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On upper bounds for positive solutions of semilinear equations ☆

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Abstract

Suppose that E is a bounded domain of class $C^{2,\lambda}$ in $\mathbb{R}^d$ and L is a uniformly elliptic operator in E. The set $\mathscr{U}$ of all positive solutions of the equation $Lu = \psi(u)$ in E was investigated by a number of authors for various classes of functions ψ. In Dynkin and Kuznetsov (Comm. Pure Appl. Math. 51 (1998) 897) we defined, for every Borel subset Γ of ∂E, two such solutions $u_\Gamma \leqslant w_\Gamma$. We also introduced a class of solutions u_ν in 1–1 correspondence with a certain class $\mathscr{N}_0$ of σ-finite measures ν on ∂E. With every $u \in \mathscr{U}$ we associated a pair (Γ, ν) where Γ is a Borel subset of ∂E and $\nu \in \mathscr{N}_0$. We called this pair the fine boundary trace of u and we denoted in $\mathrm{tr}(u)$.

Let $u \oplus v$ stand for the maximal solution dominated by $u + v$. We say that u belongs to the class $\mathscr{E}_{L,\psi}$ if the condition $\mathrm{tr}(u) = (\Gamma, \nu)$ implies that $u \leqslant w_\Gamma \oplus u_\nu$ and we say that u belongs to $\mathscr{E}^*_{L,\psi}$ if the condition $\mathrm{tr}(u) = (\Gamma, \nu)$ implies that $u \geqslant u_\Gamma \oplus u_\nu$.

It was proved in Dynkin and Kuznetsov (1998) that, under minimal assumptions on L and ψ, the class $\mathscr{E}^*_{L,\psi}$ contains all bounded domains. It follows from results of Mselati (Thése de Doctorat de l'Université Paris 6, 2002; C.R. Acad. Sci. Paris Sér. I 332 (2002); Mem. Amer. Math. Soc. (2003), to appear), that all E of the class C^4 belong to $\mathscr{E}_{\Delta,\psi}$ where Δ is the Laplacian and $\psi(u) = u^2$. [Mselati proved that, in his case, $u_\Gamma = w_\Gamma$ and therefore the condition $\mathrm{tr}(u) = (\Gamma, \nu)$ implies $u = u_\Gamma \oplus u_\nu = w_\Gamma \oplus u_\nu$.]

By modifying Mselati's arguments, we extend his result to $\psi(u) = u^\alpha$ with $1 < \alpha \leqslant 2$ and all bounded domains of class $C^{2,\lambda}$.

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We start from proving a general localization theorem: $E \in \mathscr{E}_{L,\psi}$ under broad assumptions on L, ψ if, for every $y \in \partial E$ there exists a domain $E' \in \mathscr{E}_{L,\psi}$ such that $E' \subset E$ and $\partial E \cap \partial E'$ contains a neighborhood of y in ∂E.

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1. Introduction

1.1. Equation $Lu = \psi(u)$. Before we state the results, we give a brief description of our setting (which is the same as in [Dy02]).

We consider a second-order differential operator

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x^i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u(x)}{\partial x_i} \tag{1.1}$$

in a bounded smooth domain E. (We use this name for domains of class $C^{2,\lambda}$.) We assume that $a_{ij}(x) = a_{ji}(x)$ and that:

1.1.A. There exists a constant $\kappa > 0$ such that

$$\sum a_{ij}(x) t_i t_j \geqslant \kappa \sum t_i^2 \quad \text{for all } x \in E, t_1, \dots, t_d \in \mathbb{R}.$$

1.1.B. All coefficients $a_{ij}(x)$ and $b_i(x)$ are Hölder continuous in $\bar{E}$.

Our objective is to investigate the set $\mathscr{U}$ of all positive solutions of the equation

$$Lu = \psi(u) \text{ in } E. \tag{1.2}$$

Here ψ is a function on $E \times \mathbb{R}_+$ and $\psi(u)$ is an abbreviation for $\psi(x, u(x))$. We assume that ψ satisfies conditions:

1.1.C. For every x, $\psi(x, \cdot)$ is convex and $\psi(x, 0) = 0$, $\psi(x, u) > 0$ for $u > 0$.

1.1.D. $\psi(x, u)$ is continuously differentiable.

1.1.E. ψ is locally Lipschitz continuous in u uniformly in x: for every $t \in \mathbb{R}_+$, there exists a constant c_t such that

$$|\psi(x, u_1) - \psi(x, u_2)| \leqslant c_t |u_1 - u_2| \text{ for all } x \in E, \ u_1, u_2 \in [0, t].$$

1.1.F. There is a constant a such that

$$\psi(x, 2u) \leqslant a \psi(x, u)$$

for all u and x.

1.1.G. Function $\frac{\partial \psi(x,u)}{\partial u}$ is continuously differentiable.

1.1.H. (The Keller–Osserman condition) $\int_N^\infty ds [\int_0^s \psi(u)\, du]^{-1/2} < \infty$ for some $N > 0$.

Condition 1.1.C implies that $\psi(x,u_1)\leqslant\psi(x,u_2)$ for all $0\leqslant u_1<u_2$ and the Keller–Osserman condition implies the existence of a maximal element in every class $\mathscr{U}(D)$.[1]

1.2. Singular points of a solution. We consider the tangent cone to $\mathscr{U}$ at point u which we define as the set of tangent vectors v to all smooth curves u_t in $\mathscr{U}$ with the properties:

(a) $u_0=u$ and $u_t\in\mathscr{U}$ for $0\leqslant t<\varepsilon$;
(b) $u_t(x)$ is monotone increasing in t.

Condition (a) implies that $Lu_t=\psi(u_t)$ for $0\leqslant t<\varepsilon$ and therefore $v(x)=\partial u_t(x)/\partial t|_{t=0}$ satisfies a linear equation

$$Lv = av \tag{1.3}$$

where

$$a=\psi'(u).$$

$\left(\text{We use abbreviation } \psi'(u) \text{ for } \frac{\partial\psi(x,u)}{\partial u}\right)$. Since ψ is monotone increasing in u, $a(x)\geqslant 0$.

Condition (b) implies that $v(x)\geqslant 0$. There exists a function $k_a(x,y)$, $x\in E$, $y\in\partial E$ (the Poisson kernel) such that every positive solution of (1.3) has an integral representation

$$h(x)=\int_{\partial E}k_a(x,y)\nu(dy), \tag{1.4}$$

where ν is a finite measure on ∂E. If a is bounded, then k_a is strictly positive. However, if a blows up sufficiently fast near $y\in\partial E$, then $k_a(x,y)=0$ for all x. If this is the case, then we call y a *point of rapid growth for a*. We say that $y\in\partial E$ is a *singular point of* $u\in\mathscr{U}$ and we write $y\in\mathrm{SG}(u)$ if y is a point of rapid growth of $\psi'(u)$, i.e., if $k_{\psi'(u)}(x,y)=0$ for all $x\in E$.[2]

1.3. Solutions u_ν, u_Γ and w_Γ. Denote by $\mathscr{H}(D)$ the class of positive solutions of the equation $Lu=0$ in D and put $\mathscr{H}=\mathscr{H}(E)$. (We call elements of $\mathscr{H}$ harmonic (or L-harmonic) functions.) The formula

$$h_\nu(x)=\int_{\partial E}k(x,y)\nu(dy) \tag{1.5}$$

defines a 1–1 correspondence between $\mathscr{H}$ and the set $\mathscr{M}(\partial E)$ of all finite measures on Γ. (The function $k(x,y)$ is a special case of the kernel $k_a(x,y)$ corresponding to the value $a=0$.)

[1] $U(D)$ stands for the set of all positive solutions of the equation $Lu=\psi(u)$ in D.
[2] An equivalent probabilistic definition of singular points is given in Section 3.

Put $\nu \in \mathcal{N}_1$ if the integral equation[3]

$$u + G\psi(u) = h_\nu$$

has a positive solution (we denote it u_ν). Put $\nu \in \mathcal{N}_0$ if there exists a monotone increasing sequence of measures $\nu_n \in \mathcal{N}_1$ such that $\nu_n \uparrow \nu$. The mapping $\nu \to u_\nu$ from $\mathcal{N}_1$ to $\mathcal{U}$ can be continued to a monotone mapping from $\mathcal{N}_0$ to $\mathcal{U}$. We call a solution u_ν moderate if $\nu \in \mathcal{N}_1$ and σ-moderate if $\nu \in \mathcal{N}_0$.

For every Borel set $B \subset \partial E$, we define $\mathcal{N}_1(B)$ as the class of all measures $\nu \in \mathcal{N}_1$ that do not charge B^c. It is proved that the supremum $u(x)$ of $u_\mu(x)$ over all $\mu \in \mathcal{N}_1(B)$ belongs to $\mathcal{U}$. We denote it u_B.

For every compact subset K of ∂E there exists a maximal solution w_K of the problem

$$Lu = \psi(u) \text{ in } E,$$

$$u = 0 \text{ on } \partial E \setminus K. \tag{1.6}$$

To every Borel set $\Gamma \subset \partial E$, there corresponds a solution w_Γ equal to the supremum of w_K over all compact subsets K of Γ. We have

$$w_\Gamma \geqslant u_\nu \text{ for all } \nu \in \mathcal{N}_1(\Gamma). \tag{1.7}$$

Let $u, v \in \mathcal{U}$. We denote by $u \oplus v$ the maximal solution dominated by $u + v$. If $u \geqslant v$, then $u \ominus v$ stands for the minimal solution which dominates $u - v$.

1.4. Main results. The *fine trace* $\operatorname{tr}(u)$ of a solution u is a pair (Γ, ν) where $\Gamma = \mathrm{SG}(u)$ and ν is defined on Borel sets $B \subset \partial E$ by the formula

$$\nu(B) = \sup\{\mu(B) : \mu \in \mathcal{N}_1, \mu(\Gamma) = 0, u_\mu \leqslant u\}. \tag{1.8}$$

It is proved (see [Dy02, Section 11.7.1]) that ν is a σ-finite measure which belongs to $\mathcal{N}_0$ and that

$$u_\Gamma \oplus u_\nu \leqslant u. \tag{1.9}$$

(Moreover, $u_\Gamma \oplus u_\nu$ is the maximal σ-moderate solution dominated by u.)

