ELSEVIER Applied Mathematics and Computation 103 (1999) 1–13

APPLIED MATHEMATICS AND COMPUTATION

Classes of analytic functions associated with the generalized hypergeometric function

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Abstract

Using the generalized hypergeometric function, we introduce and study a class of analytic functions with negative coefficients. Coefficients estimates, distortion theorems, extreme points, and the radii of convexity and starlikeness for this class are given. Relevant connections of these results with those in several earlier investigations are indicated. © 1999 Elsevier Science Inc. All rights reserved.

AMS Classification: Primary: 30C45; 26A33; Secondary: 33C20

1. Introduction

Let $\mathscr{A}(p,k)$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k;\ p,k \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1}$$

which are *analytic* in $\mathscr{U} = \mathscr{U}(1)$, where $\mathscr{U}(r) = \{z\colon\ z \in \mathbb{C} \text{ and } |z| < r\}$. Also let us put

$$\mathscr{A}(p) = \mathscr{A}(p,p+1) \quad \text{and} \quad \mathscr{A} = \mathscr{A}(1).$$

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PII: S0096-3003(98)10042-5

A function f belonging to the class $\mathscr{A}(p)$ is said to be *convex* in $\mathscr{U}(r)$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad (z\in\mathscr{U}(r);\ 0<r\leqslant 1).$$

A function f belonging to the class $\mathscr{A}(p)$ is said to be *starlike of order* α in $\mathscr{U}(r)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \quad (z\in\mathscr{U}(r);\ 0<r\leqslant 1;\ 0\leqslant\alpha<p). \tag{2}$$

We denote by S_p^{c} the class of all functions in $\mathscr{A}(p)$ which are convex in $\mathscr{U}$ and by $S_p^*(\alpha)$ we denote the class of all functions in $\mathscr{A}(p)$ which are starlike of order α in $\mathscr{U}$. We also set

$$S_p^*=S_p^*(0) \quad \text{and} \quad S^*(\alpha)=S_1^*(\alpha).$$

Let $\mathscr{B}$ be a subclass of the class $\mathscr{A}$. We define the radius of starlikeness $R^*(\mathscr{B})$ and the radius of convexity $R^{\mathrm{c}}(\mathscr{B})$ for the class $\mathscr{B}$ by

$$R^*(\mathscr{B})=\inf_{f\in\mathscr{B}}(\sup\{r\in(0,1]:\ f \text{ is starlike of order 0 in } \mathscr{U}(r)\}),$$

$$R^{\mathrm{c}}(\mathscr{B})=\inf_{f\in\mathscr{B}}(\sup\{r\in(0,1]:\ f \text{ is convex in } \mathscr{U}(r)\}).$$

For analytic functions

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty}b_nz^n,$$

by $f*g$ we denote the *Hadamard product or convolution* of f and g, defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n.$$

For complex parameters $\alpha_1,\ldots,\alpha_q$ and $\beta_1,\ldots,\beta_s$ ($\beta_j\neq 0,-1,-2,\ldots;\ j=1,\ldots,s$), we define the *generalized hypergeometric function* ${}_qF_s(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;\ z)$ by

$${}_qF_s(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;\ z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{z^n}{n!}$$

$$(q\leqslant s+1;\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ z\in\mathscr{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0),\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}).\end{cases}$$

Corresponding to a function $h_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;\ z)$ defined by

$$h_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;\ z) = z^p\ {}_qF_s(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;\ z),$$

we consider a linear operator

$$H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s) : \mathscr{A}(p) \to \mathscr{A}(p),$$

defined by the convolution

$$H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z) = h_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;\ z) * f(z).$$

We observe that, for a function f of the form (1), we have

$$H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z) = z^p + \sum_{n=k}^{\infty}\Gamma_n a_n z^n, \tag{3}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-p}\cdots(\alpha_q)_{n-p}}{(\beta_1)_{n-p}\cdots(\beta_s)_{n-p}(n-p)!}. \tag{4}$$

Thus, after some calculations, we obtain

$$\alpha_1 H_p(\alpha_1+1,\alpha_2,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z) = zH_p'(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z) + (\alpha_1-p)H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z). \tag{5}$$

