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A DECOMPOSITION THEOREM FOR PLANAR HARMONIC MAPPINGS

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ABSTRACT. A necessary and sufficient condition is found for a complex-valued harmonic function to be decomposable as an analytic function followed by a univalent harmonic mapping.

In the theory of quasiconformal mappings, it is proved that for any measurable function μ with $\|\mu\|_\infty < 1$, the Beltrami equation $f_{\overline{z}} = \mu f_z$ admits a homeomorphic solution F, and every solution has the form $f = \psi \circ F$ for some analytic function ψ. (See [2], Ch. 6, §§1,2.) A complex-valued harmonic function with positive Jacobian in a domain D is known to satisfy a Beltrami equation of second kind $\overline{f_{\overline{z}}} = a f_z$, where a is an analytic function with the property $|a(z)| < 1$ in D. On the other hand, every solution of such an equation is harmonic in D. Moreover, if D is simply connected and $\|a\|_\infty < 1$ on D, then the equation admits homeomorphic solutions (see [1]). A nonconstant complex-valued harmonic function f is said to be *sense-preserving* if it satisfies $\overline{f_{\overline{z}}} = a f_z$ for some analytic function a with $|a(z)| < 1$.

Since harmonic functions are preserved under precomposition with analytic functions, the question now arises whether every sense-preserving harmonic function has the structure $f = F \circ \varphi$ for some univalent harmonic function F and some analytic function φ. In this paper we give a necessary and sufficient condition for the existence of such a decomposition.

Recall first that every harmonic function has a local representation of the form $f = h + \overline{g}$, where h and g are analytic. The Jacobian of f is $J = |h'|^2 - |g'|^2$, and its dilatation a satisfies $g' = ah'$. Thus a nonconstant function f is sense-preserving in a domain D if and only if $J(z) \geq 0$ there. According to a theorem of Lewy [3], the Jacobian of a univalent harmonic map in the plane can never vanish, so $J(z) > 0$ for sense-preserving univalent harmonic maps.

It is instructive to begin with two simple examples.

Example 1. Let f be the harmonic polynomial $f(z) = z^2 + \frac{2}{3}\overline{z}^3$. Then f has dilatation $a(z) = z$, and f is sense-preserving in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. We claim that f has no decomposition of the desired form in any neighborhood of the origin. Suppose on the contrary that $f = F \circ \varphi$, where φ is analytic near the origin and F is harmonic and univalent on the range of φ. Then F is sense-preserving because f is. We may suppose without loss of generality that $\varphi(0) = 0$.

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Then F has a representation $F = H + \overline{G}$ near the origin, where H and G are analytic and have power-series expansions

$$H(\zeta) = \sum_{n=1}^{\infty} A_n \zeta^n \quad \text{and} \quad G(\zeta) = \sum_{n=1}^{\infty} B_n \zeta^n,$$

with $|A_1| > |B_1| \geq 0$. Since the analytic part of $f(z)$ is $H(\varphi(z)) = z^2$, the function φ must have an expansion of the form

$$\varphi(z) = c_2 z^2 + c_3 z^3 + \dots, \qquad c_2 = 1/A_1.$$

It follows that $G \circ \varphi$ has an expansion starting with an even power of z. However, the given form of f shows that $G(\varphi(z)) = \frac{2}{3} z^3$; we have reached a contradiction. The conclusion is that f has no decomposition $f = F \circ \varphi$ of the required form in any neighborhood of the origin.

Example 2. Let $f(z) = z^2 + \frac{1}{2}\overline{z}^4$. Now f has dilatation $a(z) = z^2$, so it is sense-preserving in $\mathbb{D}$. But here f does have a decomposition $f = F \circ \varphi$ of the desired form in $\mathbb{D}$, with $F(\zeta) = \zeta + \frac{1}{2}\overline{\zeta}^2$ and $\varphi(z) = z^2$.

We are now ready to state our decomposition theorem.

Theorem 1. *Let f be a complex-valued nonconstant harmonic function defined on a domain $D \subset \mathbb{C}$ and let a be its dilatation function. Then in order that f have a decomposition $f = F \circ \varphi$ for some function φ analytic in D and some univalent harmonic mapping F defined on $\varphi(D)$, it is necessary and sufficient that $|a(z)| \neq 1$ on D and $a(z_1) = a(z_2)$ wherever $f(z_1) = f(z_2)$. Under these conditions the representation is unique up to conformal mapping; any other representation $f = \widetilde{F} \circ \tilde{\varphi}$ has the form $\widetilde{F} = F \circ \psi^{-1}$ and $\tilde{\varphi} = \psi \circ \varphi$ for some conformal mapping ψ defined on $\varphi(D)$.*

Note that in Example 1 the dilatation function is univalent, while f is not univalent in any neighborhood of the origin. In Example 2 the univalence of F in the disk means that $f(z_1) = f(z_2)$ if and only if $z_1^2 = z_2^2$; or if and only if $a(z_1) = a(z_2)$.