Recall that E belongs to the class $\mathcal{E}_{L,\psi}$ if:

(A) The condition $\operatorname{tr}(u) = (\Gamma, \nu)$ implies that $u \leqslant w_\Gamma \oplus u_\nu$.

We prove:

Theorem 1.1. *The following condition is sufficient for E to be in class $\mathcal{E}_{L,\psi}$:*

(B) *If $\operatorname{tr}(u) = (\Lambda, \nu)$ and if $\Lambda \subset \Gamma \subset \partial E$ and ν is concentrated on Γ, then $u \leqslant w_\Gamma$.*

[3] G is Green's operator for L in E.

Theorem 1.2. *E belongs to $\mathscr{E}_{L,\psi}$ if, for every $y \in \partial E$, there exists a domain $E' \in \mathscr{E}_{L,\psi}$ such that $E' \subset E$ and $\partial E' \cap \partial E$ contains a neighborhood of y in ∂E.*

Theorem 1.3. *All bounded smooth domains belong to $\mathscr{E}_{\Delta,\psi}$ where Δ is the Laplacian and*

$$\psi(u) = u^\alpha, \quad 1 < \alpha \leqslant 2. \tag{1.10}$$

1.5. The state of the classification program. A program to describe the set $\mathscr{U}$ of all positive solutions of Eq. (1.2) has been initiated by Dynkin in the earlier 1990s. Important results in this direction were obtained by Le Gall and by Marcus and Véron. All σ-moderate solutions were described by Dynkin and Kuznetsov [DK98] under assumptions 1.1.A–1.1.H. In the Epilogue to the monograph [Dy02] the following key problem was formulated: Is every solution σ-moderate? As a step for a solution of this problem, a question was posed: Is $w_\Gamma = u_\Gamma$ for every Borel boundary set Γ? In [Ms02a,Ms02b,Ms03], Mselati gave positive answers to both questions in the case of the equation $\Delta u = u^2$. The present paper is inspired by his results. The main probabilistic tool of Mselati is the Brownian snake introduced by Le Gall. This tool does not work even for the equation

$$\Delta u = u^\alpha \tag{1.11}$$

with $\alpha \neq 2$ and we replace it by superdiffusions.

The problem of classification of solutions u by their traces (Γ, ν) can be split into three parts:

A. Find a lower bound in terms u_Γ and u_ν.

B. Find an upper bound in terms w_Γ and u_ν.

C. Prove that $u_\Gamma = w_\Gamma$.

Problem A was settled in [DK98]. In [Ms02a,Ms03] a solution of problem C preceeds the investigation of B. We study B independently of C. We use extensively Mselati's arguments which we translate into the language of superdiffusions. We also develop some new tools to replace the Hilbert space techniques and capacities Cap_∂ used in [Ms02a,Ms03]. In Sections 2 and 3, we describe our analytic and probabilistic tools while referring for proofs, as a rule, to [Dy02]. Theorems 1.1 and 1.2 are proved in Section 4 and Theorem 1.3 is proved in Section 5.

Proof of Theorem 1.3 is based on self-similarity of Eq. (1.11) which does not hold for a more general equation (1.2). Fresh ideas are needed to make the next step and to extend Theorem 1.3 to such equation.

Problem C remains open even for Eq. (1.11) with $\alpha \neq 2$. This is the most challenging outstanding part of the classification problem. For $\alpha = 2$, problem C is solved in Chapter 3 of Mselati's memoir [Ms03]. The solution consists of two parts: upper bounds for w_Γ in terms of a boundary capacity Cap_∂ and lower bounds for certain σ-moderate solutions. Recently, Kuznetsov [Ku03] succeeded in extending the first part to the general α (with Cap_∂ replaced by the Poisson capacity). The second part still waits for its resolution.

2. Analytic tools

2.1. Operators K_D, G_D and V_D. To every open subset D of E there correspond the Poisson operator K_D and Green's operator G_D. If D is bounded and smooth, then, for every function $\varphi \in \mathscr{B}(\partial D)$,

$$K_D\varphi(x) = \int_{\partial D} k_D(x,y)\varphi(x)\gamma(dy), \tag{2.1}$$

where $k_D(x,y)$ is the Poisson kernel in D and γ is the normalized surface area on ∂D. If φ is continuous, then $h = K_D\varphi$ is a unique solution of the Dirichlet problem

$$Lh = 0 \quad \text{in } D,$$

$$h = \varphi \quad \text{on } \partial D.$$

(We write $h = \varphi$ at $\tilde{x} \in \partial D$ if $h(x) \to \varphi(\tilde{x})$ as $x \to \tilde{x}$, $x \in D$.) For every $f \in \mathscr{B}(D)$,

$$G_D f(x) = \int_D g_D(x,y) f(y)\, dy. \tag{2.2}$$

$g_D(x,y)$ is called the Green function for L in D. If f is Hölder continuous in $\bar{D}$, then $u = G_D f$ is a unique solution of the problem

$$Lu = -f \quad \text{in } D,$$

$$u = 0 \quad \text{on } \partial D.$$

By Theorem 4.3.1 in [Dy02], if ψ satisfies conditions 1.1.C and 1.1.E, then, for every $f \in b\mathscr{B}(\bar{E})$[4] and for every $D \subset E$, there exists a unique solution of the equation

$$u + G_D\psi(u) = K_D f. \tag{2.3}$$

We denote it $V_D(f)$. We have:[5]

2.1.A. $V_D(f) \leqslant K_D(f)$, in particular, $V_D(c) \leqslant c$ for every constant c.

2.1.B. If $f \leqslant \tilde{f}$, then $V_D(f) \leqslant V_D(\tilde{f})$.

2.1.C. If $f_n \uparrow f$, then $V_D(f_n) \uparrow V_D(f)$.

2.1.D. For every D and every Borel $f : \partial D \to [0, \infty]$, the function $u = V_D(f)$ belongs to $\mathscr{U}(D)$. If D is smooth and if f is continuous in a neighborhood U of $\bar{x} \in \partial D$, then $u = f$ at $\tilde{x}$.

[4] If f is a function and $\mathscr{B}$ is a σ-algebra, then writing $f \in \mathscr{B}$ means that f is $\mathscr{B}$-measurable and $f \geqslant 0$. Writing $f \in b\mathscr{B}$ means that F is, in addition, bounded.

[5] See Section 8.2.1 in [Dy02].

Properties 2.1.B–2.1.C allow us to define $V_D(f)$ for all $f \in \mathscr{B}(\bar{D})$ by the formula

$$V_D(f) = \sup_{\ell} V_D(f \wedge \ell). \tag{2.4}$$

Eq. (2.3) and conditions 2.1.B–2.1.D are preserved under this extension. The extended operators have also the properties:

2.1.E. If $f = 0$ off a Borel set $\Gamma \subset \partial E$, then

$$V_E(f) \leqslant w_\Gamma. \tag{2.5}$$

This follows from 2.1.D and the definition of w_Γ.

2.1.F. If ψ satisfies condition 1.1.C, then

$$V_D(f_1 + f_2) \leqslant V_D(f_1) + V_D(f_2) \tag{2.6}$$

for all $f_1, f_2 \geqslant 0$.

(See [Dy02], Theorem 8.2.1.)

We write $D \Subset E$ if $\bar{D} \subset E$. We say that a sequence D_n *exhausts* E if $D_1 \Subset D_2 \Subset \cdots \Subset D_n \Subset \cdots$ and E is the union of D_n.

2.1.G. If D_n exhaust a bounded smooth domain D and if f is a continuous function on $\bar{D}$, then $V_{D_n}(f)(x) \to V_D(f)(x)$.

Proof. Let $\varepsilon_n \downarrow 0$. Functions f and $u = V_D(f)$ are uniformly continuous in $\bar{D}$. Hence, there exists $\delta_n > 0$ such that $|f(x) - f(\tilde{x})| < \varepsilon_n$ and $|u(x) - u(\tilde{x})| < \varepsilon_n$ for $|x - \bar{x}| < \delta_n$. Note that $\max_{x \in \partial D_n} d(x, \partial D) \to 0$. Therefore, if n is sufficiently large, then, for every $x \in \partial D_n$, there exists $\tilde{x} \in \partial D$ such that $|x - \tilde{x}| < \delta_n$. We have $|f(x) - f(\tilde{x})| < \varepsilon_n$, $|u(x) - u(\tilde{x})| < \varepsilon_n$. By 2.1.D, $u(\tilde{x}) = f(\bar{x})$. Hence

$$f(x) \leqslant u(x) + 2\varepsilon_n, \quad u(x) \leqslant f(x) + 2\varepsilon_n \quad \text{for all } x \in \partial D_n.$$

By 2.1.F and 2.1.A, $V_{D_n}(f) \leqslant V_{D_n}(u) + 2\varepsilon_n$ and $V_{D_n}(u) \leqslant V_{D_n}(f) + 2\varepsilon_n$. □

2.2. On solutions of $Lu = \psi(u)$. The next three propositions can also be found in Section 8.2.1 of [Dy02]:

2.2.A (Mean value property) If $u \in \mathscr{U}(D)$, then, for every $U \Subset D$, $V_U(u) = u$ and therefore $u + G_U\psi(u) = K_U u$.

2.2.B. (Comparison principle) Suppose D is bounded. Then $u \leqslant v$ assuming that u, $v \in C^2(D)$,

$$Lu - \psi(u) \geqslant Lv - \psi(v) \quad \text{in } D \tag{2.7}$$

and, for every $\tilde{x} \in \partial D$,

$$\limsup [u(x) - v(x)] \leqslant 0 \quad \text{as } x \to \tilde{x}. \tag{2.8}$$

2.2.C. If D is a bounded smooth domain and if a function $f : \partial D \to [0, \infty)$ is continuous, then $u = V_D(f)$ is a unique solution of the problem

$$Lu = \psi(u) \quad \text{in } D,$$

$$u = f \quad \text{on } \partial D. \tag{2.9}$$

We also need the following property proved in [Ms02a, Proposition 1.3.3], (see also [Ms03, Proposition 1.12]).[6]

2.2.D. Every sequence $u_n \in \mathcal{U}$ contains a subsequence u_{n_i} which converges uniformly on every $D \Subset E$ to an element of $\mathcal{U}$.

