The Inner Mapping Radius of Harmonic Mappings of the Unit Disk[1]

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Abstract

The class S_H consists of univalent, harmonic, and sense-preserving functions f in the unit disk, Δ, such that $f = h + \overline{g}$ where $h(z) = z + \sum_2^\infty a_k z^k$, $g(z) = \sum_1^\infty b_k z^k$. Using a technique from Clunie and Sheil-Small, we construct a family of 1-slit mappings in S_H by varying $\omega(z) = g'(z)/f'(z)$. As $\omega(z)$ changes, the tip of the slit slides along the negative real axis from the point 0 to -1. In doing so, we establish that the inner mapping radius, $\rho(f)$ can be as large as 4. In addition, we show that the inner mapping radius for functions in S_H^0 can be as small as 1/2 and as large as 2.

1 Introduction

For $f \in S_H$, the inner mapping radius, $\rho(f)$, of the domain $f(\Delta)$ is the real number $F'(0)$, where $F(z)$ is the analytic function that maps Δ onto $f(\Delta)$ and satisfies the conditions $F(0) = 0$, $F'(0) > 0$. If $f \in S_H^O$, the inner mapping radius is denoted by $\rho_O(f)$. The lower bound for $\rho(f)$ is 0. It is conjectured that the lower bound for $\rho_O(f)$ is $\frac{2}{3}$, although it has only been proved that $\rho_O(f) \geq \frac{1}{4}$ [6]. The upper bound for $\rho(f)$ cannot be larger than 2π, because of the Koebe $\frac{1}{4}$-theorem and Hall's result [4] showing that $f(\Delta)$ omits some point on any circle of radius R, where $R \geq r = \frac{\pi}{2}$. Similarily, $\rho_O(f)$ is bounded above by $\frac{8\pi\sqrt{3}}{9} < 4.837$. Sheil-Small conjectured that $\rho(f) \leq \frac{\pi}{2}$ [1,6]. This is based upon Hall's example [4] which maps Δ onto a disk whose radius is arbitrarily close to $\frac{\pi}{2}$. As far as we know no conjecture has been made on the upper bound for $\rho_O(f)$. In an earlier paper [3], the first author presented a collection of harmonic slit mappings for which there is a function in S_H whose inner mapping radius is π and a function in S_H^O with $\rho_O(f) \approx 1.91$.

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[2]*Dedicated to A.W. Goodman on the occasion of his eightieth birthday.*

Here we will present a collection of univalent, harmonic 1-slit mappings, $f = h + \overline{g}$ with $g'(z) = \omega(z)h'(z)$, whose slit is on the negative real axis. By changing $\omega(z)$, we are able to slide the slit away from the origin. For $f \in S_H^O$, the tip of the slit can be brought as close as $-\frac{1}{6}$ and as far as $-\frac{1}{2}$. The inner mapping radius for this last function is 2. When we enlarge the class so that $f \in S_H$, the slit point can be moved from 0 to -1, and thus the inner mapping radius can be brought arbitrarily close to 4. Hence, these functions provide the largest known values for the inner mapping radius.

2 Sliding 1-slit mappings

Let $f = h + \overline{g}$ be a complex-valued harmonic function in Δ. Suppose that the following conditions hold for all $z \in \Delta$:

1. $h(z) = \frac{k(z)}{(1-z)^3}$, where $k(z) = z + a_2z^2 + a_3z^3 + \cdots$ $(a_i \in \mathbb{R})$;
2. $g'(z) = \omega(z)h'(z)$, where $|\omega(z)| < 1$; and
3. $\omega(z) = \dfrac{\frac{1}{z}h'(\frac{1}{z})}{zh'(z)}$.

We require that h be analytic in the entire plane except for a pole at $z = 1$. By the first condition, we know that

$$h'(z) \quad = \frac{1 + (2a_2 + 2)z + \cdots + ((n+1)a_{n+1} + (3-n)a_n)z^n + \cdots}{(1-z)^4}.$$

Notice $h'(0) = 1$. The second condition assures us that f is locally univalent and sense-preserving, provided $h'(z) \neq 0$ in Δ. The final condition comes from forcing the function to be constant on the arc $\{e^{i\theta} : \ 0 < \theta < 2\pi\}$, as required by a result of Hengartner and Schober [5, Corollary 2.2] if f is to be a slit mapping. By 3, since h has real coefficients, $|w(z)| = 1$ when $z = e^{i\theta}$, $0 < \theta < 2\pi$ and Hengartner and Schober's theorem implies than an arc of the circle cannot map onto a line segment.

