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Robin functions and distortion theorems for regular mappings

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Capacities of generalized condensers are applied to prove a two-point distortion theorem for conformal mappings. The result is expressed in terms of the Robin function and the Robin capacity with respect to the domain of definition of the mapping and subsets of the boundary of this domain. The behavior of Robin function under multivalent functions is studied. Some corollaries and examples of applications to distortion theorems for regular functions are given.

1 Introduction

The notion of the Robin function arises in a natural way in the theory of partial differential equations as a generalization of Green's function [2]. Because of conformal invariance these functions have wide applicability in complex analysis. The study of Robin's function and the Robin capacity associated to it has recently received a lot of attention (see e.g. [3, 8, 10–13, 25, 27, 34, 35, 37]). However, a series of questions remains open. Thus, the behavior of the Robin capacity under multivalent mappings has not been adequately studied. Furthermore in the literature not enough attention has been paid to applications of the Robin capacity to distortion theorems of regular functions (even for the case of conformal mappings). The capacity approach in the proof of such results is not restricted merely to the case of plane but it also applies to the case of higher dimensions [38]. In this paper these gaps will be partially filled. We begin with applications to conformal maps. In Section 2 a new two-point distortion theorem for regular univalent functions is proved in terms of the Robin capacity and the Robin function (Theorem 2.1). This theorem has a general character and contains many earlier results as particular cases (see Section 3). The proof of Theorem 2.1 makes use of the connection of some quadratic forms which depend on the values of the Robin function at the points under consideration and asymptotic expansion of the capacity of generalized condensers [4]. Note that a considerable part of well-known two-point distortion theorems were obtained by methods applicable to simply-connected domains for which the study of functions defined for instance in a ring domain is already quite complicated. It should be emphasized that our method applies in the same way for an arbitrary number of points and for domains of arbitrary connectivity. In Section 4 the behavior of Robin functions under multivalent functions is studied. For this purpose Lindelöf's principle involving the Green function is generalized to the case of the Robin function (Theorem 4.1). This generalization leads to Theorem 4.2 about the behaviour of the Robin capacity under regular mappings. Here we develop further the approach of Mityuk [23, 24], from which many applications follow. For these applications a crucial feature is taking into account the multiplicity of the covering. We also consider particular cases of these theorems and other results, connected with aforementioned questions. We proceed to the definition of the Robin function and to its counterpart with a boundary pole [4].

Let a domain B of the complex plane $\overline{\mathbf{C}}_z$ be bounded by a finite number of analytic curves, let γ be a nonempty closed subset of ∂B, consisting of a finite number of nondegenerated Jordan arcs, and let z_0 be a finite point of

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the set $\overline{B}\setminus\gamma$. We denote by $g(z) = g_B(z, z_0, \gamma)$ a continuous real-valued function on $\overline{B}\setminus\{z_0\}$, continuously differentiable on $\overline{B}\setminus(\gamma\cup\{z_0\})$, harmonic in $B\setminus\{z_0\}$ and satisfying the following conditions

$$g(z) = 0 \quad \text{for} \quad z \in \gamma,$$
$$\frac{\partial g}{\partial n}(z) = 0 \quad \text{for} \quad z \in (\partial B)\setminus(\gamma\cup\{z_0\}),$$

$g(z) + \log|z - z_0|$ is a harmonic function in a neighborhood of the point z_0 ($\partial/\partial n$ denotes differentiation in the direction of the interior normal to the boundary ∂B). In the case when $z_0 = \infty$, the function $g_B(z, z_0, \gamma)$ is defined in an analogical way with the only difference, that the harmonicity of the function $g_B(z, z_0, \gamma) - \log|z|$ in a neighborhood of infinity is required. A finitely connected domain $B \subset \overline{\mathbf{C}}_z$ and a closed subset γ of its boundary ∂B are called admissible, if the boundary ∂B does not have isolated points, and γ consists of a finite number of nondegerated boundary components. The definition of the function $g_B(z, z_0, \gamma)$ for general admissible domains B and sets γ takes place with the help of a conformal mapping (cf. [4]). For $z_0 \in B$ the function $g_B(z, z_0, \gamma)$ is called the Robin function of the domain B with a pole at the point z_0. In the case when $\gamma = \partial B$, the Robin function agrees with the Green function $g_B(z, z_0, \partial B) = g_B(z, z_0)$. We also introduce the notation:

$$r(B, \gamma, z_0) = \exp\left\{\lim_{z\to z_0}[g_B(z, z_0, \gamma) + \log|z - z_0|]\right\}$$

in the case of a finite point z_0 and

$$r(B, \gamma, \infty) = \exp\left\{\lim_{z\to\infty}[g_B(z, \infty, \gamma) - \log|z|]\right\}.$$

When $z_0 \in B$ and $\gamma = \partial B$ the quantity $r(B, \gamma, z_0) \equiv r(B, z_0)$ is called the inner radius of the domain B with respect to the point z_0. If furthermore B is simply connected, then $r(B, z_0)$ is its conformal radius. For $z_0 = \infty$ the conformal radius is sometimes defined as $r^{-1}(B, z_0)$. The quantity $r^{-1}(B, \infty)$ agrees with the logarithmic capacity of the complement $r^{-1}(B, \infty) = \operatorname{cap}(\overline{\mathbf{C}}_z\setminus B)$. For $z_0 = \infty$ and $z_0 \in B$ the quantity $r^{-1}(B, \gamma, z_0)$ is called *the Robin capacity* of the set γ with respect to the domain B.

Finally, when B is simply connected, γ is a boundary arc of B and $z_0 \in \partial B\setminus\gamma$, the quantity $r(B, \gamma, z_0)$ has been considered under different names and from various viewpoints in [14, 16, 22, 32].

The main method for the proof of Theorem 2.1 is the notion of the capacity of a generalized condenser [Dub1] (in what follows the word "generalization" will be omitted). Let B be a finitely connected domain on the complex plane $\overline{\mathbf{C}}_z$, and let $\overline{B}$ denote the compactification of B by means of prime ends in the sense of Caratheodory. In what follows, whenever it makes sense we will identify an element of $\overline{B}$, corresponding to an interior point of B, with the same point, and the support of an accessible boundary point and this point itself will be denoted with the same letter. A triple $C = (B, \mathcal{E}, \Delta)$ is called a condenser where $\mathcal{E} = \{E_k\}_{k=1}^n$ is the union of pairwise nonintersecting subsets E_k, $k = 1, \ldots, n$, closed in $\overline{B}$ and $\Delta = \{t_k\}_{k=1}^n$ is the union of real numbers t_k, $k = 1, \ldots, n$. The capacity $\operatorname{cap} C$ of the condenser C is defined as the infimum of Dirichlet integrals

$$I(v, B) := \iint_B |\nabla v|^2\, dx\, dy$$

taken over all functions $v(z)$ $(z = x + iy)$, continuous in $\overline{B}$, satisfying Lipschitz condition at some neighborhood of each finite point B and equal to t_k at some neighborhood of E_k, $k = 1, \ldots, n$.