The linear operator $H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)$ includes various other linear operators which were considered in earlier works. In particular, for $p = s = 1$ and $q = 2$, we obtain the linear operator:

$$\mathscr{F}(\alpha_1,\alpha_2,\beta_1)f(z) = H_1(\alpha_1,\alpha_2;\ \beta_1)f(z), \tag{6}$$

which was defined by Hohlov [1]. Putting, moreover, $\alpha_2 = 1$, we obtain the Carlson–Shaffer operator:

$$\mathscr{L}(\alpha_1,\beta_1)f(z) = H_1(\alpha_1,1;\ \beta_1)f(z), \tag{7}$$

which was introduced by Carlson and Shaffer [2].

Ruscheweyh [3] introduced an operator $\mathscr{D}^\lambda : \mathscr{A} \to \mathscr{A}$, defined by the convolution:

$$\mathscr{D}^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda \geqslant -1;\ f \in \mathscr{A}), \tag{8}$$

which implies that

$$\mathscr{D}^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \quad (n \in \mathbb{N}_0). \tag{9}$$

Next we recall the generalized Bernardi–Libera–Livingston integral operator $J_\nu : \mathscr{A} \to \mathscr{A}$, defined by (cf. [4–6])

$$J_v f(z) = \frac{v+1}{z^v} \int_0^z t^{v-1} f(t)\, \mathrm{d}t \quad (v > -1;\ f \in \mathscr{A}). \tag{10}$$

Thus we observe from Eqs. (8) and (10) that

$$\mathscr{D}^\lambda f(z) = H_1(1+\lambda, 1;\ 1) \tag{11}$$

and

$$J_v f(z) = H_1(1+v, 1;\ v+2). \tag{12}$$

Now we recall here the fractional derivative operator D_z^λ considered by Owa [7] (see also [8]).

Definition 1. *The fractional integral of order* λ is defined, for a function f, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} \mathrm{d}\zeta \quad (\lambda < 0), \tag{13}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. *The fractional derivative of order* λ is defined, for a function f, by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}\, \mathrm{d}\zeta \quad (0 \leqslant \lambda < 1), \tag{14}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n + \lambda$ is defined, for a function f, by

$$D_z^{n+\lambda} f(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} D_z^\lambda f(z) \quad (0 \leqslant \lambda < 1;\ n \in \mathbb{N}_0). \tag{15}$$

By using these definitions of fractional calculus, Srivastava and Owa [9] (see also [10–12]) defined the linear operator $\Omega^\lambda : \mathscr{A} \to \mathscr{A}$ by

$$\Omega^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \quad (\lambda \neq 2, 3, 4, \ldots;\ f \in \mathscr{A}). \tag{16}$$

Then it is easily observed that

$$\Omega^\lambda f(z) = H_1(2, 1;\ 2-\lambda) f(z). \tag{17}$$

Kim and Srivastava [13] investigated the class of functions $f \in \mathscr{A}$ such that $\mathscr{L}(a,c)f(z) \in S^*(\alpha)$, that is,

$$a\frac{\mathscr{L}(a+1,c)f(z)}{\mathscr{L}(a,c)f(z)} + 1 - a \prec \frac{1+(1-2\alpha)z}{1-z} \tag{18}$$

in terms of subordination.

Let $p,k,q,s \in \mathbb{N}$ and suppose that the parameters $\alpha_1,\ldots,\alpha_q$ and $\beta_1,\ldots,\beta_s$ are positive real numbers. Also let

$$0 \leqslant B \leqslant -1 \quad \text{and} \quad -B \leqslant A < B.$$

We denote by $V_k^p(q,s;A,B)$ the class of functions f of the form:

$$f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n \quad (a_n \geqslant 0;\ n = k, k+1, k+2, \ldots), \tag{19}$$

which also satisfy the following condition:

$$\alpha_1 \frac{H_p(\alpha_1+1,\alpha_2,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)}{H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)} + p - \alpha_1 \prec p\frac{1+Az}{1+Bz}. \tag{20}$$

In particular, for $q = s+1$ and $\alpha_{s+1} = 1$, we write

$$V_k^p(s;A,B) = V_k^p(s+1,s;A,B).$$

Classes of functions of the form given by Eq. (19) were investigated by (among others) Srivastava et al. [10,14] and Dziok [12] (see also [11]).