From condition 3, we have

$$\begin{aligned}\omega(z) &= \frac{\frac{1}{z}[1+\frac{2a_2+2}{z}+\frac{3a_3+a_2}{z^2}+\cdots](1-z)^4}{z[1+(2a_2+2)z+(3a_3+a_2)z^2+\cdots](1-\frac{1}{z})^4}\\ &= \frac{z^2+(2a_2+2)z+(3a_3+a_2)+\frac{4a_4}{z}+\cdots}{1+(2a_2+2)z+(3a_3+a_2)z^2+4a_4z^3+\cdots}.\end{aligned}$$

We want $\omega(z)$ to be analytic in Δ. Thus, $a_n = 0$ for $n \geq 4$. Hence $k(z) = z + a_2z^2 + a_3z^3$ and

$$\omega(z) = \frac{z^2+(2a_2+2)z+(3a_3+a_2)}{1+(2a_2+2)z+(3a_3+a_2)z^2} = \frac{z^2+\alpha z+\beta}{1+\alpha z+\beta z^2} \tag{1}$$

where α and β are real.

Since $\omega(0) = \beta$, the inequality $-1 < \beta < 1$ must hold. Also by conditions 1 and 2, we get

$$h'(z) = \frac{1+\alpha z+\beta z^2}{(1-z)^4}, \tag{2}$$

$$g'(z) = \frac{z^2+\alpha z+\beta}{(1-z)^4}. \tag{3}$$

Lemma 2.1. *: If w is given by (1) with α and β real then $|w(z)| < 1$ in Δ if and only if $-1 < \beta < 1$ and $|\alpha| \leq 1+\beta$. In this case, $h'(z) \neq 0$ (where h' is given by (2)).*

Proof: Notice that $|\beta z^2+\alpha z+1| = |z^2+\alpha z+\beta|$ on $\partial\Delta$ and that $|\beta z^2+\alpha z+1| > |z^2+\alpha z+\beta|$ if $z = 0$. Hence if $|\beta z^2+\alpha z+1| \neq 0$, then $|\beta z^2+\alpha z+1| > |z^2+\alpha z+\beta|$. Thus it suffices to find the conditions on α and β such that $\beta z^2+\alpha z+1 \neq 0$. Since the product of the roots is $\frac{1}{\beta}$ necessarily $\frac{1}{|\beta|} > 1$. If the roots are complex, they are conjugates and both of them are outside of Δ. On the other hand, if the roots are real, the condition $-1 < \beta < 1$ implies at least one root is outside Δ so we require only $\beta x^2+\alpha x+1 \geq 0$ when $x = \pm 1$. That is $|\alpha| \leq 1+\beta$. This concludes the proof of the lemma.

Integrating (2) and normalizing the result so that $h(0) = 0$, we have

$$h(z) = \frac{(\frac{1}{3}+\frac{1}{3}\beta-\frac{1}{6}\alpha)z^3+(\frac{1}{2}\alpha-1)z^2+z}{(1-z)^3}.$$

Repeating this for (3), we get

$$g(z) = \frac{(\frac{1}{3} + \frac{1}{3}\beta - \frac{1}{6}\alpha)z^3 + (\frac{1}{2}\alpha - \beta)z^2 + \beta z}{(1-z)^3}.$$

Therefore,

$$\begin{aligned} f(z) &= \operatorname{Re}[h(z) + g(z)] + i \operatorname{Im}[h(z) - g(z)] \\ &= \operatorname{Re}\left[\frac{(\frac{2}{3} + \frac{2}{3}\beta - \frac{1}{3}\alpha)z^3 + (\alpha - 1 - \beta)z^2 + (1+\beta)z}{(1-z)^3}\right] \\ &\qquad + i \operatorname{Im}\left[\frac{(1-\beta)z}{(1-z)^2}\right]. \end{aligned}$$

Lemma 2.2. *Let $|\beta| < 1$ and $|\alpha| \le \beta + 1$. If $\alpha + \beta + 1 \neq 0$, then $f(\Delta)$ consists of the entire plane slit along the negative real axis with the tip of the slit at $\frac{1}{6}\alpha - \frac{1}{3} - \frac{1}{3}\beta$. If $\alpha + \beta + 1 = 0$, then $f(\Delta)$ is the half plane $\operatorname{Re}(f(z)) > -\frac{1}{2} - \frac{1}{2}\beta$.*

Proof: Let $\frac{1+z}{1-z} = w = u + iv$. Then $z = \frac{w-1}{w+1}$. Substituting this into $f(z)$ and simplifying, we get:

$$\begin{aligned} f\left(\frac{w-1}{w+1}\right) &= \tfrac{1}{12} \operatorname{Re}\{(w-1)[(1+\alpha+\beta)(w^2+w) + (4 - 2\alpha + 4\beta)]\} \\ &\qquad + \tfrac{1-\beta}{4} i \operatorname{Im}\{w^2 - 1\}. \end{aligned}$$