We also need the asymptotics of the capacity of a condenser, a special case of the result in [4]. For a finite point z_0 of the complex sphere $\overline{\mathbf{C}}_z$ we denote by $E(z_0, r)$ a closed disk with the center at the point z_0 and with radius $r > 0$. In the case of the point at infinity we define $E(\infty, r) := \{z : |z| \geq 1/r\}$. Let the domain $B \subset \overline{\mathbf{C}}_z$ and let $\gamma \subset \partial B$ be admissible, let $Z = \{z_k\}_{k=1}^n$ be the union of distinct points of the domain B, let $\triangle = \{t_k\}_{k=1}^n$ be the union of real numbers, $\sum_{k=1}^n t_k^2 \neq 0$, and let $\Psi = \{\psi_k(r)\}_{k=1}^n$, where $\psi_k(r) \equiv \mu_k r^{\nu_k}$, and μ_k, ν_k, $k = 1, \ldots, n$, are positive numbers. For sufficiently small r we denote

$$C(r; B, \gamma, Z, \Delta, \Psi) := (B, \{\gamma, E(z_1, \psi_1(r)), \ldots, E(z_n, \psi_n(r))\}, \{0, t_1, \ldots, t_n\}).$$

From Theorem 7 of [4] the following asymptotic formula is obtained:

$$\begin{aligned}\operatorname{cap} C(r; B, \gamma, Z, \Delta, \Psi) &= 2\pi \left[\sum_{k=1}^{n} \frac{t_k^2}{\nu_k}\right] \left(-\frac{1}{\log r}\right) \\ &- 2\pi \left[\sum_{k=1}^{n} \frac{t_k^2}{\nu_k^2} \log \frac{r(B,\gamma,z_k)}{\mu_k} + \sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq k}}^{n} \frac{t_k}{\nu_k} \frac{t_k}{\nu_l} g_B(z_k, z_l, \gamma)\right] \left(\frac{1}{\log r}\right)^2 + o\left(\left(\frac{1}{\log r}\right)^2\right), \\ &\qquad r \to 0.\end{aligned} \tag{1.1}$$

Formula (1.1) remains valid if in the definition of the condenser $C(r; B, \gamma, Z, \Delta, \Psi)$ the disks $E(z_k, \psi_k(r))$, $k = 1, \ldots, n$, are replaced with “almost disks” [4].

2 Two-point distortion theorem for univalent functions

As well-known, quite many recent papers deal with two-point distortion theorems (cf. for example [18–21] and the literature therein). Our approach is different because we consider multiply connected domains and make use of the behavior of the function on the boundary. We first define “the weighted derivative” of a conformal mapping. Let the domain B of the plane $\overline{\mathbf{C}}_z$ and the set $\gamma \subset \partial B$ be admissible. In analogy with the hyperbolic case we consider the conformal invariant

$$\delta(\zeta, z; B, \gamma) := \exp\{-g_B(\zeta, z, \gamma)\},$$

where the points $\zeta, z \in B$. Let the function $w = f(z)$ map conformally and univalently the domain B into an admissible domain $G \subset \overline{\mathbf{C}}_w$ and let $\Gamma \subset \partial G$ be also admissible. For each point $z \in B$ we set

$$|Df(z)| := \lim_{\zeta \to z} \frac{\delta(f(\zeta), f(z); G, \Gamma)}{\delta(\zeta, z; B, \gamma)} = \frac{r(B, \gamma, z)}{r(G, \Gamma, f(z))} \, |f'(z)|.$$

This definition extends to the case of boundary points $z \in (\partial B)\backslash\gamma$ $(f(z) \in (\partial G)\backslash\Gamma)$, if the derivative $f'(z)$ is understood for example, as the angular derivative. In the particular case where $B = \{z : |z| < 1\}$, $\gamma = \partial B$ and $G = \{w : |w| < 1\}$, $\Gamma = \partial G$ we have

$$\delta(\zeta, z; B, \gamma) = \left|\frac{\zeta - z}{1 - \overline{z}\zeta}\right|$$

with a similar formula for $\delta(f(\zeta), f(z); G, \Gamma)$ and, therefore, the expression $|Df(z)|$ agrees with ”hyperbolic–hyperbolic derivative” of f at $z \in B$

$$|Df(z)| = \frac{1 - |z|^2}{1 - |f(z)|^2} \, |f'(z)|$$

[[20], p. 116]. The following two-point distortion theorem contains, as special cases, several well-known results of this kind.

Theorem 2.1 *Let $B \in \overline{\mathbf{C}}_z$ and $G \in \overline{\mathbf{C}}_w$, be domains and let the sets $\gamma \subset \partial B$ and $\Gamma \subset \partial G$ be admissible. We assume that the function f maps the domain B conformally and univalently into the domain G such that $f(\gamma) \subset \Gamma$, where the image $f(\gamma)$ is understood in the sense of boundary correspondence. Then for all points $z_1, z_2 \in B$ and all real numbers t_1 and t_2 the following inequality holds*

$$|Df(z_1)|^{t_1^2} |Df(z_2)|^{t_2^2} \geq \left[\frac{\delta(z_1, z_2; B, \gamma)}{\delta(f(z_1), f(z_2); G, \Gamma)}\right]^{2t_1 t_2} . \tag{2.1}$$

If for a conformal mapping f the inclusion $f(B) \subset G$ is valid and $f((\partial B)\backslash\gamma) \subset (\partial G)\backslash\Gamma$, then

$$|Df(z_1)|^{t_1^2} |Df(z_2)|^{t_2^2} \leq \left[\frac{\delta(z_1, z_2; B, \gamma)}{\delta(f(z_1), f(z_2); G, \Gamma)}\right]^{2t_1 t_2} \tag{2.2}$$

for all points $z_1, z_2 \in B$ and all real numbers t_1, t_2.