2.3. On relations between solutions of $Lu = \psi(u)$ and harmonic functions. We say that an element u of $\mathcal{U}$ is a *moderate solution* if $u \leqslant h$ for some $h \in \mathcal{H}$. The formula

$$u + G\psi(u) = h \tag{2.10}$$

establishes a 1–1 correspondence between the set $\mathcal{U}_1$ of moderate solutions and a subset $\mathcal{H}_1$ of $\mathcal{H}$: h is the minimal harmonic function dominating u, and u is the maximal solution dominated by h. Moderate solutions can be labelled by measures $v \in \mathcal{N}_1$: u_v is the solution corresponding to $h_v \in \mathcal{N}_1$ which is defined by (1.5). (The correspondence $v \leftrightarrow u_v$ is 1–1 and monotonic.)

We need the following properties of $\mathcal{H}_1$ and $\mathcal{U}_1$.

2.3.A. If $h \in \mathcal{H}_1$ and if $h' \leqslant h$ belongs to $\mathcal{H}$, then $h' \in \mathcal{H}_1$.[7]

2.3.B. $\mathcal{H}_1$ is a convex cone (that is it is closed under addition and under multiplication by positive numbers).[8]

2.3.C. $h_v = 0$ on a subset B of ∂E if and only if $v(B) = 0$.

This follows from (1.5) because $k(x, y) > 0$ for all $x \in E$, $y \in \partial E$.

2.3.D. *If $v \in \mathcal{N}_1$ and Γ is a closed subset of ∂E, then $u_v = 0$ on $O = \partial E \setminus \Gamma$ if and only if $v(O) = 0$.*

Proof. If $v(O) = 0$, then $h_v = 0$ on O by 2.3.C, and $u_v = 0$ on O because $u_v \leqslant h_v$ by (2.10).

On the other hand, if $u_v = 0$ on O, then $v(K) = 0$ for every closed subset K of O. Indeed, if η is the restriction of v to K, then $u_\eta = 0$ on Γ because $\eta(\Gamma) = 0$. We also have $u_\eta \leqslant u_v = 0$ on O. Hence $u_\eta = 0$ on ∂E. The Comparison principle 2.2.B implies that $u_\eta = 0$. Therefore $\eta = 0$. □

2.3.E. *If $v \in \mathcal{N}_1$ and Γ is a closed subset of ∂E, then $u_v \leqslant w_\Gamma$ if and only if $v \in \mathcal{N}_1(\Gamma)$.*[9]

[6] A bound for partials of u is used. Mselati considers the case $L = \Delta$ and he refers to Theorem 3.9 in [GT98]. For a general L we can refer to Theorem 6.2 in the same book.

[7] See Corollary 3.1 in [Dy02, Section 8.3.2].

[8] See Theorem 8.2.3 in [Dy02].

[9] Cf. Proposition 1.24 in [Ms03] or Proposition 1.3.10 in [Ms02a].

Proof. If $u_\nu \leqslant w_\Gamma$, then $u_\nu = 0$ on $O = \partial E \setminus \Gamma$ by the definition of w_Γ and $\nu(O) = 0$ by 2.3.D. If $\nu(O) = 0$, then $h_\nu = 0$ on O by 2.3.C and $u_\nu \leqslant h_\nu = 0$ on O. Hence $u_\nu \leqslant w_\Gamma$ by the definition of w_Γ. □

2.3.F. If $h \in \mathscr{H}$ and if $G\psi(h)(x) < \infty$ for some $x \in E$, then $h \in \mathscr{H}^1$. (See Proposition 12.2.1.A in [Dy02]).[10]

3. Probabilistic tools

3.1. ***L*-diffusion.** To every operator L subject to conditions 1.1.A–1.1.B there corresponds a strong Markov process $\xi = (\xi_t, \Pi_x)$ in E called L-diffusion. The path ξ_t is defined on a random interval $[0, \zeta)$. It is continuous and its limit ξ_ζ as $t \to \zeta$ belongs to ∂E.

There exists a function $p_t(x, y) > 0$, $t > 0$, x, $y \in E$ (called the transition density) such that:

$$\int_E p_s(x,z)\, dz p_t(z,y) = p_{s+t}(x,y) \text{ for all } s,t > 0, x,y \in E$$

and, for every $f \in \mathscr{B}(E)$,

$$\Pi_x f(\xi_t) = \int_E p_t(x,y) f(y) dy.$$

[Writing $f \in \mathscr{B}$ means that $f \geqslant 0$ is a Borel function on E.]

An L-diffusion has the following properties:

3.1.A If τ_D is the first exit time of ξ from a domain $D \subset E$, then, for every $f \in \mathscr{B}(E)$,

$$\Pi_x f(\xi_{\tau_D}) = K_D f(x) \text{ and } G_D f(x) = \Pi_x \int_0^{\tau_D} f(\xi_s) ds. \tag{3.1}$$

([Dy02, Sections 6.2.4–6.2.5].)

3.1.B. If a function $v \geqslant 0$ is continuous on $\bar{E}$ and satisfies Eq. (1.3), then

$$v(x) = \Pi_x v(\xi_\zeta) \exp\left[-\int_0^\zeta a(\xi_s)\, ds\right]. \tag{3.2}$$

([Dy02, 6.3.2.A].)

3.1.C. If D is a smooth subdomain of E and $\tau = \tau_D$, then

$$k_D(x,y) = k(x,y) - \Pi_x 1_{\tau < \zeta} k(\xi_\tau, y) \text{ for all } x \in E, y \in \partial E \cap \partial D. \tag{3.3}$$

([Dy02, 6.2.5.D].)

[10] Proposition 12.2.1.A is stated for the case $\psi(u) = u^\alpha$ but its proof is applicable to a general ψ.

3.2. h-transform. For every diffusion ξ, we denote by $\mathscr{F}^{\xi}_{\leqslant t}$ the σ-algebra generated by ξ_s, $s \leqslant t$ and by $\mathscr{F}^{\xi}$ the minimal σ-algebra which contains all $\mathscr{F}^{\xi}_{\leqslant t}$. Let $p_t(x,y)$ be the transition density of ξ and let $h \in \mathscr{H}$. To every $x \in E$ there corresponds a finite measure Π^h_x on $\mathscr{F}^{\xi}$ such that, for all $0 < t_1 < \cdots < t_n$ and every Borel sets $B_1, \ldots, B_n$,

$$\Pi^h_x\{\xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n\} = \int_{B_1} dz_1 \ldots \int_{B_n} dz_n \, p_{t_1}(x,z_1) p_{t_2-t_1}(z_1,z_2) \ldots p_{t_n-t_{n-1}}(z_{n-1},z_n) h(z_n). \tag{3.4}$$

Note that $\Pi^h_x(\Omega) = h(x)$ and therefore $\widetilde{\Pi}^h_x = \Pi^h_x / h(x)$ is a probability measure. $(\xi_t, \widetilde{\Pi}^h_x)$ is a strong Markov process with continuous paths and with the transition density

$$p^h_t(x,y) = \frac{1}{h(x)} p_t(x,y) h(y). \tag{3.5}$$

We use the following properties of h-transforms.

3.2.A. If $Y \in \mathscr{F}^{\xi}_{\leqslant t}$, then

$$\Pi^h_x Y = \Pi_x Y h(\xi_t).$$

(This follows immediately from (3.4).)

3.2.B. For every stopping time τ and every pre-τ positive Y,

$$\Pi^h_x Y 1_{\tau < \zeta} = \Pi_x Y h(\xi_\tau).$$

(See [Dy02], Lemma 7.3.1.)

3.3. Conditional L-diffusion. For every $x \in E$, $y \in \partial E$, we put $\Pi^z_x = \Pi^h_x$ where $h(\cdot) = k(\cdot, z)$. Put $Z = \xi_\zeta$. It follows from (2.1) and (3.1) that, for every $f \in \mathscr{B}(\partial E)$,

$$\Pi_x f(Z) = \int_{\partial E} k(x,z) \gamma(dz). \tag{3.6}$$

Therefore

$$\Pi_x k(y,Z) f(Z) = \int_{\partial E} k(x,z) k(y,z) \gamma(dz) \tag{3.7}$$

is symmetric in x, y.

Lemma 3.1. [11]*For every* $Y \in \mathscr{F}^{\xi}$ *and every* $f \in \mathscr{B}(\partial E)$,

$$\Pi_x Y f(Z) = \Pi_x f(Z) \widetilde{\Pi}^Z_x Y. \tag{3.8}$$

[11] Property (3.8) means that $\widetilde{\Pi}^z_x$ can be interpreted as the conditional probability distribution given that the diffusion started from x exists from E at point z.

Proof. It is sufficient to prove (3.8) for $Y = \widetilde{Y}1_{t<\zeta}$ where $Y \in \mathscr{F}^{\xi}_{\leqslant t}$. By 3.2.A,

$$\widetilde{\Pi}^z_x Y = k(x,z)^{-1}\Pi^z_x Y = k(x,z)^{-1}\Pi_x Yk(\xi_t, z).$$

Therefore the right part in (3.8) can be interpreted as

$$\int_{\Omega'} \Pi_x(d\omega')f(Z(\omega'))k(x, Z(\omega'))^{-1} \int_{\Omega} \widetilde{\Pi}_x(d\omega)\, Y(\omega)k(\xi_t(\omega), Z(\omega')).$$

Fubini's theorem and (3.7) yield that this expression is equal to

$$\int_{\Omega} \Pi_x(d\omega)\, Y(\omega) \int_{\Omega'} \Pi_x(d\omega')\, f(Z(\omega'))k(\xi_t(\omega), Z(\omega'))k(x, z(\omega'))^{-1}$$
$$= \int_{\Omega} \Pi_x(d\omega)\, Y(\omega) \int_{\partial E} f(z)k(\xi_t(\omega), z)\gamma(dz) = \Pi_x Y \Pi_{\xi_t} f(Z).$$

By the Markov property of ξ, this is equal to the left side in (3.8). □

Lemma 3.2. *Suppose that ξ' is the part of ξ in a smooth subdomain D of E and let $\tau = \tau_D$. Put $'\Pi^y_x = \Pi^h_x$ where $h(\cdot) = k_D(\cdot, y)$. We have*

$$'\Pi^y_x Y = \Pi^y_x Y 1_{\tau=\zeta} \tag{3.9}$$

for all $x \in D$, $y \in \partial E \cap \partial D$ and for all $Y \in \mathscr{F}^{\xi'}$.