Letting $\alpha + \beta + 1 = 0$, using $w = u + iv$, and taking the real and imaginary parts, this becomes $(-\frac{1}{2} - \frac{1}{2}\beta)(1-u) + i(\frac{1}{2} - \frac{1}{2}\beta)uv$. If we fix uv, (i.e. fix Im $f(z)$) then Re $f(z)$ take all values $> -\frac{1}{2} - \frac{1}{2}\beta$. Thus, the image is a half plane. On the other hand, if $\alpha + \beta + 1 \neq 0$,

$$\begin{aligned} f\left(\frac{w-1}{w+1}\right) &= \tfrac{1+\alpha+\beta}{12} \operatorname{Re}\{w^3 + 3(\tfrac{1-\alpha+\beta}{1+\alpha+\beta})w - \tfrac{4-2\alpha+4\beta}{1+\alpha+\beta}\} + \tfrac{1-\beta}{4} i \operatorname{Im}\{w^2 - 1\} \\ &= \tfrac{1+\alpha+\beta}{12}\left[u^3 - 3uv^2 + 3\left(\tfrac{1-\alpha+\beta}{1+\alpha+\beta}\right)u - \tfrac{4-2\alpha+4\beta}{1+\alpha+\beta}\right] + \tfrac{1-\beta}{2} iuv. \qquad (4) \end{aligned}$$

Notice that $u > 0$. If $v = 0$, then (4) is real and varies between $\frac{\alpha-2-2\beta}{6}$ and ∞. On the level curve, $uv = c \neq 0$, the imaginary part of (4) is constant while the real part is $[\frac{1+\alpha+\beta}{12}u^3 - \frac{(1+\alpha+\beta)c^2}{4u} + \frac{(1-\alpha+\beta)}{4(1+\alpha+\beta)}u + \frac{\alpha-2-2\beta}{6}]$, which varies between $-\infty$ and $+\infty$ as u varies between 0 and $+\infty$. Thus, for any $c \neq 0$, $f(\Delta)$ contains the entire line parallel to the real axis and through the point ic.

Theorem 2.1. *If $|\beta| < 1$ and $|\alpha| \leq \beta + 1$, then $f \in S_H$. If it is also true that $\beta = 0$, then $f \in S_H^O$.*

PROOF: Notice that $h(z) - g(z) = \frac{(1-\beta)z}{(1-z)^2}$ is an analytic univalent mapping of Δ onto a 1-slit domain, whose slit is on the negative real axis from $\frac{-(1-\beta)}{4}$ to $-\infty$. Hence, by a theorem of Clunie and Sheil-Small [2, theorem 5.3], f is univalent in Δ and convex in the direction of the real axis.

Therefore, under the constraints that $-1 < \beta < 1$, $|\alpha| \leq \beta + 1$, and $\alpha + \beta + 1 \neq 0$, we see that $f(\Delta)$ is a 1-slit domain with the slit on the negative real axis and the tip of the slit located at $\frac{1}{6}\alpha - \frac{1}{3} - \frac{1}{3}\beta$. As α and β vary, the tip of the slit moves along the negative real axis. If $\beta = 0$, then $-1 < \alpha \leq 1$ and $f \in S_H^O$, with the tip of the slit situated at any point from $-\frac{1}{6}$ up to, but, not including, $-\frac{1}{2}$. In particular, when $\alpha = 1$, f is the harmonic "Koebe" function for which $\rho_O(f) = \frac{2}{3}$. On the other extreme, as α approaches -1, the inner mapping radius approaches 2. If $-1 < \beta < 1$, then $f \in S_H$. For $\beta = -1 + \epsilon$ and $\alpha = \delta$, where $0 < \delta < \epsilon$, the tip of the slit approaches 0. As β increases and α decreases, the tip moves away from the origin. With $\beta = 1 - \epsilon$ and $\alpha = -2 + \delta$, where $0 < \delta < \epsilon$, the tip can be brought arbitrarily close to -1 and hence $\rho(f)$ can be made arbitrarily close to 4. We have proved the following theorem.