Proof of Theorem 1. Suppose first that $f = F \circ \varphi$ and let $A(\zeta)$ be the dilatation function of F. A direct calculation shows that $a(z) = A(\varphi(z))$ for all z in D, and it follows from Lewy's theorem that $|a(z)| \neq 1$. Furthermore, $f(z_1) = f(z_2)$ implies that $\varphi(z_1) = \varphi(z_2)$, since F is univalent; so it follows that $a(z_1) = a(z_2)$.

Conversely, suppose that $|a(z)| \neq 1$ in D and that $f(z_1) = f(z_2)$ implies $a(z_1) = a(z_2)$. We shall construct the required functions F and φ by appeal to the known theory of Beltrami equations. With no loss of generality we may suppose that $|a(z)| < 1$ in D; for otherwise we need only pass to the conjugate function $\overline{f}$.

It is easily seen that the problem reduces to finding a univalent function G on $\Omega = f(D)$ for which the composition $G \circ f$ is analytic on D. Thus our requirement is that $(G \circ f)_{\overline{z}} = 0$, which reduces to

$$(G_w \overline{a} + G_{\overline{w}}) \overline{f_z} = 0.$$

Hence $(G \circ f)_{\overline{z}} = 0$ in D if G is chosen to satisfy the Beltrami equation $G_{\overline{w}} = \mu G_w$, where

$$\mu(w) = -\overline{a(f^{-1}(w))}.$$

Although f need not be univalent and so $f^{-1}(w)$ may be multiple-valued, the hypothesis that $a(z_1) = a(z_2)$ wherever $f(z_1) = f(z_2)$ ensures that the composition

$a \circ f^{-1}$ is single-valued. Thus the function μ is well-defined, and the hypothesis that $|a(z)| < 1$ implies $|\mu(w)| < 1$ in Ω. Furthermore, $\sup_{w \in E} |\mu(w)| < 1$ for every compact subset $E \subset \Omega$.

Now let $\{D_n\}$ be an exhaustion of D by compact subsets with nonempty interiors: $D_1 \subset D_2 \subset \dots$ and $\bigcup_{n=1}^{\infty} D_n = D$. Letting $\Omega_n = f(D_n)$, define

$$\mu_n(w) = -\overline{a(f^{-1}(w))}, \qquad w \in \Omega_n;$$

and extend μ_n continuously onto the Riemann sphere $\widehat{\mathbb{C}}$ so that $\mu_n(\infty) = 0$ and

$$\max_{w \in \widehat{\mathbb{C}}} |\mu_n(w)| = \max_{w \in \Omega_n} |\mu_n(w)| = \max_{z \in D_n} |a(z)|.$$

By the general theory of quasiconformal mappings (see [2], Ch. 5), the Beltrami equation $G_{\overline{w}} = \mu_n G_w$ has a homeomorphic solution G_n in $\widehat{\mathbb{C}}$ such that $G_n(\infty) = \infty$. Fix z_0 and z_1 in D_1 such that $f(z_0) \neq f(z_1)$. This is possible because f is not constant in D_1. With $w_0 = f(z_0)$ and $w_1 = f(z_1)$, define

$$H_n(w) = \frac{G_n(w) - G_n(w_0)}{G_n(w_1) - G_n(w_0)}.$$

Then H_n is also a homeomorphic solution to the Beltrami equation, normalized to satisfy $H_n(w_0) = 0$, $H_n(w_1) = 1$, and $H_n(\infty) = \infty$. Consequently (see [2], Ch. 2, §5), some subsequence of $\{H_n(w)\}$ converges locally uniformly in Ω to a univalent function $H(w)$ that satisfies the Beltrami equation $H_{\overline{w}} = \mu H_w$ in $f(\Omega)$. It follows that $\varphi = H \circ f$ satisfies the Cauchy-Riemann equation $\varphi_{\overline{z}} = 0$, so it is analytic in D.

To see that $F = H^{-1}$ is harmonic in $\varphi(D)$, we need only remember that $f = F \circ \varphi$ was assumed to be harmonic in D. Near any point $\zeta = \varphi(z)$ where $\varphi'(z) \neq 0$, we can then conclude that $F = f \circ \varphi^{-1}$ is harmonic, where φ^{-1} is a local inverse. But F is locally bounded, so the (isolated) images of critical points of φ are removable, and F is harmonic in $\varphi(D)$. This establishes the required decomposition $f = F \circ \varphi$.

