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A UNIFIED CLASS OF ANALYTIC FUNCTIONS WITH FIXED ARGUMENT OF COEFFICIENTS*

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Abstract In this paper we introduce new classes of analytic functions with fixed argument of coefficients defined by subordination. Coefficient estimates, distortion theorems, integral means inequalities, and the radii of convexity and starlikeness are investigated. Convolution properties are also pointed out.

Key words analytic functions; fixed argument; subordination; Hadamard product

2000 MR Subject Classification 30C45

1 Introduction

Let $\widetilde{\mathcal{A}}$ denote the class of functions analytic in $\mathcal{U}=\mathcal{U}(1)$, where

$$\mathcal{U}(r)=\{z\in\mathbf{C}:|z|<r\}.$$

and let $\mathcal{A}$ denote the class of functions $f\in\widetilde{\mathcal{A}}$ normalized by $f(0)=f'(0)-1=0$. Each function $f\in\mathcal{A}$ can be expressed as

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad (z\in\mathcal{U}). \tag{1}$$

Also, by $\mathcal{T}_\theta$ $(\theta\in\mathbb{R})$ we denote the class of functions $f\in\mathcal{A}$ of the form

$$f(z)=z+\mathrm{e}^{\mathrm{i}\theta}\sum_{n=2}^{\infty}|a_n|z^n \quad (z\in\mathcal{U})\,. \tag{2}$$

The class $\mathcal{T}_\theta$ is called the class of functions with fixed argument of coefficients. For $\theta=\pi$ we obtain the class $\mathcal{T}_\pi$ of functions with non-negative coefficients.

Let $\alpha\in[0,1)\,,\ r\in(0,1]\,$. A function $f\in\mathcal{A}$ is said to be convex of order α in $\mathcal{U}(r)$ if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \ \ (z\in\mathcal{U}(r)).$$

A function $f\in\mathcal{A}$ is said to be starlike of order α in $\mathcal{U}(r)$ if

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>\alpha \ \ (z\in\mathcal{U}(r)). \tag{3}$$

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We denote by $\mathcal{S}^c(\alpha)$ the class of all functions $f \in \mathcal{A}$, which are convex of order α in $\mathcal{U}$ and by $\mathcal{S}^*(\alpha)$ we denote the class of all functions $f \in \mathcal{A}$, which are starlike of order α in $\mathcal{U}$. We also set

$$\mathcal{S}^c = \mathcal{S}^c(0) \text{ and } \mathcal{S}^* = \mathcal{S}^*(0).$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{A}$. We define the radius of starlikeness of order α and the radius of convexity of order α for the class $\mathcal{B}$ by

$$R^*_\alpha(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left\{\sup\left\{r \in (0,1] : f \text{ is starlike of order } \alpha \text{ in } \mathcal{U}(r)\right\}\right\},$$
$$R^c_\alpha(\mathcal{B}) = \inf_{f\in\mathcal{B}} \left\{\sup\left\{r \in (0,1] : f \text{ is convex of order } \alpha \text{ in } \mathcal{U}(r)\right\}\right\},$$

respectively.

We say that a function $f \in \widetilde{\mathcal{A}}$ is subordinate to a function $F \in \widetilde{\mathcal{A}}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \widetilde{\mathcal{A}}$, $|\omega(z)| \le |z| \, (z \in \mathcal{U})$, such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence:

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

For functions $f, g \in \widetilde{\mathcal{A}}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the Hadamard product (or convolution) of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

Let A, B, θ be real parameters, $-1 \le A < B \le 1$, $(\cos\theta < 0$ or $B \ne 1)$ and let $\varphi \in \mathcal{A}$ be a given function of the form

$$\varphi(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n \quad (z \in \mathcal{U};\ \alpha_n > 0,\ n = 2, 3, \cdots). \tag{4}$$

