LOGARITHMIC COEFFICIENTS OF UNIVALENT FUNCTIONS

By

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Dedicated to the memory of Zeev Nehari

In this paper we present some relations between the logarithmic coefficients of a univalent function and the mean growth of its logarithmic derivative. We also relate the logarithmic coefficients to certain geometric properties of the function.

1. Introduction

Let $f(z)$ be analytic in the unit disk $|z|<1$, and let

$$I_p(r,f)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta, \qquad 0\leqq r<1,$$

denote its integral means of order p, where $0<p<\infty$. The function f is said to belong to the Hardy space H^p if $I_p(r,f)$ remains bounded as r tends to 1. The class of all bounded analytic functions is called H^∞. Each function $f\in H^p$ has a radial limit $f(e^{i\theta})$ in almost every direction, and the boundary function belongs to L^p. A function in H^p is said to belong to the smoothness class Λ_α^p if its boundary function has an integral modulus of continuity $\omega_p(t)=O(t^\alpha)$, $0<\alpha\leqq 1$. (See [7], p. 72.) For $1\leqq p<\infty$ and $0<\alpha\leqq 1$, an analytic function f belongs to Λ_α^p if and only if

$$[I_p(r,f')]^{1/p}=O\left(\frac{1}{(1-r)^{1-\alpha}}\right), \qquad r\to 1.$$

(See [7], p. 78.) In particular, $f\in\Lambda_{1/2}^2$ if and only if

$$I_2(r,f')=O\left(\frac{1}{1-r}\right).$$

A function $f\in H^1$ is said to belong to the class BMOA if its boundary function has bounded mean oscillation. It is known that $H^\infty\subset\text{BMOA}\subset H^p$ for every $p<\infty$. On the basis of the "Carleson measure" characterization of BMOA, it is not difficult to show (see [5]) that $\Lambda^2_{1/2} \subset$ BMOA. (This may be compared with the classical result of S. Bernstein that f has an absolutely convergent power series if $f \in \Lambda_\alpha$ for some $\alpha > 1/2$.)

The class S consists of all functions f analytic and univalent in the unit disk, with $f(0)=0$ and $f'(0)=1$. It is known (see [7], p. 50) that $S \subset H^p$ for every $p < 1/2$. Associated with each $f \in S$ is a well-defined *logarithmic function*

$$g(z) = \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \qquad |z| < 1.$$

The numbers γ_n are called the *logarithmic coefficients* of f. It is clear that $g \in H^p$ for all $p < \infty$, since $f \in S \subset H^p$ for some $p > 0$. Baernstein [3] obtained the stronger result that $g \in$ BMOA for each $f \in S$, and Cima and Schober [6] found an alternate proof.

This raises the question whether $g \in \Lambda^2_{1/2}$ for each $f \in S$. This is true for certain subclasses of S, but false in general. On the positive side it is fairly easy to show, by passing to area integrals over annuli, that

$$(1) \qquad I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log M(r,f)\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right)$$

for every $f \in S$ (cf. [15], p. 130), where

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

However, Hayman [12] has constructed a function $f \in S$ for which

$$I_2\left(r, \frac{f'}{f}\right) \neq o\left(\frac{1}{1-r} \log \frac{1}{1-r}\right), \qquad r \to 1.$$

In particular, there exists $f \in S$ with logarithmic function $g \notin \Lambda^2_{1/2}$.

The problem remains to describe the subclass of functions $f \in S$ whose logarithmic functions g belong to $\Lambda^2_{1/2}$. This is true for all support points of S, as Cima and Petersen [5] observed. Indeed, each support point $f \in S$ is analytic in the closed disk $|z| \leqq 1$ apart from a double pole at one point on the boundary (see [16] or [8]), so it follows directly that $g \in \Lambda^2_{1/2}$. Allen Shields has asked (private communication) whether the same is true for all extreme points of S. By a result of Brickman [4], each extreme point $f \in S$ is a monotone slit-mapping: it maps the disk onto the complement of an arc which tends to infinity with increasing modulus. However, Hayman has produced another construction (unpublished) of a monotone slit-mapping $f \in S$ for which

$$I_2\left(r, \frac{f'}{f}\right) \neq o\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right).$$

Thus it is not true that $g \in \Lambda^2_{1/2}$ for every monotone slit-mapping $f \in S$, but the question remains open for extreme points.

2. Growth of logarithmic derivatives

The limit

$$\alpha = \lim_{r \to 1} (1-r)^2 M(r, f) \leqq 1$$

exists for each $f \in S$ and is called the *Hayman index* of f. It is known that f has a *direction of maximal growth* $e^{i\varphi}$, unique if $\alpha > 0$, in the sense that

$$\lim_{r \to 1} (1-r)^2 |f(re^{i\varphi})| = \alpha.$$

The following theorem asserts that $g \in \Lambda^2_{1/2}$ for each $f \in S$ with positive Hayman index.