To prove the uniqueness assertion, suppose there were another representation $f = \widetilde{F} \circ \tilde{\varphi}$ of the prescribed form. Let $G = \widetilde{F}^{-1}$, and observe that G is a smooth function since its inverse is harmonic. Furthermore, the composite function $G \circ f$ is analytic and nonconstant, so again G must satisfy the Beltrami equation $G_{\overline{w}} = \mu G_w$. But then by the uniqueness of quasiconformal mappings with prescribed complex dilatation (see [2], Ch. 4, §5), we may conclude that $G = \psi \circ H$ for some function ψ conformal on $\varphi(D)$. In other words, $\widetilde{F} = F \circ \psi^{-1}$, as claimed. This completes the proof. □

Theorem 1 describes the harmonic functions that have a global decomposition of the given form. What about the existence of a local decomposition? We have already mentioned that f is locally univalent near a point z_0 if and only if $J(z_0) \neq 0$. In this case, f admits locally the trivial decomposition $f = F \circ \varphi$ with $F = f$ and $\varphi(z) = z$. Hence we will assume that $J(z_0) = 0$. If $|a(z_0)| = 1$, we know by Theorem 1 that f has no decomposition of the required form near z_0, so we will suppose that $|a(z_0)| \neq 1$. Passing to the conjugate function $\overline{f}$ if necessary, we may assume that $|a(z_0)| < 1$. Since the Jacobian is $J = (1 - |a|^2)|h'|^2$, it follows that $h'(z_0) = g'(z_0) = 0$. Without loss of generality, we may take $z_0 = 0$ and $f(z_0) = 0$.

Then f has the form

$$f(z) = \sum_{n=m}^{\infty} a_n z^n + \sum_{n=m}^{\infty} \overline{b}_n \overline{z}^n, \qquad |a_m| > |b_m| \geq 0,$$

near the origin, for some $m \geq 2$. Suppose now that f has a local decomposition $f = F \circ \varphi$, where φ is analytic near 0 and F is harmonic and univalent in some neighborhood of $\varphi(0) = 0$. Then F has the local structure

$$F(\zeta) = \sum_{n=1}^{\infty} A_n \zeta^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{\zeta}^n, \qquad |A_1| > |B_1| \geq 0,$$

and so φ must have the form

$$\varphi(z) = \sum_{n=m}^{\infty} c_n z^n, \qquad c_m = a_m / A_1.$$

Comparing coefficients, we find that $a_n = A_1 c_n$ and $b_n = B_1 c_n$ for $m \leq n < 2m$, which gives $b_m a_n = a_m b_n$ for $m < n < 2m$. If in particular $b_m = 0$, then $b_n = 0$ for $m < n < 2m$, as in the discussion of Example 1. Our result may be summarized as follows.

Theorem 2. *Let f be a sense-preserving harmonic function in some neighborhood of a point z_0 where its Jacobian $J(z_0) = 0$. Suppose that f has a local decomposition $f = F \circ \varphi$ for some functions φ analytic near z_0 and F harmonic and univalent near $\zeta_0 = \varphi(z_0)$, where f, F, and φ have the structures indicated above for some $m \geq 2$. Then $b_m a_n = a_m b_n$ for $m < n < 2m$.*

Although the condition of Theorem 2 is necessary for the existence of a local decomposition, it is not sufficient, as the following example shows.

Example 3. Let $f(z) = 2z^2 + z^4 + z^5 + \overline{z}^2 + \overline{z}^4 + \overline{z}^5$. Then $m = 2$ and $a_2 = 2$, $a_3 = 0$, $b_2 = 1$, $b_3 = 0$. Thus $b_2 a_3 = a_2 b_3 = 0$. However, a further comparison of coefficients gives the contradictory relations

$$A_1 c_2 = 2B_1 c_2 = 2, \quad c_3 = 0, \quad \text{and} \quad A_1 c_5 = B_1 c_5 = 1.$$

This shows that f has no local decomposition at the origin.

We may also apply Theorem 1 to reach the same conclusion. Here

$$a(z) = \frac{2z + 4z^3 + 5z^4}{4z + 4z^3 + 5z^4},$$

and $|a(z)| < 1$ near the origin. The equation $f(z_1) = f(z_2)$ is equivalent to

$$z_1^2 - z_2^2 = 2\,\mathrm{Re}\{z_2^2 - z_1^2 + z_2^4 - z_1^4 + z_2^5 - z_1^5\},$$

which is satisfied for instance if $z_1 = -z_2 = it$, $t > 0$. On the other hand, the equation $a(z_1) = a(z_2)$ implies that

$$4(z_1^2 - z_2^2) = 5(z_2^3 - z_1^3),$$

which is not satisfied for $z_1 = -z_2 \neq 0$. Thus $f(z_1) = f(z_2)$ does not imply $a(z_1) = a(z_2)$, and it follows from Theorem 1 that f has no decomposition of the required type in any neighborhood of the origin.

It seems likely that a closer study of the given harmonic function f near a critical point will lead to a necessary and sufficient condition for the existence of a local decomposition. In this connection, the work of Lyzzaik [4] may be relevant.

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