Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2012,**32B**(2):586–604
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

HYPERHOLOMORPHIC THEORY ON KAEHLER MANIFOLDS*

Tang Dongmei (汤冬梅)
Department of Mathematics and Physics, Xiamen University of Technology, Xiamen 361024, China
E-mail: tdm1230@yahoo.com.cn
Zhong Tongde (钟同德)   *Qiu Chunhui* (邱春晖)
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

Abstract First of all, using the relations (2.3), (2.4), and (2.5), we define a complex Clifford algebra $\mathbb{W}_n$ and the Witt basis. Secondly, we utilize the Witt basis to define the operators $\bar{\partial}$ and $\bar{\partial}^{\wedge}$ on Kaehler manifolds which act on $\mathbb{W}_n$-valued functions. In addition, the relation between above operators and Hodge-Laplace operator is argued. Then, the Borel-Pompeiu formulas for $\mathfrak{W}_n$-valued functions are derived through designing a matrix Dirac operator $\overline{\mathbf{D}}$ and a 2×2 matrix–valued invariant integral kernel with the Witt basis.

Key words Kaehler manifolds; complex Clifford algebra; Witt basis; matrix Dirac operator; matrix Cauchy-Dirac kernel

2000 MR Subject Classification 30G35; 32A26

1 Introduction

Let M be a Kaehler manifold with metric

$$g=2\sum_{\alpha,\,\beta}g_{\alpha\bar{\beta}}\,dz^{\alpha}\,d\bar{z}^{\beta}.$$

For a (p,q) differential form on M

$$\varphi(z)=\frac{1}{p!\,q!}\sum\varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_q},$$

the Hodge-Laplace operator of $\varphi(z)$ is defined by

$$\Box=\bar{\partial}\bar{\partial}^{*}+\bar{\partial}^{*}\bar{\partial},\qquad\overline{\Box}=\partial\partial^{*}+\partial^{*}\partial,\qquad\Delta=d\delta+\delta d,\tag{1.1}$$

$$\Delta=2\Box=2\overline{\Box},\tag{1.2}$$

*Received October 20, 2009; revised March 18, 2011. Project supported in part by the National Natural Science Foundation of China (10771174, 10601040, 10971170) and Scientific Research Foundation of Xiamen University of Technology (700298).

where $d = \partial + \bar\partial$, $\delta = \partial^* + \bar\partial^*$ and

$$\partial\varphi = \sum_{\alpha=1}^{n} \nabla_\alpha \varphi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} dz^\alpha \wedge dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_q}, \tag{1.3}$$

$$\bar\partial\varphi = \sum_{\alpha=1}^{n} \overline{\nabla}_\alpha \varphi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} d\bar z^\alpha \wedge dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_q}, \tag{1.4}$$

$$\left(\bar\partial^*\varphi\right)_{\alpha_1,\cdots,\overline\beta_{q-1}} = (-1)^{p+1} \sum_{\beta,\gamma=1}^{n} g^{\overline\beta\gamma} \nabla_\gamma \varphi_{\alpha_1,\cdots,\alpha_p\overline{\beta}\overline\beta_1\cdots\overline\beta_{q-1}}, \tag{1.5}$$

where ∇ is the covariant derivative ([4, 5]).

Let

$$\overline{D} := \begin{pmatrix} \bar\partial & \bar\partial^* \\ \bar\partial^* & \bar\partial \end{pmatrix}, \qquad \overline{D}^* := \begin{pmatrix} \bar\partial^* & \bar\partial \\ \bar\partial & \bar\partial^* \end{pmatrix},$$

$$D := \begin{pmatrix} \partial & \partial^* \\ \partial^* & \partial \end{pmatrix}, \qquad D^* := \begin{pmatrix} \partial^* & \partial \\ \partial & \partial^* \end{pmatrix}.$$