Proof. It is sufficient to prove (3.9) for $Y = \widetilde{Y}1_{t<\zeta}$ where $\widetilde{Y} \in \mathscr{F}^{\xi'}_{\leqslant t}$. By 3.2.A, (3.3), 3.2.B and Markov property of ξ,

$$'\Pi^y_x Y = \Pi_x Y k_D(\xi'_t, y) = \Pi_x Y[k(\xi_t) - \Pi_{\xi t} 1_{\tau<\zeta} k(\xi_\tau, y)]$$
$$= \Pi_x Y k(\xi_t, y) - \Pi_x Y 1_{\tau<\zeta} k(\xi_\tau, y) = \Pi^y_x Y - \Pi^y_x Y 1_{\tau<\zeta}. \quad \square$$

3.4. **(L, ψ)-superdiffusion.** Suppose that to every open set $D \subset E$ end every $\mu \in \mathscr{M}(E)$[12] there corresponds a random measure (X_D, P_μ) on $\bar{E}$ such that, for every $f \in \mathscr{B}(\bar{E})$,

$$P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle V_D(f), \mu \rangle}. \tag{3.10}$$

We call the family $X = (X_D, P_\mu)$ an (L, ψ)-*superdiffusion*. The existence of a (L, ψ)-superdiffusion is proved for a convex class of functions ψ which contains

$$\psi(u) = u^\alpha, 1 < \alpha \leqslant 2. \tag{3.11}$$

[12] We denote by $\mathscr{M}(E)$ the set of all finite measures on E and we write $\langle f, \mu \rangle$ for the integral of f with respect to μ.

We put $P_x = P_{\delta_x}$ where δ_x is the unit mass concentrated at x. Clearly,

$$V_D f(x) = -\log P_x e^{-\langle f, X_D \rangle}. \tag{3.12}$$

Denote by $\mathfrak{M}$ the σ-algebra in $\mathscr{M}(E)$ generated by the functions $F(\mu) = \langle f, \mu \rangle$ with $f \in \mathscr{B}(E)$ and let $\mathscr{F}$ stand for the σ-algebra in Ω generated by X_D. It follows from (3.10) that $H(\mu) = P_\mu Y$ is $\mathfrak{M}$-measurable for every $\mathscr{F}$-measurable $Y \geqslant 0$. We use the following Markov property of X:[13]

Suppose that $Y \geqslant 0$ is measurable with respect to the σ-algebra $\mathscr{F}_{\subset D}$ generated by $X_{D'}$, $D' \subset D$ and $Z \geqslant 0$ is measurable with respect to the σ-algebra $\mathscr{F}_{\supset D}$ generated by $X_{D''}$, $D'' \supset D$. Then

$$P_\mu(YZ) = P_\mu(Y P_{X_D} Z). \tag{3.13}$$

3.5. Stochastic boundary values. Suppose that $u \in \mathscr{B}(E)$ and

$$\lim \langle u, X_{D_n} \rangle = Z_u \text{ a.s.} \tag{3.14}$$

for every sequence D_n exhausting E.[14] Then we say that Z_u is a *stochastic boundary value of* u and we write $Z_u = \mathrm{SBV}(u)$. Clearly, Z_u is determined by u uniquely up to equivalence. We have:

3.5.A. A stochastic boundary value Z_u exists for every $u \in \mathscr{U}$ and

$$u(X) = -\log P_x e^{-Z_u}.$$

In particular, for every $v \in \mathscr{N}_0$

$$u_v(X) = -\log P_x e^{-Z_v}$$

where

$$Z_v = \mathrm{SBV}(u_v).$$

If $v \in \mathscr{N}_1$, then $P_x Z_v = h_v(x)$ and

$$u_v + G\psi(u_v) = h_v.$$

3.5.B. Put $Z \in \mathfrak{Z}$ if $Z = \mathrm{SBV}(u)$ for some $u \in \mathscr{U}$. If $Z_1, Z_2 \in \mathfrak{Z}$, then $Z_1 + Z_2 \in \mathfrak{Z}$ and

$$-\log P_x e^{-Z_1 - Z_2} \leqslant -\log P_x e^{-Z_1} - \log P_x e^{-Z_2}. \tag{3.15}$$

3.5.C. If $Z_1, \ldots, Z_n, \ldots \in \mathfrak{Z}$ and if $Z_n \to Z$ a.s., then $Z \in \mathfrak{Z}$.

3.5.D. If $Z \in \mathfrak{Z}$ and if $h(x) = P_x Z$ is finite at some point $x \in E$, then $h \in \mathscr{H}_1$ and $u(x) = -\log P_x e^{-Z}$ is a moderate solution.

[13] See [Dy02, 3.1.3.D].

[14] Writing "a.s." means "P_x-a.s. for all $x \in E$".

These properties follow from the results of Chapter 9 in [Dy02] (see Theorems 9.1.1–9.1.3 and propositions in Section 9.2.2).

We also need the following lemma.

Lemma 3.3. *Suppose that f is a continuous function on the closure $\bar{E}$ of a smooth domain E. If $u = V_E(f)$, then $Z_u = \langle u, X_E \rangle$ a.s.*

Proof. Consider a sequence D_n exhausting E and put $Y_n = \langle u, X_{D_n} \rangle$, $Y = \langle u, X_E \rangle$. We have

$$P_x(e^{-Y_n} - e^{-Y})^2 = P_x e^{-2Y_n} + P_x e^{-2Y} - 2P_x E^{-Y_n - Y}.$$

By (3.10), the first two terms are equal to $e^{-V_{D_n}(2u)(x)}$ and $e^{-V_E(2u)(x)}$. By (3.13) and (3.10), the third term is equal to

$$P_x e^{-Y_n} P_{X_{D_n}} e^{-Y} = P_x e^{-\langle F, X_{D_n} \rangle} = e^{-V_{D_n}(F)(x)},$$

where $F = f + V_E(u)$ is a continuous function on $\bar{E}$. By 2.1.E, $V_{D_n}(2u) \to V_E(2u) = v_1$ and $V_{D_n}(F) \to V_E(F) = v_2$. Note that $u = V_E(u) = f$ on ∂E. By 2.1.D, $v_1 = v_2 = 2f$ on ∂E. By 2.2.B, $v_1 = v_2$ in E. Hence, $e^{-Y_n} \to e^{-Y}$ in $L^2(P_x)$. On the other hand, $e^{-Y_n} \to e^{-Z_u} P_x$-a.s. Therefore $Y = Z_u$ a.s. □

3.6. On solutions w_Γ. The range of X is a minimal closed set $\mathscr{R}$ such that, for every $D \subset E$ and every $\mu \in \mathscr{M}(\bar{E})$, the measure X_D is concentrated, P_μ-a.s., on $\mathscr{R}$. We have

$$w_\Gamma(x) = -\log P_x\{\mathscr{R} \cap \Gamma = \emptyset\}. \tag{3.16}$$

(See [Dy02], Theorem 10.1.3)

We need the following properties of w_Γ:

3.6.A. For every Borel set $\Gamma \subset \partial E$, $w_\Gamma(x)$ is equal to the infimum of $w_O(x)$ over all open subsets O of ∂E.

(This follows from relations (3.1) and (3.6) in Chapter 10 of [Dy02].)

3.6.B. If $\Gamma \subset A \cup B$, then $w_\Gamma \leqslant w_A + w_B$.

(see [Dy02], 10.1.3.A and 10.1.3.E.)

3.7. A relation between superdiffusions and conditional diffusions. [15] For every $u \in \mathscr{U}$ and every $v \in \mathscr{N}_0$,

$$P_x Z_v e^{-Z_u} = e^{-u(x)} \int_{\partial E} \Pi_x^y e^{-\Phi(u)} v(dy), \tag{3.17}$$

[15] See [Dy02, Section 9.3].

where

$$\Phi(u) = \int_0^\zeta \psi'[u(\xi_t)]dt. \tag{3.18}$$

This relation can be used to prove that the condition

$$\Pi_x^y\{\Phi(u) < \infty\} = 0 \text{ for some } x \in E$$

is sufficient and the condition

$$\Pi_x^y\{\Phi(u) < \infty\} = 0 \text{ for all } x \in E$$

is necessary for $y \in \partial E$ to be singular for u.[16] (Recall that an analytic definition of singular points was given in the Introduction.)

4. Proof of Theorems 1.1 and 1.2

4.1. Proof of Theorem 1.1. We need to prove that condition (B) implies condition (A). Let $\mathrm{tr}(u) = (\Gamma, v)$. We will prove that $u \leqslant w_\Gamma \oplus u_v$ by applying (B) to $v = u \ominus u_v$. (It follows from (1.8) that $u_v \leqslant u$.) It is easy to check that $v \oplus u_v = u$.

Let $\mathrm{tr}(u \ominus u_v) = (\Lambda, \mu)$. Clearly $\Lambda \subset \Gamma$. If we show that $\mu(\Gamma^c) = 0$, then (B) will imply that $v \leqslant w_\Gamma$ and therefore $u = v \oplus u_v \leqslant w_\Gamma \oplus u_v$.

It remains to prove that $\mu(\Gamma^c) = 0$. By the definition of the trace,

$$\mu(\Gamma^c) = \sup\{\lambda(\Gamma^c) : \lambda \in \mathcal{N}_1, \lambda(\Gamma) = 0, u_\lambda \leqslant v\}. \tag{4.1}$$

If $\lambda \in \mathcal{N}_1$, $\lambda(\Gamma) = 0$, $u_\lambda \leqslant v$, then $(\lambda + v)(\Gamma) = 0$, $\lambda + v \in \mathcal{N}_1$ by 2.3.B and $u_{\lambda+v} = u_\lambda \oplus u_v \leqslant v \oplus u_v = u$. By (1.8), $\lambda + v \leqslant v$. Hence $\lambda = 0$ and $\mu(\Gamma^c) = 0$ by (4.1). □

4.2. Preparation for proving Theorem 1.2. Suppose that E' is a smooth subdomain of a bounded smooth domain E. Put $A = \partial E \cap \partial E'$ and denote by A_0 the set of $y \in A$ such that $d(y, \partial E' \setminus A) > 0$.