The main object of this paper is to investigate the coefficients estimates, distortion theorems, extreme points, and the radii of convexity and starlikeness for the class $V_k^p(q,s;\ A,B)$.

2. Coefficient estimates

We begin by listing two lemmas, which will be useful later on. By Eq. (5) for $A = -B = 1$ and a function f of the form (19), the condition (20) is equivalent to

$$H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z) \in S_p^*. \tag{21}$$

Thus we have the following lemma.

Lemma 1. *If* $\alpha_j = \beta_j$ $(j = 1,\ldots,s)$, *then*

$$V_k^p(s;1,-1) \subset S_p^*.$$

By the definition of the class $V_k^p(q,s;\ A,B)$, we have the following lemma.

Lemma 2. *If* $A_1 \leqslant A_2$ *and* $B_1 \geqslant B_2$, *then*

$$V_k^p(q,s;A_1,B_1) \subset V_k^p(q,s;A_2,B_2) \subset V_k^p(q,s;1,-1). \tag{22}$$

Theorem 1. *A function* f *of the form (19) belongs to the class* $V_k^p(q,s;\ A,B)$ *if and only if*

$$\sum_{n=k}^{\infty} C_n a_n \leqslant M \quad (C_n = \{(B+1)n-(A+1)p\}\Gamma_n;\ M = p(B-A)), \tag{23}$$

where Γ_n *is defined by Eq. (4).*

Proof. Let a function f of the form (19) belong to the class $V_k^p(q,s;\ A,B)$. By Eq. (20) and the definition of subordination, we have

$$\alpha_1 \frac{H_p(\alpha_1+1,\alpha_2,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z)}{H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f(z)} + p - \alpha_1 = p\frac{1+A\omega(z)}{1+B\omega(z)},$$

where $\omega(0)=0$ and $|\omega(z)|<1$ for $z \in \mathscr{U}$. Thus we obtain (for $z \in \mathscr{U}$)

$$\left| \frac{\alpha_1\{H_{p,q,s}(\alpha_1+1)f(z) - H_{p,q,s}(\alpha_1)f(z)\}}{\alpha_1 B H_{p,q,s}(\alpha_1+1)f(z) - (Ap+(\alpha_1-p)B)H_{p,q,s}(\alpha_1)f(z)} \right| < 1, \tag{24}$$

where, for convenience,

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s).$$

Hence, by Eq. (3), we have

$$\left| \frac{\sum_{n=k}^{\infty}(n-p)\Gamma_n a_n z^{n-p}}{M - \sum_{n=k}^{\infty}(Bn-Ap)\Gamma_n a_n z^{n-p}} \right| < 1 \quad (z \in \mathscr{U}),$$

where Γ_n is defined by Eq. (4). Putting $z=r$ $(0 \leqslant r < 1)$, we obtain

$$\sum_{n=k}^{\infty}(n-p)\Gamma_n a_n r^{n-p} < M - \sum_{n=k}^{\infty}(Bn-Ap)\Gamma_n a_n r^{n-p},$$

which, upon letting $r \to 1-$, readily yields the assertion (23).

In order to prove the converse, let a function f of the form (19) satisfy the condition (23). Then, in view of Eq. (24), it is sufficient to prove that

$$\begin{aligned} &\left|\alpha_1\left(H_{p,q,s}(\alpha_1+1)f(z) - H_{p,q,s}(\alpha_1)f(z)\right)\right| \\ &\quad - \left|\alpha_1 B H_{p,q,s}(\alpha_1+1)f(z) - (Ap+(\alpha_1-p)B)H_{p,q,s}(\alpha_1)f(z)\right| < 0 \quad (z \in \mathscr{U}). \end{aligned} \tag{25}$$