Using the idea of invariant integral kernel of [3, 5], the authors introduce the Cauchy kernels for the theory of **m.v.d.f**. from the null-sets of the corresponding operators $\overline{D}$ and $\overline{D}^*$, by the formulas

$$K_{\overline{D}}(\zeta, z) := 2\frac{(n-1)!}{\pi^n} \sum_{q=1}^{n} \begin{pmatrix} \sum_{\alpha=1}^{n} g^{\bar q\alpha} \frac{\widehat{u}_\alpha}{|u|_g^{2n}} \widehat{d\bar z^q}; & \frac{u_q}{|u|_g^{2n}} d\bar z^q \\ \frac{u_q}{|u|_g^{2n}} d\bar z^q; & \sum_{\alpha=1}^{n} g^{\bar q\alpha} \frac{\widehat{u}_\alpha}{|u|_g^{2n}} \widehat{d\bar z^q} \end{pmatrix},$$

$$K_{\overline{D}^*}(\zeta, z) := 2\frac{(n-1)!}{\pi^n} \sum_{q=1}^{n} \begin{pmatrix} \frac{u_q}{|u|_g^{2n}} d\bar z^q; & \sum_{\alpha=1}^{n} g^{\bar q\alpha} \frac{\widehat{u}_\alpha}{|u|_g^{2n}} \widehat{d\bar z^q} \\ \sum_{\alpha=1}^{n} g^{\bar q\alpha} \frac{\widehat{u}_\alpha}{|u|_g^{2n}} \widehat{d\bar z^q}; & \frac{u_q}{|u|_g^{2n}} d\bar z^q \end{pmatrix},$$

and similarly for D and D^*.

From [10], we get, given a point ζ on a Kaehler manifold, there exists a complex coordinate system normal at ζ, that is to say, $g^{\bar\beta\alpha}(\zeta) = \delta_{\bar\beta\alpha}$. For a normal coordinate system at ζ, the kernels $K_{\overline{D}}(\zeta, z)$, $K_{\overline{D}^*}(\zeta, z)$, $K_D(\zeta, z)$, and $K_{D^*}(\zeta, z)$ refer to (3.23), (3.24), (3.25), and (3.26) in [5]. It is worth while to remark that when we treat the problem at a normal coordinate system, then the problem becomes more simply, especially can save many calculations. Using kernels $K_{\overline{D}}(\zeta, z)$ and $K_D(\zeta, z)$, the authors extended hyperholomorphic theory in $\mathbb{C}^n$ developed by R.Rocha-Chávez, M.Shapiro, and F.Sommen [3] to the Kaehler manifold. First, the authors obtained the Borel-Pompeiu (or Cauchy-Green) formula for smooth differential matrix-forms ([5] Th.3.1) and then investigated the hyperholomorphic $\overline{D}$-problem on Kaehler manifolds ([5] Th.4.1, Th4.2).

In the discussion of [3, 5], there appear the differential forms $d\bar z^j$ and the operators $\widehat{dz^j}$; they are in different natures, and thus, formally, do not belong to the same algebra. To include into the same algebra all complex differential forms and the complex algebra generated by the operators $\widehat{dz^j}$, so that the equalities

$$\Delta = 2\Box = 2\overline{\Box}$$

hold, we may consider the following complex algebra generated by $i^1, \cdots, i^n, \widehat{i^1}, \cdots, \widehat{i^n}$, with the following rules of multiplication:

$$\begin{cases} i^j \cdot i^k = -i^k \cdot i^j, \\ i^j \cdot i^j = 0, \end{cases}$$

$$\begin{cases} \widehat{i^j} \cdot \widehat{i^k} = -\widehat{i^k} \cdot \widehat{i^j}, \\ \widehat{i^j} \cdot \widehat{i^j} = 0, \end{cases}$$

$$\begin{cases} \widehat{i^j} \cdot i^k = -i^k \cdot \widehat{i^j}, \\ \widehat{i^j} \cdot i^j + i^j \cdot \widehat{i^j} = 1, \end{cases}$$

where $k, j = 1, \cdots, n$, with $k \neq j$ ([3] Chapter 9).

This complex algebra is associate, distributive, noncommutive, with zero divisors and with identity. We will denote this algebra by $\mathbb{W}_n$, and the basis is called the Witt basis.