We need the following lemmas.

Lemma 4.1. *Denote by k' the Poisson kernel of L in E'. If v is a finite measure on A and if*

$$h(x) = \int_A k(x, y)v(dy),$$

[16] See [Dy02, Section 11.1.3].

then

$$h(x) = \int_A k'(x,y)\nu'(dy), \tag{4.2}$$

where

$$\nu'(dy) = a(y)\nu(dy) \ \ on\, A. \tag{4.3}$$

Proof. The restriction of $k_y(x) = k(x,y)$ to E' is harmonic in E' and therefore it has a unique representation in the form

$$k_y(x) = \int_{\partial E'} k'(x,z)\sigma_y(dz).$$

If $y \in A$ and Γ is a closed subset of $A \subset \{y\}$, then $q(x) = \int_\Gamma k'(x,y)\sigma_\nu(dy) \leqslant k_y(x)$ is a harmonic function in E' vanishing on $\partial E' \backslash \Gamma$. It vanishes also on Γ because so does $k_y(x)$. We conclude that $\sigma_y(A\backslash\{y\}) = 0$.

$$h(x) = \int_A \nu(dy)k_y(x) = \int_{\partial E'} k'(x,z)\sigma_\nu(dz) \text{ for } x \in E' \tag{4.4}$$

where $\sigma_\nu = \int_A \nu(dy)\sigma_y$. By comparing (4.2) with (4.4) we conclude $\nu' = \sigma_\nu$. Since σ_ν does not charge $A\,\backslash\{y\}$, we get (4.3) with $a(y) = \sigma_\nu\{y\}$. □

A class $\mathcal{N}_1$ of finite measures on ∂E which is in a 1–1 correspondence with the class of moderate solutions in E was introduced in Section 2.3. We denote by $\mathcal{N}_1'$ an analogous class for the subdomain E'.

Lemma 4.2. *Suppose that $K \subset A$ is at a distance $\beta > 0$ from $\partial E'\,\backslash A$ and that $\nu \in \mathcal{M}(K)$. Let u be the maximal element of $\mathcal{U}$ dominated by*

$$h(x) = \int_K k(x,y)\nu(dy)$$

and let $\tilde{u}$ be the maximal element of $\mathcal{U}(E')$ dominated by

$$\tilde{h}(x) = \int_K k'(x,y)\nu(dy). \tag{4.5}$$

Then $u \geqslant \tilde{u}$ on E'.

Proof. Consider

$$E_\varepsilon = \{x \in E : d(x,K) > \varepsilon\}, \quad E'_\varepsilon = \{x \in E' : d(x,K) > \varepsilon\}.$$

If $\varepsilon < \beta$, then $E'_\varepsilon \subset E_\varepsilon$, $\partial E_\varepsilon \cap \mathscr{E}'_\varepsilon \supset B_\varepsilon = \{x \in \mathscr{E} : d(x, K) = \varepsilon\}$ and $\partial E'_\varepsilon \setminus B_\varepsilon \subset E_\varepsilon$. Let $u_\varepsilon = V_{E_\varepsilon}(h), \tilde{u}_\varepsilon = V_{E'_\varepsilon}(\tilde{h})$. We have

$$u_\varepsilon + G_{E_\varepsilon}\psi(u_\varepsilon) = h. \tag{4.6}$$

Hence $u_\varepsilon \geqslant h$. By the Mean value property 2.2.A and by 2.1.B, if $\varepsilon > \varepsilon_2$, then

$$u_{\varepsilon_2} = V_{E_{\varepsilon_1}}(u_{\varepsilon_2}) \leqslant V_{E_{\varepsilon_1}}(h) = u_{\varepsilon_1}$$

and (4.6) implies that $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$. Similarly, $\tilde{u}_\varepsilon \downarrow \tilde{u}$. The lemma will be proved if we demonstrate that $u_\varepsilon \geqslant \tilde{u}_\varepsilon$ in E'_ε. This follows from the Comparison principle 2.2.B because

$$\tilde{u}_\varepsilon = \tilde{h} \leqslant h = u_\varepsilon \quad \text{on } B_\varepsilon$$

and

$$\tilde{u}_\varepsilon = 0 \leqslant u_\varepsilon \quad \text{on } \partial E' \setminus B_\varepsilon. \qquad \square$$

Lemma 4.3. *If a measure $\nu \in \mathscr{N}'_1$ is concentrated on A_0, then $\nu \in \mathscr{N}_1$.*

Proof. If the restriction of ν to any compact subset K of A_0 belongs to $\mathscr{N}_1$, then $\nu \in \mathscr{N}_0$ and, if it is finite, then $\nu \in \mathscr{N}_1$. Thus we can assume that ν is concentrated on a compact set $K \subset A_0$. Consider functions $h, \tilde{h}, u, \tilde{u}$ introduced in Lemma 4.2. The minimal harmonic function h^* dominating u has a representation

$$h^*(x) = \int_K k(x, y)\mu(dy).$$

By Lemma 4.1,

$$h^*(x) = \int_K k'(x, y)\mu'(dy), \tag{4.7}$$

where

$$\mu'(dy) = a(y)\mu(dy).$$

By Lemma 4.2, $\tilde{u} \leqslant u$ and, since $u \leqslant h^*$, we have $\tilde{u} \leqslant h^*$. This implies $\tilde{h} \leqslant h^*$ (because $\tilde{h}$ is the minimal harmonic function dominating $\tilde{u}$). By comparing (4.5) and (4.7), we get $\nu(dy) \leqslant a(y)\mu(dy)$. Since $\mu \in \mathscr{N}_1$, we deduce from 2.3.A and 2.3.B that $\nu \in \mathscr{N}_1$. $\square$

It follows from Lemma 4.3 that a moderate solution u_η in E and a moderate solution u'_η in E' correspond to every $\eta \in N'_1(A)$.

For $C\subset A$ both functions $w_C\in\mathscr{U}$ and $w'_C\in\mathscr{U}'$ are defined. Clearly,

$$w'_C(x)\leqslant w_C(x) \quad \text{and for all } x\in E'. \tag{4.8}$$

Lemma 4.4. *If a measure $\eta\in\mathscr{N}'_1$ is concentrated on a closed subset K of A_0, then, for every $y\in A_0$,*

$$\lim_{x\to y}[u_\eta(x)-u'_\eta(x)]=0. \tag{4.9}$$

Proof. It follows from 3.1.C that

$$h_\eta(x)=h'_\eta(x)+\Pi_x 1_{\tau<\zeta}h_\eta(\xi_\tau). \tag{4.10}$$

This implies $h_\eta\geqslant h'_\eta$ and

$$h_\eta(x)-h'_\eta(x)\to 0 \quad \text{as } x\to y. \tag{4.11}$$

Eq. (4.9) will be proved if we show that

$$0\leqslant u_\eta-u'_\eta\leqslant h_\eta-h'_\eta \quad \text{in } E' \tag{4.12}$$

Note that

$$u_\eta+G\psi(u_\eta)=h_\eta \quad \text{in } E \tag{4.13}$$

and

$$u'_\eta+G'\psi(u'_\eta)=h'_\eta \quad \text{in } E'. \tag{4.14}$$

Hence

$$u_\eta-u'_\eta=h_\eta-h'_\eta-G\psi(u_\eta)+G'\psi(u'_\eta). \tag{4.15}$$

On the other hand,

$$u_\eta+G'\psi(u_\eta)=\widetilde{h} \quad \text{in } E', \tag{4.16}$$

where $\widetilde{h}$ is the minimal harmonic majorant of u_η and E'. By using the Markov property of ξ, we get from (4.13) and (4.16)

$$\widetilde{h}(x)=h_\eta(x)-\Pi_x\int_\tau^\zeta \psi(u_\eta(\xi_s))ds=h_\eta(x)-\Pi_x 1_{\tau<\zeta}G\psi(u_\eta)(\xi_\tau) \quad \text{for } x\in E'. \tag{4.17}$$

Since $G\psi(u_\eta)\leqslant h_\eta$ by (4.13), we get from (4.17) and (4.10),

$$\widetilde{h}(x)\geqslant h_\eta(x)-\Pi_x 1_{\tau<\zeta}\geqslant h_\eta(\xi_\tau)=h'_\eta(x) \quad \text{in } E'$$

which implies $u_\eta \geqslant u'_\eta$ in E'. Since ψ is monotonic and $g' \leqslant g$, we have $G'\psi(u'_\eta) \leqslant G'\psi(u_\eta) \leqslant G\psi(u_\eta)$. Formula (4.12) follows from (4.15). □

Lemma 4.5. *Suppose that u' is the restriction of $u \in \mathscr{U}$ to E' and let*

$$\operatorname{tr}(u) = (\Lambda, v), \quad \operatorname{tr}(u') = (\Lambda', v'). \tag{4.18}$$

We have

$$\Lambda' = \Lambda \cap A, \tag{4.19}$$

and $v'(C) = 0$ for every $C \subset A_0$ such that $C \cap \Lambda' = \emptyset$.

Proof. 1°. To prove (4.19) we consider the part ξ' of the L-diffusion ξ in E'. If $y \in \partial E' \setminus A$, then, Π_x^y-a.s., $u'(\xi'_t)$ is bounded and therefore $\Phi(u') < \infty$. Hence, $\Lambda' \subset A$.

For all $x \in E', y \in A$, by Lemma 3.2,

$${}^{\prime}\Pi_x^y\{\Phi(u') < \infty\} = \Pi_x^y\{\Phi(u') < \infty, \tau = \zeta\} = \Pi_x^y\{\Phi(u) < \infty, \tau = \zeta\} \tag{4.20}$$

which implies $\Lambda \cap A \subset \Lambda'$.

If $y \in \Lambda'$, then $y \in A$. Hence $\Pi_x^y\{\tau \neq \zeta\} = 0$ for all $x \in E'$ and, by (4.20), $\Pi_x^y\{\Phi(u) < \infty\} = 0$. Therefore $\Lambda' \subset \Lambda$.