By $\mathcal{W}(\varphi; A, B)$ we denote the class of functions $f \in \mathcal{A}$ such that

$$\frac{(\varphi * f)(z)}{z} \prec \frac{1 + Az}{1 + Bz}. \tag{5}$$

Now, we define the classes of functions with fixed argument of coefficients related to the class $\mathcal{W}(\varphi; A, B)$. Let us denote

$$\mathcal{W}_\theta(\varphi; A, B) := \mathcal{T}_\theta \cap \mathcal{W}(\varphi; A, B).$$

The families $\mathcal{W}(\varphi; A, B)$ and $\mathcal{W}_\theta(\varphi; A, B)$ unifies various new and well-known classes of analytic functions. In particular, the class

$$\mathcal{W}(t; \varphi; A, B) := \mathcal{W}(tz\varphi'(z) + (1-t)\varphi(z); A, B) \quad (0 \le t \le 1)$$

contains functions $f \in \mathcal{A}$ such that

$$t\,(\varphi * f)'\,(z) + (1-t)\,\frac{(\varphi * f)\,(z)}{z} \prec \frac{1+Az}{1+Bz}.$$

Let

$$\varphi(z) := z\ {}_{s+1}F_s(\alpha_1, \cdots, \alpha_{s+1}; \beta_1, \cdots, \beta_s; z),$$

and ${}_qF_s$ be the generalized hypergeometric function (see for details [1, 2]). Recently, many classes of functions were defined by using the linear operator $Hf\,(z) := (\varphi * f)\,(z)\ \ (f \in \mathcal{A})$ introduced by Dziok and Srivastava [3] (see also [4–7]). In particular the class $V_1^2(s; A, B; t) := \mathcal{W}\,(t; \varphi; A, B)$was studied in [8]. Many other classes were also particular cases of the classes investigated here, see for example [9–14].

In the present paper we obtain coefficient estimates, distortion theorems, integral means inequalities, and the radii of convexity and starlikeness for the class $\mathcal{W}_\theta\,(\varphi; A, B)$. We also derive convolution properties for the class.

2 Coefficient Estimates

Before stating and proving coefficient estimates for the class $\mathcal{W}\,(\varphi; A, B)$ we need the following lemma.

Lemma 1 [15] Let $f$be a function of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

analytic in $\mathcal{U}$. If $f \prec g$ and $g \in \mathcal{S}^c$, then $|a_n| \leq 1\ \ (n \in \mathbb{N})$.

Theorem 1 If a function f of form (1) belongs to the class $\mathcal{W}\,(\varphi; A, B)$, then

$$|a_n| \leq \frac{B-A}{\alpha_n} \quad (n = 2, 3, \cdots). \tag{6}$$

The result is sharp.

Proof Let a function f of form (1) belong to the class $\mathcal{W}\,(\varphi; A, B)$ and let us put

$$g(z) = \frac{(\varphi * f)\,(z) - z}{(A-B)\,z} \quad \text{and} \quad h(z) = \frac{z}{1+Bz}.$$

Then, by (5), we have $g \prec h$. Since the function g is given by

$$g(z) = \sum_{n=2}^{\infty} \frac{\alpha_n}{A-B} a_n z^{n-1}$$

and the function h belongs to the class $\mathcal{S}^c$, by Lemma 1, we obtain

$$\frac{\alpha_n}{B-A}\,|a_n| \leq 1 \ \ (n = 2, 3, \cdots). \tag{7}$$

Thus we have (6). Equality in (7) holds true for the functions g_n of the form

$$g_n(z) = h(z^{n-1}) = z^{n-1} + \sum_{k=n}^{\infty} b_k z^k \ \ (k = 2, 3, \cdots)$$

for some b_k $(k = 2, 3, \cdots)$. Consequently, equality in (6) holds true for the functions f_n of the form

$$f_n(z) = z + \frac{A-B}{\alpha_n} z^n + \sum_{k=n+1}^{\infty} \frac{A-B}{\alpha_k} b_{k-1} z^k \quad (n = 2, 3, \cdots).$$