P r o o f. We may assume that $t_1^2 + t_2^2 \neq 0$, and the domains B and G are bounded by analytic Jordan arcs. Suppose that the inclusion $f(\gamma) \subset \Gamma$ holds. For sufficiently small $r > 0$ we consider the condenser

$$C(r; f(B), f(\gamma), W, \Delta, \Psi),$$

where $W = \{w_k\}_{k=1}^2$, $w_k = f(z_k)$, $k = 1, 2$; $\Delta = \{t_1, t_2\}$; $\Psi = \{|f'(z_k)|r\}_{k=1}^2$. We shall prove that the capacity of this condenser does not exceed the capacity of the condenser

$$C(r; G, \Gamma, W, \Delta, \Psi).$$

Indeed, if a function v is continuous in $\overline{G}$, satisfies the Lipschitz condition in a neighborhood of every finite point of the domain G, is equal to zero in a neighborhood of Γ and for $k = 1, 2$ is equal to t_k in a neighborhood of the set $E(w_k, |f'(z_k)|r)$, , then it enjoys the afore listed properties, if the domain G is replaced with $f(B)$, and the set Γ with $f(\gamma)$. Thus

$$I(v, G) \geq I(v, f(B)) \geq \operatorname{cap} C(r; f(B), f(\gamma), W, \Delta, \Psi).$$

Taking here the infimum over all possible functions v, we obtain

$$\operatorname{cap} C(r; G, \Gamma, W, \Delta, \Psi) \geq \operatorname{cap} C(r; f(B), f(\gamma), W, \Delta, \Psi).$$

On the other hand, from the conformal invariance of the capacity it follows that

$$\operatorname{cap} C(r; f(B), f(\gamma), W, \Delta, \Psi) = \operatorname{cap}(B, \mathcal{E}_1, \Delta_1),$$

where

$$\mathcal{E}_1 = \{\gamma, f^{-1}(E(w_1, |f'(z_1)|r)), f^{-1}(E(w_2, |f'(z_2)|r))\},$$

$\Delta_1 = \{0, t_1, t_2\}$. The sets $f^{-1}(E(w_k, |f'(z_k)|r))$, $k = 1, 2$, form “almost disks”. By virtue of formula (1.1)

$$\begin{aligned}\operatorname{cap}(B, \mathcal{E}_1, \Delta_1) = 2\pi(t_1^2 + t_2^2)\left(-\frac{1}{\log r}\right)& \\ - 2\pi\big[t_1^2 \log r(B, \gamma, z_1) + t_2^2 \log r(B, \gamma, z_2) + 2t_1 t_2 g_B(z_1, z_2, \gamma)\big]&\left(\frac{1}{\log r}\right)^2 \\ + o\left(\left(\frac{1}{\log r}\right)^2\right), \quad r \to 0.&\end{aligned}$$

We applied here the symmetry of the Robin function: $g_B(z_1, z_2, \gamma) = g_B\,(z_2, z_1, \gamma)$. Again by formula (1.1) we have

$$\begin{aligned}&\operatorname{cap} C(r; G, \Gamma, W, \Delta, \Psi) \\ &\quad = 2\pi\big(t_1^2 + t_2^2\big)\left(-\frac{1}{\log r}\right) \\ &\quad - 2\pi\left[t_1^2 \log\frac{r(G, \Gamma, w_1)}{|f'(z_1)|} + \, + t_2^2 \log\frac{r(G, \Gamma, w_2)}{|f'(z_2)|} + 2t_1 t_2 g_G(w_1, w_2, \Gamma)\right]\left(\frac{1}{\log r}\right)^2 \\ &\quad + o\left(\left(\frac{1}{\log r}\right)^2\right), \quad r \to 0.\end{aligned}$$

Summing up the above relations for the capacities of condensers we arrive at the inequality (2.1).

We now assume that the inclusion $f((\partial B)\backslash\gamma) \subset (\partial G)\backslash\,\Gamma$ holds, and let the function u be continuous at the points of $\overline{f(B)}$, let it satisfy the Lipschitz condition in a neighborhood of every finite point of the domain $f(B)$, let it equals zero in a neighborhood of $f(\gamma)$ and let it equals t_k in a neighborhood of the sets $E(w_k, |f'(z_k)|r)$, $k = 1, 2$. The function $\widetilde{u}$, defined as an extension of u by

$$\widetilde{u}(w) = \begin{cases} u(w), & w \in \overline{f(B)}, \\ 0 & \in \overline{G} \backslash \overline{f(B)}, \end{cases}$$

satisfies the aforementioned properties with $f(B)$ in place of G and $f(\gamma)$ in place of Γ. Therefore

$$I(u, f(B)) = I(\widetilde{u}, G) \geq \operatorname{cap} C(r; G, \Gamma, W, \Delta, \Psi).$$

In view of the arbitrary choice of the function u, we have

$$\operatorname{cap} C(r; f(B), f(\gamma), W, \Delta, \Psi) \geq \operatorname{cap} C(r; G, \Gamma, W, \Delta, \Psi).$$

Therefore, unlike in the first case, now also an inequality in the other direction is valid. Repeating again a corresponding part of the proof of inequality (2.1), we arrive at the inequality (2.2). The theorem is proved. □

Remark 2.2 From the result of the paper [7] it follows that the sign of the equality in (2.1) is attained only in the case when $\overline{f(B)} = \overline{G}$, $f(\gamma) = \Gamma$ and the set $G \cap \partial f(B)$ consists of a finite number of piecewise smooth curves, and at the interior points of the curves the normal derivative of the function $t_1 g_G(w, w_1, \Gamma) + t_2 g_G(w, w_2, \Gamma)$ is equal to zero. The equality in (2.2) hold if and only if $\overline{f(B)} = \overline{G}$ and $t_1 g_G(w, w_1, \Gamma) + t_2 g_G(w, w_2, \Gamma) = 0$ on $G \cap f(\gamma)$.

3 Particular cases

First of all we observe that for $t_1 = 1$, $t_2 = 0$. Theorem 2.1 has the character of a majorization principle: if $f(\gamma) \subset \Gamma$, then for every point z in the domain B the following

$$|Df(z)| \geq 1, \tag{3.1}$$

whereas in the case $f((\partial B) \backslash \gamma) \subset (\partial G) \backslash \Gamma$ we have

$$|Df(z)| \leq 1, \tag{3.2}$$

for every point $z \in B$. In particular, if $\gamma = \partial B$, $\Gamma = \partial G$ the inequality (3.2) is well-known as the monotonicity property of the inner radius of the domain. If $\gamma \neq \partial B$, $\Gamma \neq \partial G$, then the inequalities (3.1) and (3.2) can be extended to a boundary point $z \in (\partial B) \backslash \gamma$ under the condition of the existence of the corresponding limit of the derivative f'. Let us now consider the particular cases of the inequalities (2.1), (2.2), (3.1), (3.2), when $B = \{z : |z| < 1\}$ and $G = \{w : |w| < 1\}$.