Indeed, letting $|z| = r$ $(0<r<1)$, we have

$$
\begin{aligned}
&\left|\alpha_1\left(H_{p,q,s}(\alpha_1+1)f(z)-H_{p,q,s}(\alpha_1)f(z)\right)\right| \\
&\quad -\left|\alpha_1 B H_{p,q,s}(\alpha_1+1)f(z)-(Ap+(\alpha_1-p)B)H_{p,q,s}(\alpha_1)f(z)\right| \\
&= \left|\sum_{n=k}^{\infty}(n-p)\Gamma_n a_n z^n\right| - \left|M-\sum_{n=k}^{\infty}(Bn-Ap)\Gamma_n a_n z^n\right| \\
&\leqslant \sum_{n=k}^{\infty}(n-p)\Gamma_n a_n r^n - \left(M-\sum_{n=k}^{\infty}(Bn-Ap)\Gamma_n a_n r^n\right) \\
&= r^p\left(\sum_{n=k}^{\infty}C_n a_n r^{n-p}-M\right) < \sum_{n=k}^{\infty}C_n a_n - M \leqslant 0,
\end{aligned}
$$

so that $f \in V_k^p(q,s;\ A,B)$.

Since the expression C_n defined with Eq. (23) is a decreasing function with respect to β_j $(j=1,\ldots,s)$ and an increasing function with respect to α_l $(l = 1,\ldots,q)$, from Theorem 1 we obtain:

Corollary 1. *If $l \in \{1,\ldots,q\},\ j\in\{1,\ldots,s\},\ 0\leqslant \alpha_l' \leqslant \alpha_l$, and $\beta_j' \geqslant \beta_j$, then the class $V_k^p(q,s;\ A,B)$ (for the parameters $\alpha_1,\ldots,\alpha_q$ and $\beta_1,\ldots,\beta_s$) is included in the class $V_k^p(q,s;\ A,B)$ for the parameters*

$$\alpha_1,\ldots,\alpha_{l-1},\alpha_l',\alpha_{l+1},\ldots,\alpha_q \text{ and } \beta_1,\ldots,\beta_{j-1},\beta_j',\beta_{j+1},\ldots,\beta_s.$$

By Theorem 1, we also have the following corollary.

Corollary 2. *If a function f of the form (19) belongs to the class $V_k^p(q,s;\ A,B)$, then*

$$a_n \leqslant \frac{M}{C_n} \quad (n=k,k+1,k+2,\ldots),$$

where C_n and M are defined with Eq. (23). The result is sharp, the functions f_n of the form:

$$f_n(z) = z^p - \frac{M}{C_n}z^n \quad (n=k,\ k+1,k+2,\ldots) \tag{26}$$

being the extremal functions.

3. Distortion theorems and extreme points

Theorem 2. *Let a function f of the form (19) belong to the class $V_k^p(q,s;\ A,B)$. If the sequence $\{C_n\}$ is nondecreasing, then*

$$r^p - \frac{M}{C_k}r^k \leqslant |f(z)| \leqslant r^p + \frac{M}{C_k}r^k \quad (|z| = r < 1). \tag{27}$$

If the sequence $\{(C_n/n)\}$ is nondecreasing, then

$$pr^{p-1} - \frac{kM}{C_k} r^{k-1} \leqslant |f'(z)| \leqslant pr^{p-1} + \frac{kM}{C_k} r^{k-1} \quad (|z| = r < 1), \tag{28}$$

where C_n and M are defined with Eq. (23). The result is sharp, with the extremal function f_k of the form (26).

Proof. Let a function f of the form (19) belong to the class $V_k^p(q, s;\ A, B)$. If the sequence $\{C_n\}$ is nondecreasing and positive, by Corollary 2 we have

$$\sum_{n=k}^{\infty} a_n \leqslant \frac{M}{C_k}, \tag{29}$$

and if the sequence $\{C_n/n\}$ is nondecreasing and positive, by Corollary 2 we have

$$\sum_{n=k}^{\infty} n a_n \leqslant \frac{kM}{C_k}. \tag{30}$$

Making use of conditions (29) and (30), in conjunction with the definition (19), we readily obtain the assertions (27) and (28) of Theorem 2.

Corollary 3. *Let a function f of the form (19) belong to the class $V_k^p(s;\ A, B)$. If $\alpha_1 \leqslant \beta_1 + 1$, and $\alpha_j \leqslant \beta_j$ $(j = 2, \ldots, s)$, then the assertion (27) holds true. Moreover, if $\alpha_1 \leqslant \beta_1$, then the assertion (28) holds true.*

Proof. If $q = s$, $\alpha_1 \leqslant \beta_1 + 1$, and $\alpha_j \leqslant \beta_j$ $(j = 2, \ldots, s)$, then the sequence $\{C_n\}$ is nondecreasing. Moreover, if $\alpha_1 \leqslant \beta_1$, then the sequence $\{C_n/n\}$ is nondecreasing. Thus, by Theorem 2, we have Corollary 3.