In this article, using this algebra proceeded as in [3] Chapter 9, we develop a hyperholomorphic theory on Kaehler manifolds in complex Clifford analysis (with Witt basis) closed related to the above hyperholomorphic theory on Kaehler manifolds in complex analysis, so that we obtain the Borel–Pompeiu formula for smooth differential matrix–forms in complex Clifford analysis (with Witt basis), etc.

2 Hyperholomorphic Theory on Kaehler Manifolds in Complex Clifford Analysis

First, we introduce some notations [3]. Let Ω be a domain in the complex manifold M, denote by $\overline{\mathcal{G}}_k^q(\Omega)$ the set of all $(0,q)$-forms on Ω of class $C^{(k)}$, and by $\mathcal{G}_k^p(\Omega)$ the set of all $(p,0)$-forms of the same class, and set $\overline{\mathcal{G}}_k(\Omega) := \bigcup\limits_{q=0}^{n} \overline{\mathcal{G}}_k^q(\Omega)$, $\mathcal{G}_k(\Omega) := \bigcup\limits_{p=0}^{n} \mathcal{G}_k^p(\Omega)$.

An object of this article is the set of 2×2 matrices with entries from $\overline{\mathcal{G}}(\Omega)$. Occasionally we shall consider its symmetric image, replacing $\overline{\mathcal{G}}$ by $\mathcal{G}$. We use the following notations:

$$\overline{\mathfrak{G}}_k^q(\Omega) := \begin{pmatrix} \overline{\mathcal{G}}_k^q(\Omega) \ \overline{\mathcal{G}}_k^q(\Omega) \\ \overline{\mathcal{G}}_k^q(\Omega) \ \overline{\mathcal{G}}_k^q(\Omega) \end{pmatrix} := \left\{ \begin{pmatrix} \varphi^{11} \ \varphi^{12} \\ \varphi^{21} \ \varphi^{22} \end{pmatrix} \middle| \left\{\varphi^{ij}\right\} \subset \overline{\mathcal{G}}_k^q(\Omega) \right\}, \tag{2.1}$$

$$\overline{\mathfrak{G}}_k(\Omega) := \begin{pmatrix} \overline{\mathcal{G}}_k(\Omega) \ \overline{\mathcal{G}}_k(\Omega) \\ \overline{\mathcal{G}}_k(\Omega) \ \overline{\mathcal{G}}_k(\Omega) \end{pmatrix} := \left\{ \begin{pmatrix} \varphi^{11} \ \varphi^{12} \\ \varphi^{21} \ \varphi^{22} \end{pmatrix} \middle| \left\{\varphi^{ij}\right\} \subset \overline{\mathcal{G}}_k(\Omega) \right\}. \tag{2.2}$$

The same for $\mathfrak{G}_k^p(\Omega)$ and $\mathfrak{G}_k(\Omega)$. We call them as sets of matrix-valued differential forms, and we will use sometimes the abbreviation **m.v.d.f.** .

In [3, 5], there appear the differentials $d\bar{z}^j$ and the operators $\widehat{dz^j}$, in different natures, and thus, formally, do not belong to the same algebra. To include into the same algebra all complex differential forms and the complex algebra generated by the operators $\widehat{dz^j}$, such that the equalities $\Delta = 2\Box = 2\overline{\Box}$ hold, we may consider the following complex algebra generated by $i^1, \cdots, i^n; \widehat{i^1}, \cdots, \widehat{i^n}$, with the following rules of multiplication:

$$\begin{cases} i^j \cdot i^k = -i^k \cdot i^j, \\ i^j \cdot i^j = 0, \end{cases} \tag{2.3}$$

$$\begin{cases} \widehat{i^j} \cdot \widehat{i^k} = -\widehat{i^k} \cdot \widehat{i^j}, \\ \widehat{i^j} \cdot \widehat{i^j} = 0, \end{cases} \tag{2.4}$$

$$\begin{cases} \widehat{i^j} \cdot i^k = -i^k \cdot \widehat{i^j}, \\ \widehat{i^j} \cdot i^j + i^j \cdot \widehat{i^j} = 1, \end{cases} \tag{2.5}$$

where $k, j = 1, \cdots, n$, with $k \neq j$.