Theorem 2 If a function f of form (2) belongs to the class $\mathcal{W}_\theta(\varphi; A, B)$, then

$$\sum_{n=2}^{\infty} \alpha_n |a_n| \leq \delta(\theta; A, B), \tag{8}$$

where

$$\delta(\theta; A, B) := \frac{B-A}{\sqrt{1 - B^2 \sin^2 \theta} - B \cos \theta}. \tag{9}$$

Proof Let a function f belong to the class $\mathcal{W}_\theta(\varphi; A, B)$. Then, by (5) and the definition of subordination, we have

$$\frac{(\varphi * f)(z)}{z} = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathcal{U}$. Thus, we obtain

$$\left| \frac{(\varphi * f)(z) - z}{B(\varphi * f)(z) - Az} \right| < 1 \quad (z \in \mathcal{U}).$$

Hence, by (2), we have

$$\left| \sum_{n=2}^{\infty} \alpha_n |a_n| z^{n-1} \right| < \left| B - A + B\mathrm{e}^{\mathrm{i}\theta} \sum_{n=2}^{\infty} \alpha_n |a_n| z^{n-1} \right| \quad (z \in \mathcal{U}). \tag{10}$$

Putting $z = r$ $(0 \leq r < 1)$, we find that

$$|w| < |B - A + Bw\mathrm{e}^{\mathrm{i}\theta}|, \tag{11}$$

where, for convenience,

$$w = \sum_{n=2}^{\infty} \alpha_n |a_n| r^{n-1}.$$

Since w is a real number, by (11) we have

$$(1 - B^2)w^2 - [2B(B-A)\cos\theta]\, w - (B-A)^2 < 0.$$

Solving this inequality with respect to w, we obtain

$$\sum_{n=2}^{\infty} \alpha_n |a_n| r^{n-1} < \delta(\theta; A, B),$$

which, upon letting $r \to 1^-$, readily yields assertion (8) of Theorem 1.

Theorem 3 A function f of form (2) belongs to the class $\mathcal{W}_\pi(\varphi; A, B)$ if and only if

$$\sum_{n=2}^{\infty} \alpha_n |a_n| \leq \frac{B-A}{1+B}. \tag{12}$$

Proof By virtue of Theorem 1, we only need to show that condition (12) is a sufficient condition. Let a function f of form (2) satisfy condition (12). Then, in view of (10), it is sufficient to prove that

$$\left|\sum_{n=2}^{\infty}\alpha_n|a_n|z^{n-1}\right|-\left|B-A-B\sum_{n=2}^{\infty}\alpha_n|a_n|z^{n-1}\right|<0 \quad (z\in\mathcal{U}).$$

Indeed, letting $|z|=r$ $(0<r<1)$, we have

$$\begin{aligned}&\left|\sum_{n=2}^{\infty}\alpha_n|a_n|z^{n-1}\right|-\left|B-A-B\sum_{n=2}^{\infty}\alpha_n|a_n|z^{n-1}\right|\\ \le&\left(\sum_{n=2}^{\infty}\alpha_n|a_n|r^{n-1}\right)-\left(B-A-B\sum_{n=2}^{\infty}\alpha_n|a_n|r^{n-1}\right)\\ <&(1+B)\sum_{n=2}^{\infty}\alpha_n|a_n|-(B-A)\le 0,\end{aligned}$$

which implies that $f\in\mathcal{W}_\pi(\varphi;A,B)$.

Theorem 2 readily yields

Corollary 1 If a function f of form (2) belongs to the class $\mathcal{W}_\theta(\varphi;A,B)$, then

$$|a_n|\le\frac{\delta(\theta;A,B)}{\alpha_n} \quad (n=2,3,\cdots), \tag{13}$$

where $\delta(\theta;A,B)$ is defined by (9). The result is sharp for $\theta=\pi$. Then the functions f_n of the form

$$f_n(z)=z-\frac{B-A}{(1+B)\,\alpha_n}z^n \quad (z\in\mathcal{U};n=2,3,\cdots) \tag{14}$$

are the extremal functions.