Corollary 3.1 *Let the function f be regular and univalent in the disk $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$, for $|z| < 1$. We suppose that the function f maps in the sense of boundary correspondence a set α, consisting of a finite number of open arcs of the circle $|z| = 1$, into the circle $|w| = 1$. Then*

$$\sqrt{|f'(0)|} \operatorname{cap} \{w : |w| = 1,\ w \notin f(\alpha)\} \leq \operatorname{cap} \{z : |z| = 1,\ z \notin \alpha\},$$

where $\operatorname{cap}(\cdot)$ *denotes the logarithmic capacity.*

P r o o f. We may assume that the sets $\gamma = \{z : |z| = 1, z \notin \alpha\}$ and $\Gamma = \{w : |w| = 1,\ w \notin f(\alpha)\}$ are admissible. The inequality (3.2) gives

$$r(\{z : |z| < 1\}, \gamma, 0)|f'(0)| \leq r(\{w : |w| < 1\}, \Gamma, 0).$$

It remains to verify that

$$r(\{z : |z| < 1\}, \gamma, 0) = (\operatorname{cap} \gamma)^{-2}. \tag{3.3}$$

In view of the symmetry with respect to the circle $|z| = 1$ we have

$$g_U(z, 0, \gamma) = g_{\overline{\mathbf{C}}_z \setminus \gamma}(z, 0) + g_{\overline{\mathbf{C}}_z \setminus \gamma}(z, \infty),$$

where $U = \{z : |z| < 1\}$. Therefore

$$\log r(U, \gamma, 0) = \log r(\overline{\mathbf{C}}_z \setminus \gamma, 0) + g_{\overline{\mathbf{C}}_z \setminus \gamma}(0, \infty).$$

On the other hand

$$g_{\overline{\mathbf{C}}_z \setminus \gamma}(z, 0) + \log |z| = g_{\overline{\mathbf{C}}_z \setminus \gamma}(z, \infty)$$

and therefore

$$\log r(\overline{\mathbf{C}}_z \setminus \gamma, 0) = g_{\overline{\mathbf{C}}_z \setminus \gamma}(0, \infty). \tag{3.4}$$

Summing up the above equalities we get

$$r(U, \gamma, 0) = (r(\overline{\mathbf{C}}_z \setminus \gamma, 0))^2,$$

which is equivalent to (3.3). The corollary is proved. □

Applying the inequality (3.1) instead of (3.2) we arrive as above to the following result of Pommerenke.

Corollary 3.2 ([28, p. 217]) *If the function f is regular and univalent in the disk $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$, and if a closed set γ on the circle $|z| = 1$ is mapped in the sense of boundary correspondence under the mapping f onto a closed set Γ on the circle $|w| = 1$, then*

$$\sqrt{|f'(0)|}\, \operatorname{cap} \Gamma \geq \operatorname{cap} \gamma.$$

Corollary 3.3 ([26]) *Let the function f map the disk $|z| < 1$ conformally and univalently into the disk $|w| < 1$. Then for all points z_1, z_2 of the disk $|z| < 1$ and for all real numbers t_1, t_2 the following inequality is valid*

$$\left[\frac{(1 - |z_1|^2)|f'(z_1)|}{1 - |f(z_1)|^2}\right]^{t_1^2} \left[\frac{(1 - |z_2|^2)|f'(z_2)|}{1 - |f(z_2)|^2}\right]^{t_2^2} \leq \left|\frac{(z_1 - z_2)(1 - \overline{f(z_1)} f(z_2))}{(1 - \overline{z}_1 z_2)(f(z_1) - f(z_2))}\right|^{2t_1 t_2}.$$

This inequality follows immediately from (2.2). For some applications of this inequality see [20, p. 125]. For $t_1 = 1, t_2 = 0$ we obtain the inequality of Pick

$$|f'(z)|(1 - |z|^2) \leq 1 - |f(z)|^2, \quad |z| < 1,$$

which, as well-known, is valid for all funtions $w = f(z)$, $|f(z)| < 1$, for $|z| < 1$ regular in the disk $|z| < 1$ (not necessarily univalent). Setting in Corollary 3.3 $z_1 = 0$, $z_2 = re^{i\varphi}$, $t_1 = -t_2 = 1$, and letting $r \to 0$, we arrive after simple calculations at the following statement for the Schwarzian derivatives $S_f(z)$.

Corollary 3.4 *If the function $f(z) = c_1 z + \dots$ is regular and univalent in the disk $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$, then*

$$|S_f(0)| \leq 6(1 - |c_1|^2).$$

Corollary 3.5 *Let the function f be regular and univalent in the disk $|z| < 1$, let $|f(z)| < 1$ for $|z| < 1$, and let $f(\gamma) \subset \Gamma$, where γ is a closed subset of the circle $|z| = 1$, be different from the whole circle $|z| = 1$, and let Γ be a similar subset of $|w| = 1$. Then for all z_1, z_2 in $|z| < 1$ and all real numbers t_1, t_2 the following inequality holds*

$$\prod_{k=1}^{2} \left[\frac{r^2(\overline{\mathbf{C}}_z \setminus \gamma, z_k)(1 - |f(z_k)|^2)|f'(z_k)|}{r^2(\overline{\mathbf{C}}_w \setminus \Gamma, f(z_k))(1 - |z_k|^2)}\right]^{t_k^2}$$
$$\geq \left\{\frac{\exp[g_{\overline{\mathbf{C}}_w \setminus \Gamma}(f(z_1), f(z_2)) + g_{\overline{\mathbf{C}}_w \setminus \Gamma}(f(z_1), 1/\overline{f(z_2)})]}{\exp[g_{\overline{\mathbf{C}}_z \setminus \gamma}(z_1, z_2) + g_{\overline{\mathbf{C}}_z \setminus \gamma}(z_1, 1/\overline{z}_2)]}\right\}^{2t_1 t_2}.$$

P r o o f. According to Theorem 2.1

$$\prod_{k=1}^{2}\left[\frac{r(U_z,\gamma,z_k)|f'(z_k)|}{r(U_w,\Gamma,f(z_k))}\right]^{t_k^2} \ge \left\{\frac{\exp[g_{U_w}(f(z_1),f(z_2),\Gamma)]}{\exp[g_{U_z}(z_1,z_2,\gamma)]}\right\}^{2t_1t_2},$$

where $U_z = \{z : |z| < 1\}$ and $U_w = \{w : |w| < 1\}$. Applying the inequality

$$g_{U_z}(\zeta,z,\gamma) = g_{\overline{\mathbf{C}_z}\setminus\gamma}(\zeta,z) + g_{\overline{\mathbf{C}_z}\setminus\gamma}(\zeta,1/\overline{z}), \quad \zeta,z\in U_z,$$

we conclude that the right-hand side of the above inequality agrees with the right-hand side of the inequality of Corollary 3.5. Furthermore, from this we conclude that

$$\log r(U_z,\gamma,z_k) = \log r(\overline{\mathbf{C}}_z\setminus\gamma,z_k) + g_{\overline{\mathbf{C}}_z\setminus\gamma}(z_k,1/\overline{z}_k), \quad k=1,2.$$

Let the function $\varphi(z) := (z-z_k)/(1-\overline{z}_k z)$ and $B := \varphi(\overline{\mathbf{C}}_z\setminus\gamma)$. From the conformal invariance of the Green function and the inequality (3.4) it follows that

$$\begin{aligned} g_{\overline{\mathbf{C}_z}\setminus\gamma}(z_k,1/\overline{z}_k) &= g_B(0,\infty) = \log r(B,0)\\ &= \log[r(\overline{\mathbf{C}}_z\setminus\gamma,\ z_k)|\varphi'(z_k)|\,] = \log[r(\overline{\mathbf{C}}_z\setminus\gamma,z_k)/(1-|z_k|^2)\,], \quad k=1,2. \end{aligned}$$

