$$r\leq\left(\frac{(1-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n+1}}\quad(n\in\mathbb{N})\,.$$

It follows that each function $f\in\mathcal{TW}_{\eta}^{0}(\phi,\varphi;\gamma,k)$ is meromorphically starlike of order α in the disk $\mathcal{D}\,(r)$, where $r=R^{*}\left(\mathcal{TW}_{\eta}^{0}(\phi,\varphi;\gamma,k)\right)$ is defined by (16).

Theorem 6 The radius of convexity of order α for the class $\mathcal{TW}_{\eta}^{1}(\phi,\varphi;\gamma,k)$ is given by

$$R_{\alpha}^{c}\left(\mathcal{TW}_{\eta}^{1}(\phi,\varphi;\gamma,k)\right)=\inf_{n\in\mathbb{N}}\left(\frac{(1-\alpha)\,d_n}{n\,(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n+1}},$$

where d_n is defined by (7).

The proof is analogous to that of Theorem 4, and we omit it.

From Theorems 5 and 6, we obtain the following two corollaries.

Corollary 4 The radius of starlikeness of order α for the class $\mathcal{TW}^{0}(\phi,\varphi;\gamma,k)$ is given by

$$R_{\alpha}^{*}\left(\mathcal{TW}^{0}(\phi,\varphi;\gamma,k)\right)=\inf_{n\in\mathbb{N}}\left(\frac{(1-\alpha)\,d_n}{(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n+1}},$$

where d_n is defined by (7).

Corollary 5 The radius of convexity of order α for the class $\mathcal{TW}^{1}(\phi,\varphi;\gamma,k)$ is given by

$$R_{\alpha}^{c}\left(\mathcal{TW}_{\eta}^{1}(\phi,\varphi;\gamma,k)\right)=\inf_{n\in\mathbb{N}}\left(\frac{(1-\alpha)\,d_n}{n\,(n-\alpha)\,(1-\gamma)}\right)^{\frac{1}{n+1}},$$

where d_n is defined by (7).

5 Partial Sums

Let $f\in\mathcal{H}$ be a function of the form (1). Due to Silverman [5] and Silvia [6] (see also [7]), we define the partial sums f_m by

$$f_m(z)=\frac{1}{z}+\sum_{n=1}^{m}a_n z^n\quad(m\in\mathbb{N}).\tag{18}$$

In this section, we consider partial sums of functions of the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi,\varphi;\gamma,k)$ and obtain sharp lower bounds for the real parts of ratios of f to f_m and f' to f'_m.

Theorem 7 Let $m\in\mathbb{N}$ and let the sequence $\{d_n\}$, defined by (7), satisfy the inequalities

$$1-\gamma\le d_n\le d_{n+1}\quad(m\in\mathbb{N}).\tag{19}$$

If a function f belongs to the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi,\varphi;\gamma,k)$, then,

$$\mathrm{Re}\left\{\frac{f(z)}{f_m(z)}\right\}\ge 1-\frac{1-\gamma}{d_{m+1}}\quad(z\in\mathcal{D})\tag{20}$$

and

$$\mathrm{Re}\left\{\frac{f_m(z)}{f(z)}\right\}\ge\frac{d_{m+1}}{1-\gamma+d_{m+1}}\quad(z\in\mathcal{D}).\tag{21}$$

The bounds are sharp, with the extremal function $f_{m+1,\eta}$ of the form (11).

Proof Let a function f of form (1) belong to the class $\mathcal{TW}_{\eta}^{\varepsilon}(\phi,\varphi;\gamma,k)$. Then, by (19) and Theorem 2, we have

$$\sum_{n=1}^{m}|a_n|+\frac{d_{m+1}}{1-\gamma}\sum_{n=m+1}^{\infty}|a_n|\le\sum_{n=1}^{\infty}\frac{d_n}{1-\gamma}|a_n|\le 1.\tag{22}$$

If put

$$g(z)=\frac{d_{m+1}}{1-\gamma}\left\{\frac{f(z)}{f_m(z)}-\left(1-\frac{1-\gamma}{d_{m+1}}\right)\right\}$$
$$=1+\frac{\frac{d_{m+1}}{1-\gamma}\sum\limits_{n=m+1}^{\infty}a_nz^{n+1}}{1+\sum\limits_{n=1}^{m}a_nz^{n+1}}\quad(z\in\mathcal{D}),$$

then it suffices to show that

$$\mathrm{Re}g(z)\geq 0\quad(z\in\mathcal{D}),$$

or

$$\left|\frac{g(z)-1}{g(z)+1}\right|\leq 1\quad(z\in\mathcal{D}).$$

Applying (22), we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right|\leq\frac{\frac{d_{m+1}}{1-\gamma}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{n}|a_n|-\frac{d_{m+1}}{1-\gamma}\sum\limits_{n=m+1}^{\infty}|a_n|}\leq 1\quad(z\in\mathcal{D}),$$

which readily yields the assertion (20). To see that $f=f_{m+1,\eta}$ gives the result sharp, we observe that, for $z=re^{\mathrm{i}\eta}$, we have

$$\frac{f(z)}{f_m(z)}=1-\frac{(1-\gamma)\,r^{m+2}}{d_{m+1}}\overset{r\to1^-}{\longrightarrow}1-\frac{1-\gamma}{d_{m+1}}.$$