This complex algebra is associative, distributive, noncommutative, with zero divisors, and with identity. We will denote this algebra by $\mathbb{W}_n$.

Immediate consequences of rules (2.3), (2.4), and (2.5) are that, for any $j = 1, \cdots, n$, the following equalities hold:

$$\widehat{i^j} \cdot i^j \cdot \widehat{i^j} \cdot i^j = \widehat{i^j} \cdot i^j, \qquad i^j \cdot \widehat{i^j} \cdot i^j \cdot \widehat{i^j} = i^j \cdot \widehat{i^j}.$$

Each element $\mathbf{a}$ of $\mathbb{W}_n$ is of the form

$$\mathbf{a} = \sum_{\mathbf{j}\,\mathbf{k}} \mathbf{a_{jk}}\ \widehat{i^{\mathbf{j}}} \cdot i^{\mathbf{k}}, \tag{2.6}$$

where $\mathbf{a_{jk}} \in \mathbb{C}$. $\mathbf{j}, \mathbf{k}$ are, respectively, strictly increasing $|\mathbf{j}|$-tuple, $|\mathbf{k}|$-tuple, in $\{1, \cdots, n\}$ with $|\mathbf{j}|, |\mathbf{k}| = 0, \cdots, n$, and $\widehat{i^{\mathbf{j}}} := \widehat{i^{j_1}} \cdot \cdots \cdot \widehat{i^{j_{|\mathbf{j}|}}}$, $\quad i^{\mathbf{k}} := i^{k_1} \cdot \cdots \cdot i^{k_{|\mathbf{k}|}}$.

Note that $\mathbf{j}$ or $\mathbf{k}$ may be equal to $\emptyset$, the empty set; in this case, $\widehat{i^{\mathbf{j}}}$ or $i^{\mathbf{k}}$ are not included in the expression $\widehat{i^{\mathbf{j}}} \cdot i^{\mathbf{k}}$.

The defining equalities (2.3), (2.4), and (2.5) say that two Grassmann algebras, one generated by $\{i^k\}$ and the other by $\{\widehat{i^k}\}$, are mixed by the crucial conditions (2.5) into a single object.

Then, we give a classical definition of a complex Clifford algebra.

Let $Cl_{0,2n}$ be a complex Clifford algebra with generators $e_1, e_2, \cdots, e_{2n}$. This means that

$$e_k^2 = -1 =: -e_0, \qquad k \in \{1, \cdots, 2n\},$$

$$e_k e_q + e_q e_k = 0, \qquad k \neq q.$$

Any element $a \in Cl_{0,2n}$ is of the form $a = \sum\limits_A a_A e_A$, where $A = (\alpha_1, \cdots, \alpha_p)$ with $1 \leq \alpha_1 < \cdots < \alpha_p \leq 2n$, $\{a_A\} \subset \mathbb{C}$, $e_A := e_{\alpha_1} \cdots e_{\alpha_p}$.

Consider the following elements of $Cl_{0,2n}$:

$$f_j := \frac{1}{2}(e_{2j} + i\, e_{2j-1}) \qquad \text{and} \qquad \bar{f}_j := \frac{1}{2}(e_{2j} - i\, e_{2j-1}),$$

where $j = 1, \cdots, n$. Note that the $2n$-tuple $(f_1, \cdots, f_n, -\bar{f}_1, \cdots, -\bar{f}_n)$ is an underlying space basis of $Cl_{0,2n}$ in the sense of a complex vector space. But, alternatively,

$$\begin{cases} f_j \cdot f_k = -f_k \cdot f_j, \\ (-\bar{f}_j) \cdot f_k = -f_k \cdot (-\bar{f}_j), \\ (-\bar{f}_j) \cdot (-\bar{f}_k) = -(-\bar{f}_k) \cdot (-\bar{f}_j), \end{cases} \qquad \text{for all } k, j = 1, \cdots, n, \text{ with } k \neq j;$$

$$\begin{cases} f_j \cdot f_j = 0, \\ (-\bar{f}_j) \cdot f_j + f_j \cdot (-\bar{f}_j) = 1, \\ (-\bar{f}_j) \cdot (-\bar{f}_j) = 0, \end{cases} \qquad \text{for all } j = 1, \cdots, n. \tag{2.7}$$

Hence, we can make the following identifications: $i^j \equiv f_j$ and $\widehat{i^j} \equiv -\bar{f}_j$.