Therefore

$$r(U_z,\gamma,z_k) = \frac{r^2(\overline{\mathbf{C}}_z\setminus\gamma,z_k)}{1-|z_k|^2}, \quad k=1,2.$$

Similar representation holds for $r(U_w,\Gamma,f(z_k))$, $k=1,2$, which completes the proof of Corollary 3.5. □

Corollary 3.6 *Under the assumptions of Corollary 3.5 we assume furthermore that at two points $z_k, |z_k| = 1$, $z_k \notin \gamma$ the angular limits $f(z_k)$ exist with $|f(z_k)| = 1$. Then for the angular derivatives $f'(z_k)$ the following inequality holds*

$$\prod_{k=1}^{2}\left[\frac{r(\overline{\mathbf{C}}_z\setminus\gamma,z_k)}{r(\overline{\mathbf{C}}_w\setminus\Gamma,f(z_k))}|f'(z_k)|\right]^{t_k^2} \ge \exp\left\{2t_1t_2[g_{\overline{\mathbf{C}}_w\setminus\Gamma}(f(z_1),f(z_2)) - g_{\overline{\mathbf{C}}_z\setminus\gamma}(z_1,z_2)\,]\right\}$$

for all real t_1 and t_2.

The proof follows from Corollary 3.5 with a limiting passage of points in $|z| < 1$ to boundary points (cf. [28, pp. 79–83]). We next represent a corollary of Theorem 2.1 for functions defined in an annulus.

Corollary 3.7 *Let the function f be regular and univalent in the annulus $R = \{z : \rho < |z| < 1\}$, whose image $f(R)$ lies in the disk $|w| < 1$ and $|f(z)| = 1$ for $|z| = 1$ and let β be a closed subset of the circle $|z| = 1$, consisting of a finite number of nondegenerate arcs. Then for all points z_1 and z_2, in the set $R \cup \{z : |z| = 1$, $z \notin \beta\}$, the following inequality holds*

$$\prod_{k=1}^{2}\frac{r(R,\gamma,z_k)|f'(z_k)|}{r(U_w,f(\beta),f(z_k))} \le \exp\left\{2[g_R((z_1,z_2,\gamma) - g_{U_w}(f(z_1),f(z_2),f(\beta))\,]\right\},$$

where $\gamma = \beta\cup\{z : |z| = \rho\}$, $U_w = \{w : |w| < 1\}$.

P r o o f. Setting in Theorem 2.1 $B = R$, $G = U_w$, $\Gamma = f(\beta)$ and $t_1 = -t_2 = 1$, we obtain from (2.2) the desired relation. □

The particular case of Corollary 3.7, when $\beta = \emptyset$, $|z_1| = |z_2| = 1$ was considered by A. Yu. Solynin [33, p. 135]. For $\beta = \{z : |z| = 1\}$ this corollary coincides with Theorem 1.4 of [6], which in the methodic sense goes back to Nehari [26]. Distortion theorems for functions described in Corollary 3.7 were also earlier considered in [9, 17].

Corollary 3.8 ([1, 31]) *Under the hypotheses of Corollary 3.7 let the set β coincide with the circle $|z| = 1$. Then for every point z of the ring R we have the inequality*

$$\left|\frac{1}{6}S_f(z) + \pi l(z,z)\right| \le \pi K(z,\overline{z}) - \frac{|f'(z)|^2}{(1-|f(z)|^2)^2},$$

where $K(\cdot,\cdot)$ and $l(\cdot,\cdot)$ are Bergman kernels of the first and second kind of the domain R with respect to the class of all functions, regular in R and with square integrable modulus in R.

P r o o f. Let z_0 be an arbitrary fixed point of the annulus R and let φ be a real number. Applying Corollary 3.7 in the case of the points $z_1 = z_0 + \rho e^{i\varphi}$ and $z_2 = z_0 - \rho e^{i\varphi}$, we get

$$\begin{aligned} &\frac{r(R, z_0+\rho e^{i\varphi}) r(R, z_0-\rho e^{i\varphi})}{4\rho^2 \exp[2g_R(z_0+\rho e^{i\varphi}, z_0-\rho e^{i\varphi})]} \cdot \frac{|f'(z_0+\rho e^{i\varphi}) f'(z_0-\rho e^{i\varphi})| 4\rho^2}{|f(z_0+\rho e^{i\varphi}) - f(z_0-\rho e^{i\varphi})|^2} \\ &\quad \le \frac{(1-|f(z_0+\rho e^{i\varphi})|^2)(1-|f(z_0-\rho e^{i\varphi})|^2)}{|1 - f(z_0+\rho e^{i\varphi})\overline{f(z_0-\rho e^{i\varphi})}|^2}. \end{aligned} \tag{3.5}$$

Making use of well-known relations between Bergman kernels and Green functions (see, e.g. [30]), as well as the Taylor expansion of the function f

$$f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + c_3(z-z_0)^3 + \ldots,$$

we conclude that the first quotient of the left side of the inequality (3.5) is equal to

$$1 - 4\pi\left\{K(z_0,\overline{z}_0) - \mathrm{Re}\left[e^{2i\varphi} l(z_0,z_0)\right]\right\}\rho^2 + o(\rho^2), \quad \rho \to 0.$$

The second quotient of the left side of (3.5) has the form

$$\left|1 + 4\left(\frac{c_3}{c_1} - \frac{c_2^2}{c_1^2}\right) e^{2i\varphi}\rho^2 + o(\rho^2)\right| = \left|1 + \frac{2}{3}S_f(z_0) e^{2i\varphi}\rho^2 + o(\rho^2)\right|, \quad \rho \to 0.$$

Finally the right side of (3.5) after simple transformations can be expressed in the form

$$1 - \frac{4|c_1|^2}{(1-|c_0|^2)^2}\,\rho^2 + o(\rho^2), \quad \rho \to 0.$$

Adding together the above expressions we arrive at the required inequality for $z = z_0$ because φ was arbitrary. □

4 Majorization principles for regular functions

We begin with the analogue of Lindelöf's principle which expresses the behavior of the Robin function under a regular mapping. The proof of this statement resembles in many respects the proof of the Lindelöf principle itself for the Green function [36, Chap. Y, §2].