3 Distortion Theorems

From Theorem 2 we have the following lemma.

Lemma 2 Let a function f of form (2) belong to the class $\mathcal{W}_\theta(\varphi;A,B)$. If the sequence $\{\alpha_n\}$ defined by (4) satisfies the inequalities

$$\alpha_2\le\alpha_n \quad (n=2,3,\cdots), \tag{15}$$

then

$$\sum_{n=2}^{\infty}a_n\le\frac{\delta(\theta;A,B)}{\alpha_2}.$$

Moreover, if

$$n\alpha_2\le 2\alpha_n \quad (n=2,3,\cdots), \tag{16}$$

then

$$\sum_{n=2}^{\infty}na_n\le\frac{2\delta(\theta;A,B)}{\alpha_2}.$$

Theorem 4 Let a function f belong to the class $\mathcal{W}_\theta(\varphi; A, B)$. If the sequence $\{\alpha_n\}$ defined by (4) satisfies (15), then

$$r - \frac{\delta(\theta; A, B)}{\alpha_2} r^2 \le |f(z)| \le r + \frac{\delta(\theta; A, B)}{\alpha_2} r^2 \quad (|z| = r < 1). \tag{17}$$

Moreover, if (16) holds, then

$$1 - \frac{2\delta(\theta; A, B)}{\alpha_2} r \le |f'(z)| \le 1 + \frac{2\delta(\theta; A, B)}{\alpha_2} r \quad (|z| = r < 1). \tag{18}$$

The result is sharp for $\theta = \pi$, with the extremal function f_2 of form (14).

Proof Let a function f of form (2) belong to the class $\mathcal{W}_\theta(\varphi; A, B)$, $|z| = r < 1$. Since

$$|f(z)| = \left| z + \mathrm{e}^{\mathrm{i}\theta} \sum_{n=2}^{\infty} a_n z^n \right| \le r + \sum_{n=2}^{\infty} |a_n| r^n = r + r^2 \sum_{n=2}^{\infty} |a_n| r^{n-2} \le r + r^2 \sum_{n=2}^{\infty} |a_n|$$

and

$$|f(z)| = \left| z + \mathrm{e}^{\mathrm{i}\theta} \sum_{n=2}^{\infty} a_n z^n \right| \ge r - \sum_{n=2}^{\infty} |a_n| r^n = r - r^2 \sum_{n=2}^{\infty} |a_n| r^{n-2} \ge r - r^2 \sum_{n=2}^{\infty} |a_n|,$$

then by Lemma 2 we have (17). Analogously we may prove (18).

4 Integral Means Inequalities

Due to Littlewood [16], we obtain integral means inequalities for the functions of the class $\mathcal{W}_\theta(\varphi; A, B)$.

Lemma 3 [16] Let f, $g \in \widetilde{\mathcal{A}}$. If $f \prec g$, then

$$\int_0^{2\pi} \left| f(r\mathrm{e}^{\mathrm{i}t}) \right|^\lambda \mathrm{d}t \le \int_0^{2\pi} \left| g(r\mathrm{e}^{\mathrm{i}t}) \right|^\lambda \mathrm{d}t \quad (0 < r < 1,\ \lambda > 0). \tag{19}$$

Silverman [17] found that the function $g(z) = z - \frac{z^2}{2}$ $(z \in \mathcal{U})$ is often extremal over the family of functions with negative coefficients. He applied this function to resolve integral means inequality, conjectured in [18] and settled in [19], that (19) holds true for all functions f with negative coefficients. In [19] he also proved his conjecture for some subclasses of $\mathcal{T}_\pi$.

Applying Lemma 3 and Theorem 2 we prove the following result.