This means that the complex algebra $\mathbb{W}_n$ is nothing more than the complex Clifford algebra $Cl_{0,2n}$, but with the other basis fixed.

This basis is called the Witt basis and later we will see that it is important to study the Grassmann algebra as a part of the Clifford algebra. We shall use the notation $\mathbb{W}_n$ when we want to use the representation (2.6) of a complex Clifford number.

From now on, we introduce some differential operators on $\mathbb{W}_n$-valued functions. Let Ω be a domain on the Kaehler manifold M. We shall consider $\mathbb{W}_n$–valued functions defined in Ω:

$$\mathbf{f} : \Omega \longrightarrow \mathbb{W}_n.$$

On the set $C^1(\Omega; \mathbb{W}_n)$, define the operators $\bar{\partial}$ and $\bar{\partial}^\wedge$, respectively, by the equalities (ref. (1.2) and (1.3))

$$\bar{\partial}[\ \ ] := \sum_{\alpha=1}^{n} i^\alpha \nabla_{\bar{\alpha}}[\ \ ], \tag{2.8}$$

$$\bar{\partial}^\wedge[\ \ ] := \sum_{\mu,\,\beta=1}^{n} \widehat{i^\mu} \cdot g^{\bar{\mu}\beta} \cdot \nabla_\beta[\ \ ], \tag{2.9}$$

where $\nabla_{\bar{\alpha}}$ and ∇_β are covariant derivatives. For a scalar function $\mathbf{f}$, we have

$$\bar{\partial}[\mathbf{f}] := \sum_{\alpha=1}^{n} i^{\,\alpha} \nabla_{\bar{\alpha}}[\mathbf{f}] = \sum_{\alpha=1}^{n} i^{\,\alpha} \frac{\partial \mathbf{f}}{\partial \bar{z}^{\,\alpha}},$$

$$\bar{\partial}^\wedge[\mathbf{f}] := \sum_{\mu,\,\beta=1}^{n} \widehat{i^\mu} \cdot g^{\bar{\mu}\beta} \cdot \nabla_\beta[\mathbf{f}] = \sum_{\mu,\,\beta=1}^{n} \widehat{i^\mu} \cdot g^{\bar{\mu}\beta} \cdot \frac{\partial f}{\partial z^\beta},$$

as the covariant derivative of a scalar function $\mathbf{f}$ is defined to be its ordinary derivative. For the metric tensor $g^{\bar{\nu}\gamma}$, we have

$$\bar{\partial}[g^{\bar{\nu}\gamma}] = \sum_{\alpha=1}^{n} i^\alpha \nabla_{\bar{\alpha}} g^{\bar{\nu}\gamma} = 0, \qquad \bar{\partial}^\wedge[g^{\bar{\nu}\gamma}] = \sum_{\mu,\,\beta=1}^{n} \widehat{i^\mu} \cdot g^{\bar{\mu}\beta} \cdot \nabla_\beta g^{\bar{\nu}\gamma} = 0.$$

$\mathbb{W}_n$-valued function $\bar{\partial}[\mathbf{f}]$ can be interpreted as a specific "Clifford product" of a $\mathbb{W}_n$–valued function $\mathbf{f}$ with the "$\mathbb{W}_n$–valued function" whose coordinates are covariant derivations (not partial derivatives), that is, $\bar{\partial} := \sum\limits_{\alpha=1}^{n} i^{\,\alpha}\nabla_{\bar{\alpha}}$; with what is more, in this sense $\mathbf{f}$ is multiplied by $\bar{\partial}$ on the left-hand side:

$$\bar{\partial}\cdot\mathbf{f} := \bar{\partial}[\mathbf{f}]. \tag{2.10}$$

Of course, it is assumed here that a scalar–valued function commutes with the generators of $\mathbb{W}_n$. The same interpretations are valid for $\bar{\partial}^{\wedge}$.