Theorem 4.1 *Let the domains B and G, lie in the planes $\mathbf{C}_z$ and $\mathbf{C}_w$, respectively, and let also the subsets $\gamma \subset \partial B$ and $\Gamma \subset \partial G$ be admissible. We assume that the function f is regular in the domain B, $f(B) \subset G$ and $f((\partial B)\backslash\gamma) \subset (\partial G)\backslash\Gamma$ (i.e. for each sequence of points $\zeta_n \in B$, approaching the set $(\partial B)\backslash\gamma$, the corresponding sequence $f(\zeta_n) \to (\partial G)\backslash\Gamma$). Let w_0 be a point of $f(B)$, let $\{z_\nu\}$ $(\nu = 0, 1, \ldots)$ be in B, with $f(z_\nu) = w_0$ and let n_ν be the orders at the points z_ν of the zeros of $f(z) - w_0$. Then*

$$g_G(f(z), w_0, \Gamma) \ge \sum_{\nu \ge 0} n_\nu g_B(z, z_\nu, \gamma) \tag{4.1}$$

for all $z \in B$. Equality in (4.1) *for a single point $z \in B \setminus \bigcup_\nu \{z_\nu\}$ implies that* (4.1) *holds at every point B.*

P r o o f. Applying, if necessary, a conformal and univalent mapping we may suppose without loss of generality that the domains B and G are bounded by a finite number of circles and that the sets γ and Γ consist of a finite number of arcs on these circles. In this case the function f is defined on $B \cup (\partial B)\backslash\gamma$. We fix a natural number N, and consider the function

$$I_N(z) = g_G(f(z), w_0, \Gamma) - \sum_{\nu=0}^{N} n_\nu g_B(z, z_\nu, \gamma),$$

defined on the set $B\backslash\bigcup_{\nu\geq 0} z_\nu$. Applying the expansion of the function f in a neighborhood of the points z_ν we have

$$f(z) = w_0 + c_{n_\nu}(z - z_\nu)^{n_\nu} + \ldots, \quad c_{n_\nu} \neq 0,$$

and also the representation of the Robin function in a neighborhood of a pole, we easily conclude that the points z_ν, $\nu \leq N$, are removable singularities of the function $I_N(z)$. This function $I_N(z)$ is harmonic in the domain $B\backslash\bigcup_{\nu>N} z_\nu$, approaches $+\infty$ when $z \to z_\nu$, $\nu > N$, is nonnegative on the boundary points at γ, and at the points $z \in (\partial B)\backslash\gamma$ satisfies the condition $\partial I_N(z)/\partial n = 0$. By the maximum principle of E. Hopf we conclude that $I_N(z) \geq 0$ in the domain $B\backslash\bigcup_{\nu>N} z_\nu$, (cf., for instance, [29]). In view of the arbitrariness of N, the inequality (4.1) holds for all $z \in B$. At the same time we proved that the series $\sum_{\nu\geq 0} n_\nu g_B(z, z_\nu, \gamma)$ converges and that the function

$$g_G(f(z), w_0, \Gamma) - \sum_{\nu\geq 0} n_\nu g_B(z, z_\nu, \gamma) \tag{4.2}$$

is nonnegative and harmonic in B. By virtue of the maximum principle if the function is zero at a point of the domain B then it vanishes identically. The theorem is proved. □

In the papers [23,24] Mityuk considered a theorem about the change of the interior radius of the domain under a regular mapping and proved the effectiveness of this result with symmetrization methods. Following Mityuk, we consider the corresponding majorazation principle for the quantity $r(B, \gamma, z_0)$, which also may be considered as a distortion theorem.

Theorem 4.2 *Under the hypotheses of Theorem 4.1 suppose that in a neighborhood of the point z_0 we have the expansion*

$$f(z) = w_0 + c_n(z - z_0)^n + \ldots, \quad c_n \neq 0,$$

$(n = n_0)$. *Then*

$$r(G, \Gamma, w_0) \geq |c_n| r^n(B, \gamma, z_0) \exp\left\{\sum_{\nu\geq 1} n_\nu g_B(z_0, z_\nu, \gamma)\right\}. \tag{4.3}$$

If the mapping f satisfies $f(\gamma) = \Gamma$ and $f((\partial B)\backslash\gamma) = (\partial G)\backslash\Gamma$, then equality holds in the formula (4.3).

P r o o f. From inequality (4.1) it follows that in a small enough neighborhood of z_0

$$\begin{aligned} &- \log|c_n(z - z_0)^n + \ldots| + \log r(G, \Gamma, w_0) + o(1) \\ &\quad \geq -n\log|z - z_0| + n\log r(B, \gamma, z_0) + \sum_{\nu\geq 1} n_\nu g_B(z, z_\nu, \gamma), \quad z \to z_0. \end{aligned}$$

Therefore

$$\log r(G, \Gamma, w_0) \geq \log|c_n| + \log|1 + o(1)| + \log r^n(B, \gamma, z_0) + \sum_{\nu\geq 1} n_\nu g_B(z_0, z_\nu, \gamma) + o(1), \quad z \to z_0.$$

Passing to the limit when $z \to z_0$, we obtain the inequality (4.3). If $f(\gamma) = \Gamma$ and $f((\partial B)\backslash\gamma) = (\partial G)\backslash\Gamma$, then by the maximum principle of E. Hopf we conclude that the function (4.2) vanishes identically. Consequently, in (4.1) and also in (4.3) the equality sign holds. The theorem is proved. □

From Theorems 4.1 and 4.2 there follows a statement concerning the behavior of the quadratic form

$$\sum_{k=1}^{n} t_k^2 \log r(B,\gamma,z_k) + \sum_{k=1}^{n}\sum_{\substack{l=1\\ l\neq k}}^{n} t_k t_l g_B(z_k,z_l,\gamma)$$

under a regular mapping f in the case when the points z_k of the set $Z=\{z_k\}_{k=1}^n$ are located in the domain B, and the real numbers t_k of the set $\Delta=\{t_k\}_{k=1}^n$, have the same sign. For arbitrary real numbers t_k and p-valent functions f the following counterpart of the majorization principle of [5] holds.

Theorem 4.3 *We suppose that in the hypotheses of Theorem 4.1 the function $w=f(z)$ is p-valent in B. Let $w_l,\ l=1,\dots,m$, be distinct points of the domain $f(B)$, each of which has exactly p preimages in B taking into account multiplicity, and let $z_{jl},\ j=1,\dots,p$, be preimages of the points $w_l,\ l=1,\dots,m$ (each zero of the function $f(z)-w_l$ occurs as many times as its multiplicity indicates). Then for all real numbers $t_l, l=1,\dots,m$, the following inequality holds*

$$\begin{aligned} &p\left\{\sum t_l^2 \log r(G,\Gamma,w_l) + \sum t_k t_l g_G(w_k,w_l,\Gamma)\right\}\\ &\quad \geq \sum t_l^2[\log|c_{jl}| + p_{jl}\log r(B,\gamma,z_{jl})] + \sum t_k t_l g_B(z_{ik},z_{jl},\gamma),\end{aligned} \tag{4.4}$$

where c_{jl} and p_{jl} are defined from the expansion

$$f(z)-w_l = c_{jl}(z-z_{jl})^{p_{jl}}+\dots,\quad c_{jl}\neq 0,$$

$j=1,\dots,p,\ l=1,\dots,m$ (here and later the symbol $\sum$ denotes summation, over all indices appearing from the context, excluding those for which the summand is either ∞ or not defined). If, furthermore, $f(\gamma)=\Gamma$, $f((\partial B)\backslash\gamma)=(\partial G)\backslash\Gamma$ and the mapping f defines a p-valent covering of the domain G onto B, then in the inequality (4.4) the equality sign holds.