Theorem 3. *Let C_n and M be defined with Eq. (23) and let us put*

$$f_{k-1}(z) = z^p \quad \text{and} \quad f_n(z) = z^p - \frac{M}{C_n} z^n \ \ (n = k,\ k+1, k+2, \ldots). \tag{31}$$

A function f belongs to the class $V_k^p(q, s;\ A, B)$ if and only if it is of the form:

$$f(z) = \sum_{n=k-1}^{\infty} \gamma_n f_n(z) \quad (z \in \mathcal{U}), \tag{32}$$

where

$$\sum_{n=k-1}^{\infty} \gamma_n = 1 \quad (\gamma_n \geqslant 0;\ n = k-1, k, k+1, \ldots).$$

Proof. Let a function f of the form (19) belong to the class $V_k^p(q, s;\ A, B)$. Setting

$$\gamma_n = \frac{C_n}{M} a_n \quad (n = k, k+1, k+2, \ldots) \quad \text{and} \quad \gamma_{k-1} = 1 - \sum_{n=k}^{\infty} \gamma_n,$$

we see that

$$\gamma_n \geqslant 0 \quad (n = k, k+1, k+2, \ldots).$$

Since $\gamma_{k-1} \geqslant 0$, by Eq. (23), we thus have

$$\begin{aligned}
\sum_{n=k-1}^{\infty} \gamma_n f_n(z) &= \left(1 - \sum_{n=k}^{\infty} \frac{C_n}{M} a_n\right) z^p + \sum_{n=k}^{\infty} \left(z^p - \frac{M}{C_n} z^n\right) \frac{C_n}{M} a_n \\
&= z^p - \sum_{n=k}^{\infty} \frac{C_n}{M} a_n z^p + \sum_{n=k}^{\infty} \frac{C_n}{M} a_n z^p - \sum_{n=k}^{\infty} a_n z^n \\
&= z^p - \sum_{n=k}^{\infty} a_n z^n = f(z),
\end{aligned}$$

and the condition holds true.

Next let a function f satisfy the condition (32). Then we have

$$\begin{aligned}
f(z) &= \sum_{n=k-1}^{\infty} \gamma_n f_n(z) = \gamma_{k-1} f_{k-1} + \sum_{n=k}^{\infty} \gamma_n f_n(z) \\
&= \left(1 - \sum_{n=k}^{\infty} \gamma_n\right) z^p + \sum_{n=k}^{\infty} \gamma_n \left(z^p - \frac{M}{C_n} z^n\right) \\
&= z^p - \sum_{n=k}^{\infty} \gamma_n \frac{M}{C_n} z^n.
\end{aligned}$$

Thus the function f is of the form (19), where

$$a_n = \frac{M\gamma_n}{C_n} \quad (n = k, k+1, k+2, \ldots).$$

It is sufficient to prove that the condition (23) holds true. Since

$$\sum_{n=k}^{\infty} C_n a_n = \sum_{n=k}^{\infty} M\gamma_n = M(1 - \gamma_{k-1}) \leqslant M,$$

the required condition is indeed true.

By Theorem 3, we obtain the following corollary.

Corollary 4. *The class* $V_k^p(q, s;\ A, B)$ *is convex. The extremal points are functions of the form (31).*

4. The radii of convexity and starlikeness

Theorem 4. *The radius of starlikeness for the class* $V_k^p(q,s;\ A,B)$ *is given by*

$$R^*(V_k^p(q,s;A,B)) = \inf_{n\geqslant k}\left(\frac{p}{n}\frac{C_n}{M}\right)^{1/(n-p)}, \tag{33}$$

where C_n *and* M *are defined with Eq. (23). The result is sharp.*

Proof. By Eq. (2), the function f of the form (19) is starlike in the disk $\mathscr{U}(r)$, if

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p \quad (z\in\mathscr{U}(r);\ 0<r\leqslant 1). \tag{34}$$