We define now

$$\bar{\partial}_r[\mathbf{f}] := \mathbf{f}\cdot\bar{\partial} := \sum_{\alpha=1}^{n}\nabla_{\bar{\alpha}}\mathbf{f}\cdot i^{\,\alpha},$$

$$\bar{\partial}_r^{\wedge}[\mathbf{f}] := \mathbf{f}\cdot\bar{\partial}^{\wedge} := \sum_{\mu,\,\beta=1}^{n} g^{\,\bar{\mu}\beta}\cdot\nabla_{\beta}\mathbf{f}\cdot\widehat{i^{\,\mu}}.$$

Note that $\bar{\partial}_r[\mathbf{f}]$ and $\bar{\partial}_r^{\wedge}[\mathbf{f}]$ are $\mathbb{W}_n$-valued functions, which differ greatly from the definition of $\bar{\partial}_r^{*}$ in [3]. Mention that now one can use the notation $\mathbf{f}\cdot\bar{\partial}^{\wedge}$ without possible confusions (compare with [3]), but one needs to be careful with the fact that the multiplication "·", when the factors $\bar{\partial}$ and $\bar{\partial}^{\wedge}$ are included, is not associative, that is, in general:

$$(\mathbf{f}\cdot\bar{\partial})\cdot\mathbf{g}\neq\mathbf{f}\cdot(\bar{\partial}\cdot\mathbf{g}), \qquad (\mathbf{f}\cdot\bar{\partial}^{\wedge})\cdot\mathbf{g}\neq\mathbf{f}\cdot(\bar{\partial}^{\wedge}\cdot\mathbf{g}).$$

From the definitions of $\mathbf{f}\cdot\bar{\partial}$, $\bar{\partial}\cdot\mathbf{g}$, $\mathbf{f}\cdot\bar{\partial}^{\wedge}$ and $\bar{\partial}^{\wedge}\cdot g$, we know that the above two inequalities hold.

On $\mathbb{W}_n$-valued functions of class C^2, it is natural to define the Hodge-Laplace operator on a Kaehler manifold M as

$$\Delta[\mathbf{f}] := 2\sum_{\alpha,\beta=1}^{n} g^{\,\bar{\alpha}\beta}\nabla_{\bar{\alpha}}\nabla_{\beta}[\mathbf{f}] = 2\sum_{\alpha,\beta=1}^{n} g^{\,\bar{\alpha}\beta}\frac{\partial^2\mathbf{f}}{\partial\bar{z}^{\alpha}\partial z^{\beta}}. \tag{2.11}$$

Remark 2.1 On Kaehler manifolds, we have

$$\nabla_{\alpha}\nabla_{\bar{\beta}}[\mathbf{f}] = \nabla_{\alpha}\frac{\partial\mathbf{f}}{\partial\bar{z}^{\beta}} = \frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial\bar{z}^{\beta}}.$$

Proof By the definition of the covariant derivative of a covariant vector, we have

$$\nabla_{\alpha}\nabla_{\bar{\beta}}[\mathbf{f}] = \nabla_{\alpha}\frac{\partial\mathbf{f}}{\partial\bar{z}^{\beta}} = \frac{\partial}{\partial z^{\alpha}}\left(\frac{\partial\mathbf{f}}{\partial\bar{z}^{\beta}}\right) - \sum_{\gamma=1}^{n}\frac{\partial\mathbf{f}}{\partial\bar{z}^{\gamma}}\Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}}\,,$$

where ([11])

$$\Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}} = \frac{1}{2}g^{\bar{\gamma}\varepsilon}\left(\frac{\partial g_{\varepsilon\bar{\beta}}}{\partial z^{\alpha}} - \frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\varepsilon}}\right).$$

By the Kaehler condition,

$$\frac{\partial g_{\varepsilon\bar{\beta}}}{\partial z^{\alpha}} = \frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\varepsilon}},$$

we have

$$\Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}} = 0.$$

It follows that

$$\nabla_{\alpha}\nabla_{\bar{\beta}}[\mathbf{f}] = \nabla_{\alpha}\frac{\partial\mathbf{f}}{\partial\bar{z}^{\beta}} = \frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial\bar{z}^{\beta}}.$$