Theorem 1. *For each $f \in S$ with Hayman index $\alpha > 0$,*

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r}\right), \qquad r \to 1. \tag{2}$$

The proof uses an equivalent formulation of the condition (2) in terms of the logarithmic coefficients of f.

Lemma 1. *Let γ_n be the logarithmic coefficients of a function $f \in S$. Then* (2) *holds if and only if*

$$\sum_{n=1}^{N} n^2 |\gamma_n|^2 = O(N), \qquad N \to \infty. \tag{3}$$

Corollary. *If $\gamma_n = O(1/n)$, then* (2) *holds.*

Proof of Lemma. The condition (2) is equivalent to

$$I_2(r, g') = O\left(\frac{1}{1-r}\right),$$

or

$$\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 r^n = O\left(\frac{1}{1-r}\right). \tag{4}$$

If (4) holds, then

$$r^N \sum_{n=1}^{N} n^2 |\gamma_n|^2 \leqq \sum_{n=1}^{N} n^2 |\gamma_n|^2 r^n \leqq \frac{C}{1-r}$$

for each N and for all $r < 1$. Choosing $r = 1 - 1/N$, we obtain (3). Conversely, a summation by parts in (4) shows that (3) implies (4), which is equivalent to (2).

Lemma 2. *A function $f \in S$ satisfies*

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right)$$

if and only if its logarithmic coefficients γ_n satisfy

$$\sum_{n=1}^{N} n^2 |\gamma_n|^2 = O(N \log N).$$

The proof is similar to that of Lemma 1 and is omitted. We remark that these lemmas suggest the possibility of an analytic approach to the result of Hayman [12] proving the existence of a function $f \in S$ with $g \notin \Lambda^2_{1/2}$. Hayman's geometric construction is quite difficult.

Proof of Theorem. Let

$$s_n = \sum_{k=1}^{n} \left(k |\gamma_k|^2 - \frac{1}{k}\right), \qquad n = 1, 2, \cdots.$$

According to Milin's lemma [13, 15, 8], $s_n \leqq \delta < 0.312$ for all n, and for arbitrary Hayman index α. Milin ([13], Ch. 3, §2) has shown that if $\alpha > 0$, then s_n is also bounded from below. In fact, under the assumption that $e^{i\varphi} = 1$ is the direction of maximal growth,

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} k\left|\gamma_k - \frac{1}{k}\right|^2 + \log\alpha \leqq \frac{1}{2}\log\alpha.$$

Thus a summation by parts gives

$$\begin{aligned}\sum_{n=1}^{N} n^2|\gamma_n|^2 &= N + \sum_{n=1}^{N} n\left(n|\gamma_n|^2 - \frac{1}{n}\right)\\ &= N + (N+1)s_N - \sum_{n=1}^{N} s_n\\ &= O(N).\end{aligned}$$

Corollary. *If $f \in S$ has Hayman index $\alpha > 0$, then*

$$\sum_{n=1}^{\infty} n|\gamma_n|r^n = O\left(\frac{1}{1-r}\right).$$

This follows from the Schwarz inequality and the equivalence of (2) and (4). It gives a partial answer to a question raised by Aharonov ([1], p. 141).

The key to the proof of the theorem is Milin's result that s_n is bounded below if $\alpha > 0$. This is *never* true if $\alpha = 0$. In fact, Grinspan [10] has shown that

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k \to -\infty$$

for every function $f \in S$ with $\alpha = 0$. Nevertheless, the estimate (2) remains valid in certain cases where $\alpha = 0$. For instance, the Herglotz representation easily gives $|\gamma_n| \leqq 1/n$ for all starlike functions (and more generally for all spirallike functions). The corollary to Lemma 1 therefore shows that (2) holds for all starlike functions. This complements the theorem because, as Eenigenburg and Keogh [9] have shown, $\alpha = 0$ for all starlike functions other than the rotations of the Koebe function.

We remark also that in view of (1) it follows that (2) and therefore (3) holds for all *bounded* univalent functions, for which it is clear that $\alpha = 0$. Nevertheless, it is implicit in work of Pommerenke ([14]; [15], p. 130) that there exist bounded univalent functions with $\gamma_n \neq O(n^{-0.83})$; in particular, $\gamma_n \neq O(1/n)$.

More generally, we can show that (2) holds if the image of f has finite area.