Since

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{\sum_{n=k}^{\infty}(n-p)a_n z^n}{z^p + \sum_{n=k}^{\infty}a_n z^n}\right| \leqslant \left|\frac{\sum_{n=k}^{\infty}(n-p)a_n z^{n-p}}{1-\sum_{n=k}^{\infty}a_n z^{n-p}}\right|,$$

putting $|z| = r$, the condition (34) is true if

$$\sum_{n=k}^{\infty}\frac{n}{p}a_n r^{n-p}\leqslant 1. \tag{35}$$

By Theorem 1, we have

$$\sum_{n=k}^{\infty}\frac{C_n}{M}a_n\leqslant 1,$$

where C_n and M are defined with Eq. (23). Thus the condition (35) is true if

$$\frac{n}{p}r^{n-p}\leqslant\frac{C_n}{M}\quad (n=k,k+1,k+2\ldots),$$

that is, if

$$r\leqslant\left(\frac{p}{n}\frac{C_n}{M}\right)^{1/(n-p)}\quad (n=k,k+1,k+2,\ldots).$$

It follows that any function $f\in V_k^p(q,s;\ A,B)$ is starlike in the disk $\mathscr{U}\ (R^*(V_k^p(q,s;A,B)))$, where $R^*(V_k^p(q,s;A,B))$ is defined by Eq. (33).

Corollary 5.

$$R^*(V_k^p(s;A,B)) = \begin{cases} 1 & (\alpha_j\geqslant\beta_j;\ j=1,\ldots,s),\\ \min\limits_{n\geqslant k}\left(\frac{p}{n}\frac{C_n}{M}\right)^{1/(n-p)} & (\alpha_j<\beta_j;\ j=1,\ldots,s),\end{cases} \tag{36}$$

where C_n *and* M *are defined with Eq. (23). The result is sharp.*

Proof. By Corollary 1 and Lemmas 1 and 2, we have

$$V_k^p(s;\ A,B)\subset S_p^*\quad (\alpha_j\geqslant\beta_j,\ j=1,\ldots,s).$$

By Theorem 4, any function $f \in V_k^p(s;\ A,B)$ is starlike in the disk $\mathscr{U}(r)$, where

$$r = \inf_{n \geqslant k} (d_n)^{1/(n-p)} \qquad \left(d_n = \frac{p}{n}\frac{C_n}{M} \right).$$

Since, for $\alpha_j < \beta_j$ $(j = 1, \ldots, s)$, we have

$$\lim_{n\to\infty} d_n = d < 1,\ \lim_{n\to\infty} (d_n)^{1/(n-p)} = 1 \quad \text{and} \quad d_n > 0$$
$$(n = k, k+1, k+2, \ldots),$$

the infimum of the set $\left\{ (d_n)^{1/(n-p)}:\ n \geqslant k \right\}$ is realized for an element of this set for some $n = n_0$. Moreover, the function

$$f_{n_0}(z) = z^p - \frac{M}{C_{n_0}} z^{n_0},$$

belongs to the class $V_k^p(s;\ A,B)$, and for $z = \left(d_{n_0}\right)^{1/(n_0-p)}$ we have

$$\mathrm{Re}\left(\frac{z f'_{n_0}(z)}{f_{n_0}(z)} \right) = 0.$$

Thus the result is sharp.

From Theorem 4 we can obtain direct estimation of the radius of starlikeness for the class $V_k^p(s;\ A,B)$ with $\alpha_j \leqslant \beta_j$ $(j = 1, \ldots, s)$.

Corollary 6. *If a function f belongs to the class $V_k^p(s;\ A,B)$ with $\alpha_j \leqslant \beta_j$ $(j = 1, \ldots, s)$, then f is starlike in the disk $\mathscr{U}(r^*)$, where*