Theorem 5 Let the sequence $\{\alpha_n\}$ defined by (4) satisfy inequality (15). If $f \in \mathcal{W}_\theta(\varphi; A, B)$ then

$$\int_0^{2\pi} \left| f(r\mathrm{e}^{\mathrm{i}t}) \right|^\lambda \mathrm{d}t \le \int_0^{2\pi} \left| g(r\mathrm{e}^{\mathrm{i}t}) \right|^\lambda \mathrm{d}t \quad (0 < r < 1,\ \lambda > 0), \tag{20}$$

where

$$g(z) = z + \mathrm{e}^{\mathrm{i}\theta} \frac{\delta(\theta; A, B)}{\alpha_2} z^2 \quad (z \in \mathcal{U}).$$

Proof For function f of form (2), inequality (20) is equivalent to

$$\int_0^{2\pi} \left| 1 + \mathrm{e}^{\mathrm{i}\theta} \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\lambda \mathrm{d}t \le \int_0^{2\pi} \left| 1 + \mathrm{e}^{\mathrm{i}\theta} \frac{\delta(\theta; A, B)}{\alpha_2} z \right|^\lambda \mathrm{d}t.$$

By Lemma 3, it suffices to show that

$$\sum_{n=2}^{\infty} |a_n|\, z^{n-1} \prec \frac{\delta(\theta; A, B)}{\alpha_2} z. \tag{21}$$

Setting

$$w(z) = \sum_{n=2}^{\infty} \frac{\alpha_2}{\delta(\theta; A, B)} a_n z^{n-1} \quad (z \in \mathcal{U})$$

and using (15) and Theorem 2, we obtain

$$|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\alpha_2}{\delta(\theta; A, B)} a_n\ z^{n-1} \right| \le |z| \sum_{n=2}^{\infty} \frac{\alpha_n}{\delta(\theta; A, B)} |a_n| \le |z| \quad (z \in \mathcal{U})\,.$$

Since

$$\sum_{n=2}^{\infty} a_n z^{n-1} = \frac{\delta(\theta; A, B)}{\alpha_2} w(z) \quad (z \in \mathcal{U})\,,$$

by definition of subordination we have (21), and this completes the proof.

5 The Radii of Convexity and Starlikeness

Theorem 6 If a function f belongs to the class $\mathcal{W}_\theta\,(\varphi; A, B)$, then f is starlike of order α in the disk $\mathcal{U}(r^*)$, where

$$r^* := \inf_{n \ge 2} \left(\frac{(1-\alpha)\,\alpha_n}{(n-\alpha)\,\delta(\theta, A, B)} \right)^{\frac{1}{n-1}} \tag{22}$$

and $\delta(\theta, A, B)$, $\{\alpha_n\}$ are defined by (9) and (4), respectively. For $\theta = \pi$, the result is sharp, that is

$$R^*\left(\mathcal{W}_\pi\,(\varphi; A, B)\right) = r^*.$$

Proof A function $f \in \mathcal{A}$ of form (2) is starlike of order α in the disk $\mathcal{U}(r)$, $0 < r \le 1$, if and only if it satisfies condition (3). Since

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{e^{i\theta} \sum\limits_{n=2}^{\infty} (n-1) a_n z^n}{z + e^{i\theta} \sum\limits_{n=2}^{\infty} a_n z^n} \right| \le \frac{\sum\limits_{n=2}^{\infty} (n-1)\,|a_n|\,|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} |a_n|\,|z|^{n-1}},$$

putting $|z| = r$ condition (3) is true if

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha,$$

or equivalently

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n|\, r^{n-1} \le 1. \tag{23}$$