2°. Denote by $\mathscr{K}$ the class of compact subsets of A_0 such that the restriction of v' to K belongs to $\mathscr{N}'_1$. To prove the second statement of the lemma, it is sufficient to prove that

$$\text{If } K \in \mathscr{K}, \eta \leqslant v' \text{ add } \eta \text{ is concentrated on } K, \text{ then } \eta(C) = 0. \tag{4.21}$$

Indeed, there exist $\mu_n \in \mathscr{N}'_1$ such that $\mu_n \uparrow v'$ and there are compact sets K_m such that $K_m \uparrow A_0$. Denote by $\eta_{m,n}$ the restriction of μ_n to K_m. If (4.21) is true, then $\eta_{m,n}(C) = 0$ which implies that $v'(C) = 0$.

To prove (4.21) it is sufficient to establish that:

$$\eta \in \mathscr{N}_1, \quad \eta(\Lambda) = 0, \quad u_\eta \leqslant u. \tag{4.22}$$

Indeed, by (1.8), properties (4.22) imply that $\eta \leqslant v$ and therefore $\eta(C) \leqslant v(C) = 0$.

It remains to prove (4.22).

The definition of $\mathscr{K}$ implies that $\eta \in \mathscr{N}'_1$. By Lemma 4.3, $\eta \in \mathscr{N}_1$.

By (4.19), $\Lambda \subset \Lambda' \cup (\partial E \setminus A)$. Hence $\eta(\Lambda) = 0$ because $\eta(\Lambda') \leqslant v'(\Lambda') = 0$ and η is concentrated on $K \subset A$.

We have $u'_\eta \leqslant u'_{v'} \leqslant u'_{\Lambda'} \oplus u'_{v'}$ and therefore, by (1.9), $u'_\eta \leqslant u'$. Since $u_\eta(x) \leqslant h_\eta(x)$, we have

$$\lim_{x \to y} u_\eta(x) = 0 \leqslant u(x) \quad \text{for } y \in \partial E \setminus K.$$

By Lemma 4.4,

$$\limsup_{x\to y}[u_\eta(x) - u(x)] = \limsup_{x\to y}[u'_\eta(x) - u'(x)] \leqslant 0 \quad \text{for } y \in A_0$$

By the Comparison principle 2.2.B, this implies $u_\eta \leqslant u$ in E. □

4.3. Proof of Theorem 1.2. By Theorem 1.1, it is sufficient to check condition (B). We need to prove that, if $\mathrm{tr}(u) = (\Lambda, v)$ and if $v(\Gamma) = 0$ where $\Lambda \subset \Gamma \subset \partial E$, then $u \leqslant w_\Gamma$.

The main step is to show that

$$\limsup_{x\to y}[u(x) - 2w_\Gamma(x)] \leqslant 0 \quad \text{for all } y \in \partial E. \tag{4.23}$$

Fix y and consider a domain $E' \in \mathscr{E}_{L,\psi}$ such that $\partial E' \cap \partial E$ contains a neighborhood of y. We use the notation introduced in Lemma 4.5. Clearly, $y \in A_0$. By the definition of $\mathscr{E}_{L,\psi}$,

$$u' \leqslant w'_{\Lambda'} \oplus u'_{v'} \leqslant w'_{\Lambda'} + u'_{v'}. \tag{4.24}$$

By (4.19), $\Lambda' = \Lambda \cap A \subset \Gamma$ and therefore, by (4.8),

$$w'_{\Lambda'} \leqslant w_{\Lambda'} \leqslant w_\Gamma. \tag{4.25}$$

Note that $\partial E' = A_0 \cup B$ where B is the closure of $\partial E' \cap E$. By Lemma 4.5, $v'(A_0 \setminus \Lambda') = 0$. Hence v' is concentrated on $\Lambda' \cup B$. By (1.7) and 3.6.B,

$$u'_{v'} \leqslant w'_{\Lambda' \cup B} \leqslant w'_{\Lambda'} + w'_B. \tag{4.26}$$

By (4.24)–(4.26) and (4.8),

$$u' \leqslant 2w_\Gamma + w_B \quad \text{on } E'.$$

Therefore

$$\limsup_{x\to y}[u(x) - 2w_\Gamma(x)] = \limsup_{x\to y}[u'(x) - 2w_\Gamma(x)] \leqslant \limsup_{x\to y} w_B(x) = 0.$$

By the Comparison principle 2.2.B, (4.23) implies $u \leqslant 2w_\Gamma$ in E. Hence $Z_u \leqslant 2Z_\Gamma$. Since $Z_\Gamma = 0$ or ∞, we get $Z_u \leqslant Z_\Gamma$. Therefore $u \leqslant w_\Gamma$. □

5. Proof of Theorem 1.3 for star domains

5.1. A domain E is called a star domain relative to a point c if, for every $x \in E$, the line segment $[c, x]$ connecting c and x is contained in E. Theorem 1.3 will follow from Theorem 1.2 if we prove that all bounded smooth star domains E belong to the class

$\mathscr{E}_{\Delta,u}$ where u is given by (1.10). We relay again on Theorem 1.1. After some preparations, we demonstrate that condition (B) in this theorem is satisfied in our case. Without loss of generality, we can assume that $c = 0$.

We use the self-similarity of the equation

$$\Delta u = u^{\alpha} \quad \text{in } E. \tag{5.1}$$

Let $0<r\leqslant 1$. Put $E_r = rE, \beta = 2/(\alpha - 1)$ and

$$f_r(x) = r^{\beta} f(rx) \quad \text{for } x \in E, f \in \mathscr{B}(E). \tag{5.2}$$

If $u \in \mathscr{U}$, then u_r also belongs to $\mathscr{U}$. Moreover, for $r<1, u_r$ is continuous on $\bar{E}$ and $u_r \to u$ uniformly on each $D \Subset E$ as $r \uparrow 1$. If f is continuous, then

$$V_E(f_r)(x) = r^{\beta} V_{E_r}(f)(rx) \quad \text{for all } x \in E. \tag{5.3}$$

This is trivial for $r = 1$. For $r<1$ this follows from 2.2.C because both parts of (5.3) are solutions of Eq. (5.1) with the same boundary condition $u = f_r$ on ∂E.

5.2. Preparations.

Lemma 5.1. *Put*

$$Y_r = \exp(-Z_{u_r}). \tag{5.4}$$

For every $\gamma \geqslant 1$,

$$P_0 |Y_r - Y_1|^{\gamma} \to 0 \quad \text{as } r \uparrow 1. \tag{5.5}$$

Proof. 1°. First, we prove that

$$\lim_{r\uparrow 1} P_0(Y_r^k - Y_1^k)^2 = 0 \tag{5.6}$$

for every positive integer k. If (5.6) does not hold, then

$$\lim P_0(Y_{r_n}^k - Y_1^k)^2 > 0 \tag{5.7}$$

for some sequence $r_n \uparrow 1$.

Note that

$$P_0(Y_r^k - Y_1^k)^2 = F_r + F_1 - 2G_r \tag{5.8}$$

where $F_r = P_0 Y_r^{2k}, G_r = P_0(Y_r Y_1)^k$.

By Lemma 3.3 and (3.12),

$$F_r = P_0 \exp[-2k\langle u_r, X_E\rangle] = \exp[-V_E(2ku_r)(0)] \quad \text{for all } 0<r<1. \tag{5.9}$$

We apply (5.3) to $f = 2ku$ and $x = 0$. Note that $f_r = 2ku_r$. Therefore, by (5.3), $V_E(2ku_r)(0) = r^\beta V_{E_r}(2ku)(0)$ and (5.9) implies that

$$F_r = \exp[-r^\beta V_{E_r}(2ku)(0)] = \exp[-V_{E_r}(2ku)(0)]^{r^\beta} = [P_0 \exp(-2k\langle u, X_{E_r}\rangle)]^{r^\beta}. \tag{5.10}$$

Since $\langle u, X_{r_n}\rangle \to Z_u$ a.s., we have

$$F_{r_n} \to F_1. \tag{5.11}$$

Put

$$v_r(x) = -\log P_x(Y_r Y_1)^k = -\log P_x \exp[-kZ_{u_r} - Z_u]. \tag{5.12}$$

By 3.5.B and 3.5.A, $v_r \in \mathscr{U}$ and $Z_{v_r} \leqslant k(Z_{u_r} + Z_u)$. Therefore by 3.5.A,

$$v_r \leqslant k(u_r + u) \quad \text{in } E. \tag{5.13}$$

By 2.2.D, we can choose a subsequence r_{k_n} of the sequence r_n such that $v_{r_{k_n}}$ converge uniformly on each $D \Subset E$ to an element v of $\mathscr{U}(E)$. By changing the notation we can assume that this subsequence coincides with the sequence r_n. By (5.12), $G_r = e^{-v_r(0)}$. By 3.5.A, $P_0 e^{-Z_v} = e^{-v(0)}$. By passing to the limit in (5.13), we get that $v \leqslant 2ku$. Therefore $P_0 e^{-Z_v} \geqslant P_0 e^{-2kZ_u}$. By (5.10) and (5.11),

$$P_0 e^{-2k\langle u, X_{E_r}\rangle} \to F_1$$

as $r \to 1$ along the sequence r_n. Since $\langle u, X_{E_r}\rangle \to Z_u$, we get that $P_0 e^{-Z_v} \geqslant F_1$.

Since $v_{r_n}(0) \to v(0)$, we have $G_{r_n} \to P_0 e^{-Z_v} \geqslant F_1$. Because of (5.8) and (5.11), this contradicts (5.7).