P r o o f. We may assume that the domains B and G are bounded by a finite number of circles and all numbers $t_l, l=1,\dots,m$, are nonzero. Set $W=\{w_l\}$, $\Delta=\{t_l\}$, $\Psi=\{\psi_l(r)\}$, $\psi_l(r)\equiv r, l=1,\dots,m$, and consider the condenser $C(r;G,\Gamma,W,\Delta,\Psi)$. Our task is to compare the capacity of this condenser with the capacity of the condenser $\widetilde{C}(r)$, defined as follows. Let $\widetilde{Z}=\{\widetilde{z}_{kl}\}$, $\widetilde{z}_{kl},\ k=1,\dots,n_l,\ l=1,\dots,m,$ be the zeros of the function$f(z)-w_l$ without counting the multiplicity $\widetilde{\Delta}=\{t_{kl}\}$, $t_{kl}=t_l/p,\ k=1,\dots,n_l$,; $\widetilde{\Psi}=\{\psi_{kl}(r)\}$, $\psi_{kl}(r)=|\widetilde{c}_{kl}|^{-1/p_{kl}}r^{1/p_{kl}}$, and let $p_{kl}, k=1,\dots,n_l$, be the multiplicities at the points $\widetilde{z}_{kl}$, $\sum_{k=1}^{n_l} p_{kl}=p,\ l=1,\dots,m$, $\widetilde{c}_{kl}$ be the coefficient, corresponding to the point $\widetilde{z}_{kl}$. The condenser $\widetilde{C}(r)$ is obtained from the condenser $C(r;B,\gamma,\widetilde{Z},\widetilde{\Delta},\widetilde{\Psi})$ by changing the disks $E(\widetilde{z}_{kl},\psi_{kl}(r))$ with almost disks $\widetilde{E}(\widetilde{z}_{kl},\psi_{kl}(r))$. Here $\widetilde{E}(\widetilde{z}_{kl},\psi_{kl}(r))$ is the connected part of the preimage of $E(w_l,r)$ under the mapping f, lying in the neighborhood of the point $\widetilde{z}_{kl}$. Let now u be the potential function of the condenser $\widetilde{C}(r)$, i.e. the real-valued function u, continuous in $\overline{B}$, harmonic in $B\backslash\bigcup_{k,l}\widetilde{E}(\widetilde{z}_{kl},\psi_{kl}(r))$, equal to 0 on γ and equal to t_{kl} on $\widetilde{E}(\widetilde{z}_{kl},\psi_{kl}(r))$ and satisfying the condition $\partial u/\partial n=0$ at the points of $(\partial B)\backslash\gamma$. On the set $f(B)$ we define the function

$$U(w)=\sum_{f(z)=w} u(z).$$

For each $w\in f(B)$ the value of the function $U(w)$ is the sum of at most p summands. From the definition of the capacity of a condenser, the convexity of the function $y=x^2$ and conformal invariance of the Dirichlet integral we have

$$\operatorname{cap} C(r;G,\Gamma,W,\Delta,\Psi)\leq \iint_{f(B)}|\nabla U|^2\,du\,dv\leq p\iint_B|\nabla u|^2\,dx\,dy.$$

Applying the Dirichlet principle we conclude that

$$\operatorname{cap} C(r;G,\Gamma,W,\Delta,\Psi)\leq p\ \operatorname{cap}\widetilde{C}(r).$$

Applying the asymptotic formula (1.1) to both condensers we have

$$p\left\{\sum t_l^2 \log r(G,\Gamma,w_l) + \sum t_k t_l g_G(w_k,w_l,\Gamma)\right\}$$
$$\geq \sum t_l^2 [p_{kl}\log|\widetilde{c}_{kl}| + p_{kl}^2 \log r(B,\gamma,\widetilde{z}_{kl})] + \sum t_k p_{ik} t_l p_{jl} g_B(\widetilde{z}_{ik},\widetilde{z}_{jl},\gamma),$$

which coincides with the inequality (4.4).

Let now the function f define a complete p-to-one covering mapping of the domain G onto B and $f(\gamma) = \Gamma$, $f((\partial B)\backslash\gamma) = (\partial G)\backslash\Gamma$. If $\omega(w)$ is the potential function of the condenser $C(r; G, \Gamma, W, \Delta, \Psi)$, then the composite function $\omega(f(z))$ is the potential for the condenser $\widetilde{C}(r)$, and by the conformal invariance of the Dirichlet integral and the p-valence of the covering mapping we have

$$\operatorname{cap} C(r; G, \Gamma, W, \Delta, \Psi) = p \operatorname{cap} \widetilde{C}(r).$$

Taking into account the formula (1.1) , this gives the equality sign in (4.4). The theorem is proved. It can be proved that in the case when the t_l's have the different signs, the p-valence in Theorem 4.3 is essential ([5, p. 538]). □

5 Examples

As an application of the majorization principles of the preceding section we now consider some corollaries to the distortion in regular mappings. Immediately from Theorem 4.1 it follows that

Corollary 5.1 *Let the domain $B \subset \mathbf{C}_z$ and $G \subset \mathbf{C}_w$, and also the sets $\gamma \subset \partial B$, $\Gamma \subset \partial G$ be admissible. We require that the function f is regular in the domain B, $f(B) \subset G$ and $f((\partial B)\backslash\gamma) \subset (\partial G)\backslash\Gamma$. Then for every pair of distinct points z and ζ of the domain B the following inequality is valid*

$$\delta(f(z), f(\zeta), G, \Gamma) \leq \delta(z, \zeta, B, \gamma). \tag{5.1}$$

The equality in (5.1) *is attained in the case of conformal and univalent functions f.*

If the domains B and G are disks, and the sets γ and Γ are the boundaries of these disks, then (5.1) coincides with the invariant form of the Schwarz lemma due to G. Pick. We give an example of the inequality in (5.1) for the case when $B = \{z : 0 < \operatorname{Im} z < \pi/2\}$, $\gamma = \{z : \operatorname{Im} z = \pi/2\}$, $G = \{w : 0 < \operatorname{Im} w < \pi/2\}$ and $\Gamma = \{w : \operatorname{Im} w = \pi/2\}$:

$$\left|\frac{\left(e^{f(z)} - e^{f(\zeta)}\right)\left(e^{f(z)} - e^{\overline{f(\zeta)}}\right)}{\left(e^{f(z)} + e^{f(\zeta)}\right)\left(e^{f(z)} + e^{\overline{f(\zeta)}}\right)}\right| \leq \left|\frac{(e^z - e^\zeta)(e^z - e^{\overline{\zeta}})}{(e^z + e^\zeta)(e^z + e^{\overline{\zeta}})}\right|,$$

for all z, ζ, $0 < \operatorname{Im} z < \pi/2$, $0 < \operatorname{Im} \zeta < \pi/2$.