By Theorem 2, we have

$$\sum_{n=2}^{\infty} \frac{\alpha_n}{\delta(\theta, A, B)} |a_n| \le 1,$$

Thus, condition (23) is true if

$$\frac{n-\alpha}{1-\alpha}r^{n-1} \le \frac{\alpha_n}{\delta(\theta, A, B)} \qquad (n=2,3,\cdots),$$

that is, if

$$r \le \left(\frac{(1-\alpha)\,\alpha_n}{(n-\alpha)\,(B-A)}\right)^{\frac{1}{n-1}} \qquad (n=2,3,\cdots).$$

It follows that each function $f \in \mathcal{W}_\theta\,(\varphi; A, B)$ is starlike of order α in the disk $\mathcal{U}\,(r^*)$, where r^* is defined by (22). For $\theta = \pi$ the functions f_n of form (14) are extremal functions.

Theorem 7 If a function f belongs to the class $\mathcal{W}_\theta\,(\varphi; A, B)$, then $f$is convex in the disk $\mathcal{U}(r^c)$, where

$$r^c := \inf_{n\ge 2}\left(\frac{(1-\alpha)\,\alpha_n}{n\,(n-\alpha)\,\delta(\theta, A, B)}\right)^{\frac{1}{n-1}}$$

and $\delta(\theta, A, B)$, $\{\alpha_n\}$ are defined by (9) and (4), respectively. For $\theta = \pi$, the result is sharp, that is,

$$R^c\left(\mathcal{W}_\pi\,(\varphi; A, B)\right) = r^c.$$

The proof is analogous to that of Theorem 4, and we omit the details.

6 Convolution Properties

Let

$$f(z) = z^p + \mathrm{e}^{\mathrm{i}\alpha}\sum_{n=k}^{\infty}|a_n|z^n, \quad g(z) = z^p + \mathrm{e}^{\mathrm{i}\beta}\sum_{n=k}^{\infty}|b_n|z^n \quad (z \in \mathcal{U})\,. \tag{24}$$

We define modified Hadamard product for the functions f, g as follows:

$$(f \circledast g)(z) = z^p - \sum_{n=k}^{\infty}|a_n||b_n|z^n \quad (z \in \mathcal{U})\,.$$

Theorem 8 If $f \in \mathcal{W}_\alpha\,(\varphi; A, B)$ and $g \in \mathcal{W}_\beta\,(\psi; C, D)\,$, then $f \circledast g \in \mathcal{W}_\pi\,(\varphi * \psi; E, F)\,$, where

$$\delta(\pi, E, F) \ge \delta(\alpha, A, B)\delta(\beta, C, D). \tag{25}$$

Proof Let functions f, g of form (24) belong to the classes $\mathcal{W}_\alpha\,(\varphi; A, B)$ and $\mathcal{W}_\beta(\psi;$ $C, D)$, respectively. From Theorem 2 we have

$$\sum_{n=2}^{\infty}\frac{\alpha_n}{\delta(\alpha; A, B)}\,|a_n| \le 1, \quad \sum_{n=2}^{\infty}\frac{\beta_n}{\delta(\beta; C, D)}\,|b_n| \le 1.$$

Thus, by (25) we obtain

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\alpha_n\beta_n}{\delta(\pi, E, F)}\,|a_n b_n| &\le \sum_{n=2}^{\infty}\frac{\alpha_n\beta_n}{\delta(\alpha; A, B)\delta(\beta; C, D)}\,|a_n|\,|b_n| \\ &\le \sum_{n=2}^{\infty}\frac{\alpha_n}{\delta(\alpha; A, B)}\,|a_n|\sum_{n=2}^{\infty}\frac{\beta_n}{\delta(\beta; C, D)}\,|b_n| \le 1.\end{aligned}$$

Applying Theorem 3 we get $f \circledast g \in \mathcal{W}_\pi\,(\varphi * \psi; E, F)\,$.