2°. If $\gamma < m$, then $(P_0|Z|^\gamma)^{1/\gamma} \leqslant (P_0|Z|^m)^{1/m}$. Therefore it is sufficient to prove (5.5) for even integers $\gamma = m > 1$. Since $0 \leqslant Y_1 \leqslant 1$, the Schwarz inequality and (5.6) imply

$$P_0|Y_r^k Y_1^{m-k} - Y_1^m| \leqslant (P_0 Y_1^{2(m-k)})^{1/2} [P_0(Y_r^k - Y_1^k)^2]^{1/2} \to 0 \quad \text{as } r \uparrow 1.$$

Therefore

$$P_0|Y_r - Y_1|^m = P_0(Y_r - Y_1)^m = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} P_0(Y_r)^k Y_1^{m-k}$$
$$\to \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} P_0 Y_1^m = 0. \quad \square$$

Lemma 5.2. *For every* $\nu \in \mathcal{N}_1$ *and for all* $x \in E$,

$$P_x Z_\nu^\gamma \leqslant 1 + c_1 h_\nu(x)^2 + c_2 G(h_\nu^\alpha)(x), \tag{5.14}$$

where $c_1 = \frac{1}{2} e\gamma/(2-\gamma)$ *and* $c_2 = \gamma/(\alpha - \gamma)$.

Proof. For every probability measure P and for every positive Z

$$PZ^\gamma = P\int_0^Z \gamma\lambda^{\gamma-1} d\lambda = \int_0^\infty P\{Z > \lambda\}\gamma\lambda^{\gamma-1} d\lambda \leqslant 1 + \int_1^\infty P\{Z > \lambda\}\gamma\lambda^{\gamma-1} d\lambda. \tag{5.15}$$

Function

$$\mathscr{E}(\lambda) = e^{-\lambda} - 1 + \lambda$$

is positive, monotone increasing and $\mathscr{E}(1) = 1/e$. For each $\lambda > 0$, by Chebyshev's inequality,

$$P\{Z > \lambda\} = P\{Z/\lambda > 1\} = P\{\mathscr{E}(Z/\lambda) > 1/e\} \leqslant eq(1/\lambda) \tag{5.16}$$

where $q(\lambda) = P\mathscr{E}(\lambda Z)$. By (5.15) and (5.16),

$$PZ^\gamma \leqslant 1 + e\int_0^1 \gamma\lambda^{-\gamma-1} q(\lambda)\, d\lambda. \tag{5.17}$$

We apply (5.17) to $P = P_x$ and to $Z = Z_\nu$. By 3.5.A,

$$\begin{aligned} q(\lambda) &= P_x e^{-\lambda Z_\nu} - 1 + \lambda P_x Z_\nu \\ &= e^{-u_{\lambda\nu}(x)} - 1 + \lambda h_\nu(x) = \mathscr{E}(\lambda h_\nu) + \lambda h_\nu - u_{\lambda\nu}. \end{aligned} \tag{5.18}$$

Since $\mathscr{E}(\lambda) \leqslant \frac{1}{2}\lambda^2$, we have

$$\mathscr{E}(u_{\lambda\nu}) \leqslant \tfrac{1}{2} u_{\lambda\nu}(x)^2 \leqslant \tfrac{1}{2}\lambda^2 h_\nu(x)^2. \tag{5.19}$$

By 3.5.A,

$$\lambda h_\nu - u_{\lambda\nu} = G(u_{\lambda\nu}^\alpha) \leqslant \lambda^\alpha G(h_\nu^\alpha). \tag{5.20}$$

Formula (5.14) follows from (5.17)–(5.19) and (5.20). □

To every $x \in E$ there corresponds a capacity (we call it the Poisson capacity) defined on Borel subsets of ∂E by the formula

$$\mathrm{Cap}_x(B) = \sup_{\nu \in \mathscr{P}(B)} [G(h_\nu^\alpha)(x)]^{-1},$$

where $\mathscr{P}(B)$ is the class of all probability measures on B. By 2.3.F, $\nu \in \mathscr{N}_1$ if $G(h_\nu^\alpha)(x) < \infty$ for some x. Therefore, if $\mathrm{Cap}_x(B) > 0$, then

$$\mathrm{Cap}_x(B) = \sup_{\nu \in \mathscr{P}'(B)} [G(h_\nu^\alpha)(x)]^{-1}, \tag{5.21}$$

where $\mathscr{P}'(B) = \mathscr{P}(B) \cap \mathscr{N}_1$. We use a bound

$$w_B(x) \leqslant C(x)\mathrm{Cap}_x(B)^{1/(\alpha-1)}, \tag{5.22}$$

where $C(x)$ does not depend on B. For $\alpha = 2$, this is an implication of Theorem 3.19 in [Ms03].[17] For $1 < \alpha \leqslant 2$, (5.22) is proved by Kuznetsov [Ku03].

Lemma 5.3. *If $w_{B_n}(0) \geqslant \gamma > 0$ then there exist $\nu_n \in \mathscr{P}'(B_n)$ such that $P_0 Z_{\nu_n}^\gamma$ are bounded for every $1 < \gamma < \alpha$ and, consequently, Z_{ν_n} are uniformly P_0-integrable. The sequence Z_{ν_n} contains a subsequence convergent weakly in $L^1(P_0)$. Its limit Z has the properties: $P_0 Z > 0$ and $u_Z(x) = -\log P_x e^{-Z}$ is a moderate solution of the equation $\Delta u = u^\alpha$ in E. There exists a sequence $\widehat{Z}_k$ which converges a.s. to Z such that each $\widehat{Z}_k$ is a convex combination of finite numbers of Z_{ν_n}.*

Proof. It follows from (5.22) that, if $w_{B_n}(0) \geqslant \gamma$, then for all n, $\mathrm{Cap}_0(B_n) > \delta = [\gamma/C(0)]^{\alpha-1}$. By (5.21), $G(h_{\nu_n}^\alpha)(0) < 2/\delta$ for some $\nu_n \in \mathscr{P}'(B_n)$.

There exists a constant c such that

$$h(0) \leqslant c[G(h^\alpha)(0)]^{1/\alpha} \tag{5.23}$$

for every positive harmonic function h. Indeed, if the distance of 0 from ∂E is equal to 2ρ, then, by the mean value property,

$$h(0) = c_1^{-1} \int_{B_\rho} h(y)dy \leqslant (c_1 c_2)^{-1} \int_{B_\rho} g(0,y)h(y)\, dy, \tag{5.24}$$

where $B_\rho = \{x : |x| < \rho\}$, c_1 is the volume of B_ρ and $c_2 = \min g(0,y)$ over B_ρ. By Hölder's inequality,

$$\int_{B_\rho} g(0,y)h(y)dy \leqslant \left[\int_{B_\rho} g(0,y)h(y)^\alpha\, dy\right]^{1/\alpha} \left[\int_{B_\rho} g(0,y)\, dy\right]^{1/\alpha'}, \tag{5.25}$$

where $\alpha' = \alpha/(\alpha-1)$. Hence (5.23) follows from (5.24) and (5.25).

[17] By this theorem, $w_B(x) \leqslant C_1(x)\mathrm{Cap}_\partial(B)$ and it can be proved that $\mathrm{Cap}_\partial(B) \leqslant C_2(x)\mathrm{Cap}_x(B)$.

By (5.23),

$$h_{\nu_n}(0) \leqslant c[G(h^{\alpha}_{\nu_n})(0)]^{1/\alpha} \leqslant c(2/\delta)^{1/\alpha}$$

and (5.14) implies that, for every $1<\gamma<\alpha$, the sequence $P_0 Z^{\gamma}_{\nu_n}$ is bounded. This is sufficient for the uniform integrability of Z_{ν_n} (see, e.g., [Me66, p. 19]).

By the Dunford–Pettis criterion (see, e.g., [Me66, p. 20]), Z_{ν_n} contains a weakly convergent subsequence. By changing notation, we can assume that this subsequence coincide with Z_{ν_n}. The limit Z satisfies the condition $P_0 Z>0$ because $P_0 Z_{\nu_n} \to P_0 Z$ and

$$P_0 Z_{\nu_n} = \int_{\partial E} k(0,y)\nu_n(dy) \geqslant \inf_{\partial E} k(0,y) > 0.$$

There exists a sequence $\widetilde{Z}_m$ which converges to Z in $L^1(P_0)$ norm such that each $\widetilde{Z}_m$ is a convex combination of a finite number of Z_{ν_n} (see, e.g., [Ru73, Theorem 3.13]). A subsequence $\widehat{Z}_k$ of $\widetilde{Z}_m$ converges to Z P_0-a.s. By Theorem 2.1 in [Dy04], all measures P_x are absolutely continuous with respect to P_0 on the σ-algebra generated by $Z \in \mathfrak{Z}$. Therefore $\widehat{Z}_k$ converges to Z P_x-a.s. for all $x \in E$. By 3.5.C, $Z \in \mathfrak{Z}$ and, by 3.5.D, u_Z is a moderate solution. □

5.3. Proof of condition (B). To prove this condition we introduce a function

$$Q_r(y) = \widetilde{\Pi}^y_0 \exp\left\{-\int_0^{\zeta} u_r(\xi_t)^{\alpha-1} dt\right\} \tag{5.26}$$

and we consider, for every $\varepsilon>0$ and every $0<r<1$, a partition of ∂E into two sets

$$A_{r,\varepsilon} = \{y \in \partial E : Q_r(y) \leqslant \varepsilon\} \quad \text{and } B_{r,\varepsilon} = \{y \in \partial E : Q_r(y) > \varepsilon\}. \tag{5.27}$$

We investigate the behavior, as $r \uparrow 1$, of functions

$$f_{r,\varepsilon} = V_E(u_r I_{r,\varepsilon}) \text{ and } g_{r,\varepsilon} = V_E(u_r J_{r,\varepsilon}). \tag{5.28}$$

We assume, as in condition (B), that

$$\operatorname{tr}(u) = (\Lambda, \mu), \Lambda \subset \Gamma \subset \partial E \text{ and } \nu \text{ is concentrated on } \Gamma \tag{5.29}$$

and we prove:

Lemma 5.4. *For every* $\varepsilon>0$,

$$s_\varepsilon \leqslant w_\Gamma, \tag{5.30}$$

where

$$s_\varepsilon(x) = \limsup_{r\uparrow 1} \; g_{r,\varepsilon}(x).$$

Lemma 5.5. *Fix a relatively open subset* O *of* ∂E *which contains* Γ *and put*

$$C_{r,\varepsilon} = A_{r,\varepsilon} \cap (\partial E \setminus O), \quad q(\varepsilon) = \liminf_{r\downarrow 0} \; w_{C_{r,\varepsilon}}(0).$$