Similarly, we have

$$\nabla_{\bar{\alpha}}\nabla_{\beta}[\mathbf{f}] = \nabla_{\bar{\alpha}}\frac{\partial \mathbf{f}}{\partial z^{\beta}} = \frac{\partial^2 \mathbf{f}}{\partial \bar{z}^{\alpha}\partial z^{\beta}}.$$

Theorem 2.2 The following operator equalities hold on $\mathbb{W}_n$-valued functions of class C^2:

$$\begin{aligned} \bar{\partial}\bar{\partial}^{\wedge} + \bar{\partial}^{\wedge}\bar{\partial} &= \frac{1}{2}\Delta, \\ \bar{\partial}_r\bar{\partial}_r^{\wedge} + \bar{\partial}_r^{\wedge}\bar{\partial}_r &= \frac{1}{2}\Delta. \end{aligned} \tag{2.12}$$

$$\begin{aligned} \bar{\partial}\bar{\partial} &= 0, \\ \bar{\partial}^{\wedge}\bar{\partial}^{\wedge} &= 0, \\ \bar{\partial}_r\bar{\partial}_r &= 0, \\ \bar{\partial}_r^{\wedge}\bar{\partial}_r^{\wedge} &= 0. \end{aligned} \tag{2.13}$$

Proof

$$\begin{aligned}
&(\bar{\partial}\bar{\partial}^{\wedge} + \bar{\partial}^{\wedge}\bar{\partial})[\mathbf{f}] \\
&= \sum_{\alpha=1}^{n} i^{\,\alpha}\nabla_{\bar{\alpha}}\Big(\sum_{\mu,\,\beta=1}^{n}\widehat{i^{\mu}}g^{\bar{\mu}\beta}\nabla_{\beta}[\mathbf{f}]\Big) + \sum_{\mu,\,\alpha=1}^{n}\widehat{i^{\,\mu}}g^{\,\bar{\mu}\alpha}\nabla_{\alpha}\Big(\sum_{\beta=1}^{n} i^{\,\beta}\nabla_{\bar{\beta}}[\mathbf{f}]\Big) \\
&= \sum_{\alpha=1}^{n}\sum_{\mu,\,\beta=1}^{n} i^{\alpha}\widehat{i^{\mu}}\nabla_{\bar{\alpha}}g^{\bar{\mu}\beta}\cdot\nabla_{\beta}[\mathbf{f}] + \sum_{\alpha=1}^{n}\sum_{\mu,\,\beta=1}^{n} i^{\,\alpha}\widehat{i^{\,\mu}}g^{\,\bar{\mu}\beta}\nabla_{\bar{\alpha}}\nabla_{\beta}[\mathbf{f}] \\
&\quad + \sum_{\mu,\,\alpha=1}^{n}\sum_{\beta=1}^{n}\widehat{i^{\mu}}i^{\beta}g^{\bar{\mu}\alpha}\nabla_{\alpha}\nabla_{\bar{\beta}}[\mathbf{f}] \\
&= \sum_{\alpha=1}^{n}\sum_{\mu,\,\beta=1}^{n} i^{\alpha}\widehat{i^{\mu}}g^{\bar{\mu}\beta}\frac{\partial^2\mathbf{f}}{\partial\bar{z}^{\alpha}\partial z^{\beta}} + \sum_{\mu,\,\alpha=1}^{n}\sum_{\beta=1}^{n}\widehat{i^{\mu}}i^{\beta}g^{\bar{\mu}\alpha}\frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial\bar{z}^{\beta}} \\
&= \sum_{\mu,\,\beta=1}^{n}\left\{\sum_{\alpha=1}^{n} i^{\alpha}\widehat{i^{\mu}} + \sum_{\alpha=1}^{n}\widehat{i^{\mu}}i^{\alpha}\right\}g^{\bar{\mu}\beta}\frac{\partial^2\mathbf{f}}{\partial\bar{z}^{\alpha}\partial z^{\beta}} \\