$$r^* = \frac{\alpha_1 \cdots \alpha_s}{\beta_1 \cdots \beta_s}. \tag{37}$$

Proof. Since

$$\frac{(B+1)n - (A+1)p}{n(B-A)} > 1 \quad (n = k, k+1, k+2, \ldots) \tag{38}$$

and, for $\alpha_j \leqslant \beta_j$ $(j = 1, \ldots, s)$,

$$\Gamma_n = \frac{(\alpha_1)_{n-p} \cdots (\alpha_s)_{n-p}}{(\beta_1)_{n-p} \cdots (\beta_s)_{n-p}} \leqslant \frac{\alpha_1^{n-p} \cdots \alpha_s^{n-p}}{\beta_1^{n-p} \cdots \beta_s^{n-p}} = \left(\frac{\alpha_1 \cdots \alpha_s}{\beta_1 \cdots \beta_s} \right)^{n-p},$$

we obtain

$$\begin{aligned} R^*(V_k^p(q,s;A,B)) &= \inf_{n \geqslant k} \left(\frac{p}{n}\frac{C_n}{M} \right)^{1/(n-p)} \\ &= \inf_{n \geqslant k} \left(\frac{(B+1)n - (A+1)p}{n(B-A)} \Gamma_n \right)^{1/(n-p)} \\ &\geqslant \frac{\alpha_1 \cdots \alpha_s}{\beta_1 \cdots \beta_s}, \end{aligned}$$

which completes the proof of Corollary 6.

Theorem 5. *The radius of convexity for the class $V_k^p(q,s;\ A,B)$ is given by*

$$R^{\mathrm{c}}(V_k^p(q,s;\ A,B)) = \inf_{n\geqslant k}\left(\frac{p^2}{n^2}\frac{C_n}{M}\right)^{1/(n-p)}, \tag{39}$$

where C_n and M are defined with Eq. (23). The result is sharp.

Proof. The proof is analogous to that of Theorem 4, and we omit the details.

From Theorem 5 we can obtain direct estimation of the radius of convexity for the class $V_k^p(s;\ A,B)$.

Corollary 7. *If a function f belongs to the class*

$$V_k^p(s;A,B) \quad (\alpha_1 \leqslant \beta_1+1;\ \alpha_1 \leqslant p+1;\ \alpha_j \leqslant \beta_j\ (j=2,\ldots,s)),$$

then f is convex in the disk $\mathcal{U}(r)$, where

$$r = \frac{(\alpha_1-1)\alpha_2\cdots\alpha_s}{\beta_1\beta_2\cdots\beta_s}. \tag{40}$$

Proof. For $\alpha_1 \leqslant p+1$, we have

$$\frac{p}{n}\left(\frac{\alpha_1+n-p-1}{\alpha_1-1}\right) \geqslant 1.$$

Since

$$\begin{aligned}\frac{p}{n}\Gamma_n &= \frac{p}{n}\left(\frac{(\alpha_1)_{n-p}\cdots(\alpha_s)_{n-p}}{(\beta_1)_{n-p}\cdots(\beta_s)_{n-p}}\right)\\ &= \frac{p}{n}\left(\frac{\alpha_1+n-p-1}{\alpha_1-1}\right)\frac{(\alpha_1-1)_{n-p}(\alpha_2)_{n-p}\cdots(\alpha_s)_{n-p}}{(\beta_1)_{n-p}\cdots(\beta_s)_{n-p}}\\ &\geqslant \frac{p}{n}\left(\frac{\alpha_1+n-p-1}{\alpha_1-1}\right)\frac{(\alpha_1-1)^{n-p}\alpha_2^{n-p}\cdots\alpha_s^{n-p}}{\beta_1^{n-p}\cdots\beta_s^{n-p}}\\ &\geqslant \left(\frac{(\alpha_1-1)\alpha_2\cdots\alpha_s}{\beta_1\cdots\beta_s}\right)^{n-p},\end{aligned}$$

by Eq. (38) we have

$$\begin{aligned}R^{\mathrm{c}}(V_k^p(q,s;A,B)) &= \inf_{n\geqslant k}\left(\frac{p^2}{n^2}\frac{C_n}{M}\right)^{1/(n-p)}\\ &= \inf_{n\geqslant k}\left(\frac{(B+1)n-(A+1)p}{n(B-A)}\frac{p}{n}\Gamma_n\right)^{1/(n-p)}\\ &\geqslant \frac{(\alpha_1-1)\alpha_2\cdots\alpha_s}{\beta_1\cdots\beta_s},\end{aligned}$$

which completes the proof of Corollary 7.

Acknowledgements

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP 0007353.

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