Theorem 9 Let the sequence $\{\alpha_n\}$ $(n \geq 2)$ defined by (4) satisfy inequalies (15). If $f, g \in \mathcal{W}_\theta(\varphi; A, B)$, then $f \circledast g \in \mathcal{W}_\pi(\varphi; C, D)$, whenever

$$(D - C)\,\alpha_2 \geq (1 + D)\,[\delta(\theta, A, B)]^2\,. \tag{26}$$

Proof Let functions f, g of form (24) belong to the class $\mathcal{W}_\alpha(\varphi; A, B)$. Then by Theorem 2, we have

$$\sum_{n=2}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)}\,|a_n| \leq 1, \quad \sum_{n=2}^{\infty} \frac{\alpha_n}{\delta(\alpha; A, B)}\,|b_n| \leq 1.$$

Thus by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=2}^{\infty} \frac{\alpha_n}{\delta(\theta, A, B)}\sqrt{|a_n b_n|} \leq 1\,. \tag{27}$$

We have to prove that

$$\sum_{k=2}^{\infty} \alpha_n \frac{1 + D}{D - C}\,|a_n b_n| \leq 1\,.$$

Therefore, by (27), it is sufficient to show that

$$\frac{1 + D}{D - C}\,|a_n b_n| \leq \frac{1}{\delta(\theta, A, B)}\sqrt{|a_n b_n|} \quad (n \geq 2)$$

or equivalently,

$$\sqrt{|a_n b_n|} \leq \frac{D - C}{(1 + D)\,\delta(\theta, A, B)} \quad (n \geq 2).$$

Note that, from (27), we have

$$\sqrt{|a_n b_n|} \leq \frac{\delta(\theta, A, B)}{\alpha_n} \quad (n \geq 2).$$

Consequently, we need only to prove that

$$\frac{D - C}{(1 + D)\,\delta(\theta, A, B)} \geq \frac{\delta(\theta, A, B)}{\alpha_n} \quad (n \geq 2),$$

and this inequality follows from (26) and (15).

We note that for functions $f \in \mathcal{W}_\alpha(\varphi; A, B)$ and $g \in \mathcal{W}_{\pi-\alpha}(\psi; C, D)$ we have $f * g = f \circledast g$. Thus from Theorem 8 are obtain the following corollary.

Corollary 2 If $f \in \mathcal{W}_\alpha(\varphi; A, B)$ and $g \in \mathcal{W}_{\pi-\alpha}(\psi; C, D)$, then $f * g \in \mathcal{W}_\pi(\varphi * \psi; E, F)$, whenever

$$\delta(\pi, E, F) \geq \delta(\alpha, A, B)\delta(\pi - \alpha, C, D).$$

Putting $\theta = \pi$ in Theorem 9 we obtain the following corollary.

Corollary 3 Let the sequence $\{\alpha_n\}$ $(n \geq 2)$ defined by (4) satisfies inequalies (15). If $f, g \in \mathcal{W}_\pi(\varphi; A, B)$, then $f \circledast g \in \mathcal{W}_\pi(\varphi; C, D)$, whenever

$$(D - C)\,(1 + B)^2\,\alpha_2 \geq (1 + D)\,(B - A)^2\,.$$

Putting $C = A$ and $D = B$ in Corollary 3, we obtain the following corollary.

Corollary 4 Let the sequence $\{\alpha_n\}$ $(n \geq 2)$ defined by (4) satisfy inequalies (15). If $f, g \in \mathcal{W}_\pi(\varphi; A, B)$, then $f \circledast g \in \mathcal{W}_\pi(\varphi; A, B)$, whenever

$$(1+B)\,\alpha_2 \geq B - A.$$

Since for $\alpha = \beta = \pi$, $E = A$ and $F = B$, condition (25) is true, then from Theorem 8 we have the following corollary.

Corollary 5 If $f \in \mathcal{W}_\pi(\varphi; A, B)$ and $g \in \mathcal{W}_\pi(\psi; C, D)$, then

$$f \circledast g \in \mathcal{W}_\pi(\varphi * \psi; A, B) \cap \mathcal{W}_\pi(\varphi * \psi; C, D).$$

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