Theorem 2. *For each function $f \in S$ whose image has finite area, the relation (2) holds.*

Proof. Let $f(z)=\Sigma a_n z^n$. The hypothesis is that $\Sigma n|a_n|^2<\infty$. The growth theorem gives $|f(z)|\geqq\delta>0$ for $|z|$ near 1, and so

$$I_2\left(r,\frac{f'}{f}\right)\leqq\delta^{-2}I_2(r,f')\leqq C\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n}$$
$$\leqq\frac{C}{1-r}\sum_{n=1}^{\infty}n|a_n|^2r^n=O\left(\frac{1}{1-r}\right),$$

where the inequality $nr^n<1/(1-r)$ has been used.

3. Behavior of logarithmic coefficients

We shall now introduce a method for relating the growth of γ_n to the boundary behavior of f. It is known (see [7], p. 51) that for each $f\in S$, the quotient $f(z)/z$ is an outer function. In other words,

$$\log\frac{f(z)}{z}=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|\,dt,$$

where $f(e^{it})$ denotes the radial limit of f. Thus

$$\gamma_n=\frac{1}{2\pi}\int_0^{2\pi}e^{-int}\log|f(e^{it})|\,dt,\qquad n=1,2,\cdots. \tag{5}$$

Observe now that the function $g(z)=\log\{f(z)/z\}$ belongs to H^1, and so

$$\int_0^{2\pi}e^{int}g(e^{it})dt=0,\qquad n=1,2,\cdots.$$

In view of (5), this gives

$$\gamma_n=\frac{i}{2\pi}\int_0^{2\pi}e^{-int}\arg\{e^{-it}f(e^{it})\}dt \tag{6}$$

and

$$\gamma_n=\frac{1}{4\pi}\int_0^{2\pi}e^{-int}g(e^{it})dt,\qquad n=1,2,\cdots. \tag{7}$$

An integration by parts in (5) or (6) leads at once to the following result.

Theorem 3. *For each* $f \in S$ *such that* $\log|f(e^{it})|$ *or* $\arg\{f(e^{it})\}$ *has bounded variation in the interval* $[0, 2\pi]$, *the logarithmic coefficients satisfy* $\gamma_n = O(1/n)$.

By way of comparison, it may be observed that if $f \in S$ maps the disk onto a Jordan domain with rectifiable boundary, then (see [7], p. 44) $f' \in H^1$ and so (see [7], p. 84) the relation (2) holds. Theorem 3 gives a stronger conclusion under a weaker hypothesis.

We close with a general result which may be of some interest.

Theorem 4. *The logarithmic coefficients* γ_n *of every function* $f \in S$ *satisfy* $\sum_{n=1}^{\infty} |\gamma_n|^2 \leqq \pi^2/6$. *This inequality is sharp.*

Proof. For $f \in S$, let

$$g(z) = \log \frac{f(z)}{z} = u(z) + iv(z) = 2 \sum_{n=1}^{\infty} \gamma_n z^n$$

and

$$G(z) = \log \frac{k(z)}{z} = U(z) + iV(z) = 2 \sum_{n=1}^{\infty} \frac{1}{n} z^n,$$

where $k(z) = z(1-z)^{-2}$ is the Koebe function. By a result of Baernstein [2], the inequalities

$$\int_0^{2\pi} \Phi\left(\log\left|\frac{f(re^{i\theta})}{r}\right|\right) d\theta \leqq \int_0^{2\pi} \Phi\left(\log\left|\frac{k(re^{i\theta})}{r}\right|\right) d\theta$$

and

$$\int_0^{2\pi} \Phi\left(\log\left|\frac{r}{f(re^{i\theta})}\right|\right) d\theta \leqq \int_0^{2\pi} \Phi\left(\log\left|\frac{r}{k(re^{i\theta})}\right|\right) d\theta$$

hold in $0 \leqq r < 1$ for every convex nondecreasing function Φ. Let Φ be chosen so that $\Phi(u) = u^2$ for $u \geqq 0$ and $\Phi(u) = 0$ for $u < 0$. Adding the two resulting inequalities, we obtain

$$I_2(r, u) \leqq I_2(r, U).$$

On the other hand, since $g(0)=0$, it follows either from Parseval's identity or from the mean-value theorem applied to $\mathrm{Re}\{[g(z)]^2\}$ that $I_2(r,u)=I_2(r,v)$. Thus

$$I_2(r,g)=I_2(r,u)+I_2(r,v)=2I_2(r,u)\leqq 2I_2(r,U)=I_2(r,G),$$

or

$$\sum_{n=1}^{\infty}|\gamma_n|^2 r^{2n}\leqq \sum_{n=1}^{\infty}\frac{1}{n^2}r^{2n}.$$

Letting r tend to 1, we now obtain

$$\sum_{n=1}^{\infty}|\gamma_n|^2\leqq \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

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