A particular case of Theorem 4.2 is the following statement.

Corollary 5.2 *In the hypotheses of Corollary* 5.1 *let the point $z_0 \in B$, $w_0 = f(z_0)$; and let z_ν, $\nu = 1, 2, \ldots$, be the zeros of the function $f(z) - w_0$, different from z_0 and let n_ν be the multiplicities of z_ν, $\nu = 1, 2, \ldots$. Then*

$$|Df(z_0)| \leq \exp\left\{-\sum_{\nu\geq 1} n_\nu\, g_B(z_0, z_\nu, \gamma)\right\} \leq 1. \tag{5.2}$$

Therefore, the inequality (3.2) holds for arbitrary regular functions (not necessarily univalent) and even allows a refined formulation taking into account the multiplicity. In particular, if the function f is regular in the unit disk $|z| < 1$, $f(0) = 0$, and $|f(z)| < 1$ for $|z| < 1$, and if f maps the set α, consisting of a finite number of open arcs of the circle $|z| = 1$, into the circle $|w| = 1$, then the inequality (5.2) in view of (3.3) gives

$$\sqrt{|f'(0)|}\operatorname{cap}\{w : |w| = 1,\ w \notin f(\alpha)\}$$
$$\leq (\operatorname{cap}\{z{:}|z|{=}1, z \notin \alpha\}) \exp\left\{-\sum_{\nu\geq 1} \frac{n_\nu}{2}\, g_U(0, z_\nu, \{z{:}|z|{=}1, z \notin \alpha\})\right\},$$

where $U = \{z : |z| < 1\}$. In the case when $B = U$, $\gamma = \partial U$, $z_0 = 0$ from the inequality (5.2) there follows Hayman's inequality [15, p. 124]

$$|f'(0)| \leq r(f(U), w_0).$$

For an arbitrary domain B and $\gamma = \partial B$ Corollary 5.2 was established by Mityuk [23]. We now give an example of the inequality (5.2), when domains B and G both are half planes. Let the function f be regular in $\mathrm{Im} z > 0$, $\mathrm{Im} f(z) > 0$ for $\mathrm{Im} z > 0$, and let f map the set $\{z = x + iy : |x| > 1, y = 0\}$ into the set $\{w = u + iv : u \notin [a, b], v = 0\}$ with $f(\infty) = \infty$, $f'(\infty) = 1$. Then

$$b - a \leq 2.$$

In fact, from (5.2) for every point ζ of the upper half plane we have

$$r(\{z : \mathrm{Im} z > 0\}, [-1, 1], \zeta)|f'(\zeta)| \leq r(\{w : \mathrm{Im} w > 0\}, [a, b], f(\zeta)).$$

Letting $\zeta \to \infty$, we get

$$r(\{z : \mathrm{Im} z > 0\}, [-1, 1], \infty) \leq r(\{w : \mathrm{Im} w > 0\}, [a, b], \infty),$$

which is equivalent to the required inequality.

Corollary 5.3 *Let the function f be regular and p-valent in the domain $B := \{z = x + iy : x > 0, y > 0\}$, $f(B) \subset B$, and $f(\alpha) \subset \alpha$, where $\alpha = \{z = x + iy : x > 0, y = 0\}$. We assume that in some neighborhood of the point $z_0 \in B$ we have the expansion*

$$f(z) = z_0 + c_p(z - z_0)^p + \ldots$$

and let for some point $\zeta \in B$, distinct from z_0, there exist exactly p distinct preimages in the domain $B : z_j, j = 1, \ldots, p$. Then the inequality

$$\left|\frac{\zeta \mathrm{Re}\zeta}{\mathrm{Im}\zeta}\right|^p \begin{bmatrix} \zeta \\ z_0 \end{bmatrix}^{2p} \geq |c_p| \left\{ \prod_{j=1}^{p} \left| \frac{f'(z_j) z_j \mathrm{Re} z_j}{\mathrm{Im} z_j} \right| \begin{bmatrix} z_j \\ z_0 \end{bmatrix}^2 \right\} \prod_{\substack{j,k=1 \\ j \neq k}}^{p} \begin{bmatrix} z_j \\ z_k \end{bmatrix}^{-1} \tag{5.3}$$

holds where the following notation is used

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left| \frac{(a - b)(a - \overline{b})}{(a + b)(a + \overline{b})} \right|.$$

If the function f defines a complete p–valent covering of B onto the domain B such that $f(\alpha) = \alpha$, $f((\partial B \backslash \alpha)) = (\partial B) \backslash \alpha$, then in the inequality (5.3) the equality sign holds.

Proof. By Theorem 4.3

$$\begin{aligned} &p \{\log r(B, (\partial B)\backslash\alpha, \zeta) - 2 g_B(\zeta, z_0, (\partial B)\backslash\alpha)\} \\ &\quad \geq \sum_{j=1}^{p} \log[\,|f'(z_j)|\, r(B, (\partial B)\backslash\alpha, z_j)] + \log|c_p| - 2 \sum_{j=1}^{p} g_B(z_j, z_0, (\partial B)\backslash\alpha) \\ &\qquad + \sum_{\substack{j,k=1 \\ j \neq k}}^{p} g_B(z_j, z_k, (\partial B)\backslash\alpha). \end{aligned}$$

This inequality coincides with (5.3), as we see by observing that for $D := \{z : \mathrm{Re} z > 0\}$ we have

$$g_B(z, \zeta, (\partial B)\backslash\alpha) = g_D(z, \zeta) + g_D(z, \overline{\zeta}) = -\log \begin{bmatrix} z \\ \zeta \end{bmatrix},$$

and, therefore,

$$\log r(B, (\partial B)\backslash\alpha, \zeta) = \lim_{z \to \zeta} [g_B(z, \zeta, (\partial B)\backslash\alpha) + \log|z - \zeta|] = \log \left| \frac{2\zeta \mathrm{Re}\zeta}{\mathrm{Im}\zeta} \right|.$$

□

6 Open problems

6.1

Prove two-point distortion theorems for functions f univalent in the disk $|z| < 1$, involving a lower estimate for the difference $|f(z_1) - f(z_2)|$ in terms of the $|z_1 - z_2|$ and the Maclaurin coefficients of f.

6.2

Known two-point distortion theorems in the disk with one point z_1 fixed and the other point z_2 approaching the boundary give in the limit either trivial estimates or classical ones (provided that boundary derivative exists). Prove two-point distortion theorems which give more interesting information when one of the points tends to the boundary.

6.3

Prove a nontrivial two-point distortion theorem for functions regular (not necessarily univalent) in the unit disk, involving the Schwarzian derivative.

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