We have

$$\lim_{\varepsilon\downarrow 0} q(\varepsilon) = 0. \tag{5.31}$$

Property (B) easily follows from these two lemmas. Indeed, $f_{r,\varepsilon}$ and $g_{r,\varepsilon}$ belong to $\mathscr{U}$ by 2.1.D. By 3.6.B, $w_{A_{r,\varepsilon}} \leqslant w_O + w_{C_{r,\varepsilon}}$ because $A_{r,\varepsilon} \subset O \cup C_{r,\varepsilon}$. It follows from Lemma 5.5 that

$$\liminf_{\varepsilon\to 0} \; \liminf_{r\uparrow 1} \; w_{A_{r,\varepsilon}} \leqslant w_O(x).$$

By 3.6.A, this implies

$$\liminf_{\varepsilon\to 0} \; \liminf_{r\uparrow 1} \; w_{A_{r,\varepsilon}} \leqslant w_\Gamma(x). \tag{5.32}$$

We have $f_{r,\varepsilon} \leqslant u_r$ and $g_{r,\varepsilon} \leqslant u_r$ by 2.1.B and

$$u_r \leqslant f_{r,\varepsilon} + g_{r,\varepsilon} \tag{5.33}$$

by 2.1.F. Finally, by 2.1.E,

$$f_{r,\varepsilon} \leqslant w_{A_{r,\varepsilon}}. \tag{5.34}$$

From the bounds (5.33), (5.34), Lemma 5.4 and (5.32) we conclude that

$$u(x) \leqslant \liminf_{\varepsilon\to 0} \; \liminf_{r\uparrow 1} \; w_{A_{r,\varepsilon}} + w_\Gamma(x) \leqslant 2 w_\Gamma(x). \tag{5.35}$$

Therefore $Z_u \leqslant 2Z_\Gamma$. Since $Z_\Gamma = \infty 1_{\{\mathbb{R} \cap \Gamma = \emptyset\}}$, we get $Z_u \leqslant Z_\Gamma$ and consequently $u \leqslant w_\Gamma$.

It remains to prove Lemmas 5.4 and 5.5.

5.4. Proof of Lemma 5.4. Consider harmonic functions $h_{r,\varepsilon} = K_E(u_r J_{r,\varepsilon})$. By (2.3), $g_{r,\varepsilon} \leqslant h_{r,\varepsilon}$. First, we prove that

$$h_{r,\varepsilon}(0) \leqslant u(0)/\varepsilon. \tag{5.36}$$

By applying 3.1.B to $v = u_r$ and $a(u) = u^{\alpha-1}$ we get

$$u_r(y) = \Pi_y u_r(\xi_\zeta) Y,$$

where

$$Y = \exp\left[-\int_0^\zeta u_r(\xi_s)^{\alpha-1} ds\right].$$

By (5.26) and Lemma 3.1,

$$u_r(y) = \Pi_y u_r(\xi_\zeta)\widetilde{\Pi}_x^{\xi_\zeta} Y = K_E(u_r Q_r)(y).$$

Since $\varepsilon J_{r,\varepsilon} \leqslant Q_r \leqslant 1$, we have

$$\varepsilon h_{r,\varepsilon} = K_E(\varepsilon u_r J_{r,\varepsilon}) \leqslant K_E(u_r) = u_r$$

and (5.36) follows because $u_r(0) = r^\beta u(0) \leqslant u(0)$.

To prove that (5.30) holds at $x \in E$, we choose a sequence $r_n \uparrow 1$ such that

$$g_{r_n,\varepsilon}(x) \to s_\varepsilon(x). \tag{5.37}$$

Bound (5.36) and well-known properties of harmonic functions (see, e.g., [Dy02, 6.1.5.B and 6.1.5.C]) imply that a subsequence of $h_{r_n,\varepsilon}$ tends to an element h_ε of $\mathscr{H}$. By 2.2.D, this subsequence can be chosen in such a way that $g_{r_n,\varepsilon} \to g_\varepsilon \in \mathscr{U}$. The bounds $g_{r,\varepsilon} \leqslant h_{r,\varepsilon}$ imply that $g_\varepsilon \leqslant h_\varepsilon$. Hence g_ε is a moderate solution and it is equal to u_μ for some $\mu \in \mathscr{N}_1$. By the definition of the fine trace, $\nu(B) \geqslant \mu'(B)$ for all $\mu' \in \mathscr{N}_1$ such that $\mu'(\Lambda) = 0$ and $u_{\mu'} \leqslant u$. The restriction μ' of μ to Γ^c satisfies these conditions. Indeed, $\mu' \in \mathscr{N}_1$ by 2.3.A; $\mu'(\Lambda) = 0$ because $\Lambda \subset \Gamma$; finally, $u_{\mu'} \leqslant u_\mu = g_\varepsilon \leqslant u$ because $g_{r,\varepsilon} \leqslant V_E(u_r) \leqslant u_r$ by 2.1.B and 2.2.A. We conclude that $\mu'(\Gamma^c) \leqslant \nu(\Gamma^c)$ and $\mu' = 0$ since $\nu(\Gamma^c) = 0$. Hence $\mu \in \mathscr{N}_1(\Gamma)$ and, by (1.7), $u_\mu(x) = g_\varepsilon(x) \leqslant w_\Gamma(x)$. By (5.37), $s_\varepsilon(x) = g_\varepsilon(x)$ which implies (5.30). □

5.5. Proof of Lemma 5.5. Clearly, $q(\varepsilon) \leqslant q(\tilde{\varepsilon})$ for $\varepsilon < \tilde{\varepsilon}$. We need to show that $q(0+) = 0$. Suppose that this is not true and put $\gamma = q(0+)/2$. Consider a sequence $\varepsilon_n \downarrow 0$. Since $q(\varepsilon_n) \geqslant 2\gamma$, there exists a sequence $r_n \uparrow 1$ such that $w_{C_{r_n,\varepsilon_n}}(0) \geqslant \gamma$. We apply Lemma 5.3 to the sets $B_n = C_{r_n,\varepsilon_n}$. A sequence Z_{ν_n} defined in this lemma contains a weakly convergent subsequence. We redefine r_n and ε_n to make this subsequence identical with the sequence Z_{ν_n}.

Put $u_n = u_{r_n}$. By (3.17)

$$\begin{aligned} P_0 Z_{\nu_n} e^{-Z_{u_n}} &= e^{-u_n(0)} \int_{\partial E} \Pi_0^y e^{-\Phi(u_n)} \nu_n(dy) \\ &\leqslant \int_{\partial E} k(0,y) \widetilde{\Pi}_0^y e^{-\Phi(u_n)} \nu_n(dy), \end{aligned} \tag{5.38}$$

where Φ is defined by (3.18). Since $\psi'(u) = \alpha u^{\alpha-1} \geqslant u^{\alpha-1}$, we have

$$\widetilde{\Pi}_0^y e^{-\Phi(u_n)} \leqslant Q_{r_n}(y).$$

Since $\nu_n \in \mathscr{P}'(B_n)$ and since $Q_{r_n} \leqslant \varepsilon$ on B_n, the right side in (5.38) does not exceed

$$\varepsilon_n \int_{\partial E} k(0, y)\nu_n(dy) = \varepsilon_n h_{\nu_n}(0).$$

Therefore

$$P_0 Z_{\nu_n} e^{-Z_{u_n}} \to 0 \quad \text{as } n \to \infty. \tag{5.39}$$

Let $1 < \gamma < \alpha$. By Hölder's inequality,

$$|P_0 Z_{\nu_n}(e^{-Z_{u_n}} - e^{-Z_u})| \leqslant (P_0 Z_{\nu_n}^{\gamma})^{1/\gamma} [P_0 |e^{-Z_{u_n}} - e^{-Z_u}|^{\gamma'}]^{1/\gamma'},$$

where $\gamma' = \gamma/(\gamma - 1) > 1$. By Lemma 5.3, the first factor is bounded. By Lemma 5.1, the second factor tends to 0. Therefore

$$P_0 Z e^{-Z_u} = \lim P_0 Z_{\nu_n} e^{-Z_u} = \lim P_0 Z_{\nu_n} e^{-Z_{u_n}} = 0 \tag{5.40}$$

by (5.39). Since all measures P_x are absolutely continuous with respect to P_0 on the σ-algebra generated by $Z \in \mathfrak{Z}$ (Theorem 2.1 in [Dy04]), we get

$$P_x Z e^{-Z_u} = 0 \quad \text{for all } x \in E.$$

Hence $Z = 0$ P_x-a.s. on $\{Z_u < \infty\}$ which implies that $P_x\{Z \leqslant Z_u\} = 1$ and $u_Z(x) = -\log P_x e^{-Z} \leqslant u(x)$.

Let $F = \partial E \setminus O$. Since $\nu_n \in \mathscr{N}_1(F)$, $w_F \leqslant u_{\nu_n}$ by (1.7) and therefore $Z_F = \mathrm{SBV}(w_F) \geqslant \mathrm{SBV}(u_{\nu_n}) = Z_{nu_n}$. By Lemma 5.3, there exists a sequence of $\widehat{Z}_k$ such that $\widehat{Z}_k \to Z$ a.s. and each $\widehat{Z}_k$ is a convex combination of a finite number of Z_{ν_n}. Therefore $Z \leqslant Z_F$ a.s. and $u_Z \leqslant w_F$. Since $w_F = 0$ on O, we conclude that $u_Z = 0$ on O. Being a moderate solution, $u_Z = u_\nu$ for some $\nu \in \mathscr{N}_1$ and thus $Z = Z_\nu$. By 2.3.F, the relation $u_\nu \leqslant w_F$ implies that ν is concentrated on F. We have $\nu \in \mathscr{N}_1, \nu(\Lambda) = 0$ and $u_\nu \leqslant u$. By (1.8), $\nu \leqslant \mu$ and, since μ is concentrated on Γ, we conclude that $\nu(\Gamma^c) = 0$. Hence, $\nu(\partial E) \leqslant \nu(O) + \nu(\Gamma^c) = 0$ and $Z = Z_\nu = 0$. This contradict the property $P_0 Z > 0$ which is a part of Lemma 5.3. Therefore (5.31) is true. □

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