Theorem 2.2. *If $0 < c < 4$, there exists $f \in S_H$ such that $\mathbb{C} \setminus f(\Delta)$ is a slit along the negative real axis starting at $-\frac{c}{4}$ so that the inner mapping radius of $f(\Delta)$ is c. If $\frac{2}{3} \leq c < 2$, then f can be chosen in S_H^0. Thus, $\rho_0(f)$ can be as large as 2.*

3 An Example

Let $K_t(z) = \frac{(1-t)z}{(1-z)^2} + \frac{tz}{1-z} = \frac{z-tz^2}{(1-z)^2}, \quad 0 \le t \le 1$. Since $K_t(e^{i\theta}) = \frac{te^{i\theta}-1}{2(1-\cos\theta)}, \quad 0 < \theta < 2\pi$, it is easy to check that $K_t(\Delta)$ is the exterior of a parabola $K_t(\Delta) = \left\{(u,v) :\; u > -\frac{(1-t)}{t^2}v^2 - \frac{t+1}{4}\right\}$ when $0 < t < 1$ while $K_1(\Delta)$ is the half plane $\left\{(u,v) :\; u > -\frac{1}{2}\right\}$ and $K_0(\Delta)$ is the slit domain $\mathbb{C} - \left\{z :\; z \le -\frac{1}{4}\right\}$. For each $t, \;\; K_t$ is convex in the direction of the real axis and hence the theorem of Clunie and Sheil-Small applies. That is, if h and g are analytic with $h - g = K_t, \; h(0) = g(0) = 0$ and $g'(z) = zh'(z)$ then $f = h + \bar{g}$ is univalent and sense-preserving.

Integrating, we find

$$h(z) = \frac{z - (\frac{1}{2}+t)z^2 + (\frac{1}{6}+\frac{1}{3}t)z^3}{(1-z)^3}$$

and

$$g(z) = \frac{\frac{1}{2}z^2 + (\frac{1}{6}-\frac{2}{3}t)z^3}{(1-z)^3}$$

Again setting $w = \frac{1+z}{1-z}$ so that $z = \frac{w-1}{w+1}$ with $\mathrm{Re}\,(w) > 0$ and using the fact that $f(z) = K_t(z) + 2\ \mathrm{Re}\ g(z)$, the result is

$$\begin{aligned} f(z) &= \frac{(1-t)(w^2-1)}{4} + \frac{t(w-1)}{2} + 2\,\mathrm{Re}\left(\frac{(w-1)^2}{8}\left(\frac{2}{3}(1-t)w + (\frac{1}{3}+\frac{2}{3}t)\right)\right) \\ &= \left(\frac{(1-t)uv}{2} + \frac{tv}{2}\right)i + \frac{1-t}{6}u^3 - \frac{(1-t)}{2}uv^2 + \frac{t}{4}(u^2-v^2) - \frac{1}{6} - \frac{t}{12} \end{aligned}$$

where $w = u + iv$.

Set $f(z) = R + iI$. If $I = 0$, then $v = 0$ and $R \ge -\frac{1}{6} - \frac{t}{12}$. If $I = c =$ constant then $v = \frac{2c}{(1-t)u+t}$. For constant $I, \;\; R$ takes all values from $-\frac{I^2}{t} - \frac{1}{6} - \frac{t}{12}$ to ∞, $t \ne 0$. Thus, $f(\Delta)$ is the exterior of the parabola $R = -\frac{I^2}{t} - \frac{1}{6} - \frac{t}{12}$. To find the inner mapping radius of $f(\Delta)$, we want to find $k > 0$ and ρ, $0 < \rho < 1$ such that $kK_\rho(\Delta) = f(\Delta)$. Then k is the inner mapping radius of $f(\Delta)$. Setting $kK_\rho(e^{i\theta}) = R + iI$, we have

$$R = -\frac{I^2(1-\rho)}{k\rho^2} - \frac{k(1+\rho)}{4}.$$

Hence, we want $\frac{1-\rho}{k\rho^2} = \frac{1}{t}$ and $\frac{k(1+\rho)}{4} = \frac{1}{6} + \frac{t}{12}$. It follows that $k = \frac{(4t+2)-\sqrt{6(2t^2+t)}}{3}$ which is smallest when $t = \frac{1}{4}$. Thus $k \geq \frac{1}{2}$ with equality when $t = \frac{1}{4}$. We have proved the following.

Theorem 3.1. *If $\frac{2}{3} > \rho \geq \frac{1}{2}$, there exists $f \in S_H^0$ such that $f(\Delta)$ is the exterior of a parabola and the inner mapping radius of $f(\Delta)$ is ρ.*

It is interesting to observe that contrary to the situation in many cases, although $\left|\frac{g'(z)}{h'(z)}\right| = |z| = 1$ when $|z| = 1$ for the mappings above, we still have $\{f(e^{i\theta}) : \ 0 < \theta < 2\pi\}$ is the boundary curve for $f(\Delta)$. This is only possible because the boundary curve is concave with respect to $f(\Delta)$ in conformity with the result of Hengartner and Schober [5, Corollary 2.2].

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