Similarly, if we take

$$h(z)=(1-\gamma+d_{m+1})\left\{\frac{f_m(z)}{f(z)}-\frac{d_{m+1}}{1-\gamma+d_{m+1}}\right\}\quad(z\in\mathcal{D}),$$

and making use of (22), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right|\leq\frac{(1+\frac{d_{m+1}}{1-\gamma})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n|-(1-\frac{d_{m+1}}{1-\gamma})\sum\limits_{n=m+1}^{\infty}|a_n|}\leq 1\quad(z\in\mathcal{D}),$$

which leads us immediately to assertion (21). The bound in (21) is sharp for each $m\in N$ with the extremal function $f=f_{m+1,\eta}$, given by (11).

Theorem 8 Let $m\in N$ and let the sequence $\{d_n\}$ defined by (7) satisfy inequalities (19). If a function f belongs to the class $\mathcal{TW}_\eta^\varepsilon(\phi,\varphi;\gamma,k)$, then,

$$\mathrm{Re}\left\{\frac{f'(z)}{f_m(z)}\right\}\geq 1-\frac{(1-\gamma)\,(m+1)}{d_{m+1}}\quad(z\in\mathcal{D})\tag{23}$$

and

$$\mathrm{Re}\left\{\frac{f'_m(z)}{f(z)}\right\}\geq\frac{d_{m+1}}{(1-\gamma)\,(m+1)+d_{m+1}}\quad(z\in\mathcal{D}).\tag{24}$$

The bounds are sharp, with the extremal function $f_{m+1,\eta}$ of form (11).

The proof is analogous to that of Theorem 7, and we omit it.

From Theorems 7 and 8, we obtain the following corollary.

Corollary 6 Let $m \in N$ and let the sequence $\{d_n\}$ defined by (7) satisfy the inequalities (19). If a function f belongs to the class $\mathcal{TW}^{\varepsilon}(\phi, \varphi; \gamma, k)$, then the bounds (20), (21), (23), and (24) hold true. The result is sharp, with the extremal functions $f_{m+1,\eta}$ $(\eta \in \mathbb{R})$ of the form (11).

6 Concluding Remarks

We conclude this article by observing that, in view of the subordination relation (7), choosing the functions ϕ and φ, and the parameters k, γ, we can define classes of functions, which were investigated in earlier works.

In particular, the class $\mathcal{G}(\varphi, \phi; \gamma) := \mathcal{W}_0^0(\varphi, \phi; \gamma, 0)$ was studied by Raina and Srivastava [8]. The class

$$\mathcal{W}(\varphi; \gamma, k) := \mathcal{W}_0^0(-z\varphi'(z), \varphi(z); \gamma, k),$$

contains functions $f \in \mathcal{T}_0^0$, such that

$$\mathrm{Re}\left\{-\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} - \gamma\right\} > k\left|\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} + 1\right| \quad (z \in \mathcal{U}).$$

If we set

$$\varphi(z) := z^{-1}\,{}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) \quad (z \in \mathcal{U}),$$

where ${}_qF_s$ is the generalized hypergeometric function, then the convolution $\varphi * f$ can be used to represent the Dziok–Srivastava operator [9] (see also [10] and [11]). Then, we denote the class $\mathcal{W}(\varphi; \gamma, k)$ by $\mathcal{W}_{q,s}(\gamma, k)$. The class $\mathcal{W}_{q,s}(\gamma) := \mathcal{W}_{q,s}(\gamma, 0)$ was investigated intensively by (among others) Srivastava et al [10] and [12], Aouf et al [13], and [14]. Moreover, if we put $s = 1$, $q = 2$, and $\alpha_2 = 1$, we obtain the class $\mathcal{W}_{2,1}(\gamma)$ related to the Carlson-Shaffer operator, which was studied by Liu and Srivastava [11].

The classes

$$k\text{-}\mathcal{W}(\gamma) := \mathcal{W}\left(\frac{1}{z(1-z)}; \gamma, k\right),$$

$$k\text{-}\mathcal{V}(\gamma) := \mathcal{W}\left(\frac{1-2z}{z(1-z)^2}; \gamma, k\right),$$

defined by the following conditions

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\frac{zf'(z)}{f(z)} + 1\right| \quad (z \in \mathcal{U}),$$

$$\mathrm{Re}\left\{-1 - \frac{zf''(z)}{f'(z)} - \gamma\right\} > k\left|\frac{zf''(z)}{f'(z)} + 2\right| \quad (z \in \mathcal{U}),$$

are the classes related to the classes of k-starlike functions and k-uniformly convex functions, respectively (see [3, 4]). In particular, we have

$$0\text{-}\mathcal{W}(\gamma) = \mathcal{MS}^*(\gamma), \quad 0\text{-}\mathcal{V}(\gamma) = \mathcal{MS}^c(\gamma).$$

Many other classes of functions are related to the class investigated here, see, for example, [15–17].

If we apply the results presented here to the classes discussed above, we can obtain several additional results. Some of them were obtained in earlier works, see, for example, [8, 10–14].

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