&= \sum_{\alpha,\,\beta=1}^{n}\left(i^{\alpha}\widehat{i^{\alpha}} + \widehat{i^{\alpha}}i^{\alpha}\right)g^{\bar{\alpha}\beta}\frac{\partial^2\mathbf{f}}{\partial\bar{z}^{\alpha}\partial z^{\beta}} + \sum_{\substack{\alpha,\,\mu,\,\beta=1,\cdots,n\\ \alpha\neq\mu}}\left(i^{\alpha}\widehat{i^{\mu}} + \widehat{i^{\mu}}i^{\alpha}\right)g^{\bar{\mu}\beta}\frac{\partial^2\mathbf{f}}{\partial\bar{z}^{\alpha}\partial z^{\beta}} \\
&= \sum_{\alpha,\,\beta=1}^{n} g^{\bar{\alpha}\beta}\frac{\partial^2\mathbf{f}}{\partial\bar{z}^{\alpha}\partial z^{\beta}} \\
&= \frac{1}{2}\Delta[\mathbf{f}].
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}^{\wedge}\bar{\partial}^{\wedge}[\mathbf{f}] &= \sum_{\nu,\alpha=1}^{n}\sum_{\mu,\beta=1}^{n}\widehat{i^{\nu}}g^{\bar{\nu}\alpha}\nabla_{\alpha}\left(\widehat{i^{\mu}}g^{\bar{\mu}\beta}\nabla_{\beta}[\mathbf{f}]\right) \\
&= \sum_{\nu,\alpha=1}^{n}\sum_{\mu,\beta=1}^{n}\widehat{i^{\nu}}\widehat{i^{\mu}}g^{\bar{\nu}\alpha}g^{\bar{\mu}\beta}\left(\frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial z^{\beta}} - \frac{\partial\mathbf{f}}{\partial z^{\gamma}}\Gamma^{\gamma}_{\alpha\beta}\right) \\
&= \sum_{\mu=1}^{n}\sum_{\alpha,\,\beta=1}^{n}\widehat{i^{\mu}}\widehat{i^{\mu}}g^{\bar{\mu}\alpha}g^{\bar{\mu}\beta}\left(\frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial z^{\beta}} - \frac{\partial\mathbf{f}}{\partial z^{\gamma}}\Gamma^{\gamma}_{\alpha\beta}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\alpha,\,\beta=1}^{n}\sum_{\substack{\mu,\nu=1,\cdots,n\\ \mu<\nu}}\left\{\widehat{i^{\nu}}\,\widehat{i^{\mu}}g^{\bar\nu\alpha}g^{\bar\mu\beta}+\widehat{i^{\mu}}\,\widehat{i^{\nu}}g^{\bar\mu\alpha}g^{\bar\nu\beta}\right\}\left(\frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial z^{\beta}}-\frac{\partial\mathbf{f}}{\partial z^{\gamma}}\Gamma^{\gamma}_{\alpha\beta}\right)\\
&=\sum_{\alpha,\,\beta=1}^{n}\sum_{\substack{\mu,\nu=1,\cdots,n\\ \mu<\nu}}\left\{-\widehat{i^{\mu}}\,\widehat{i^{\nu}}g^{\bar\mu\beta}g^{\bar\nu\alpha}+\widehat{i^{\mu}}\,\widehat{i^{\nu}}g^{\bar\mu\beta}g^{\bar\nu\alpha}\right\}\left(\frac{\partial^2\mathbf{f}}{\partial z^{\alpha}\partial z^{\beta}}-\frac{\partial\mathbf{f}}{\partial z^{\gamma}}\Gamma^{\gamma}_{\alpha\beta}\right)\\
&=0.
\end{aligned}
$$

The same for the rest of equalities in (2.12) and (2.13).

Now, we need the set of 2×2 matrices with entries from $\mathbb{W}_n$. We use the following denotations:

$$\mathfrak{W}_n := \begin{pmatrix}\mathbb{W}_n\ \mathbb{W}_n\\ \mathbb{W}_n\ \mathbb{W}_n\end{pmatrix} := \left\{ \begin{pmatrix}\mathbf{A}^{11}\ \mathbf{A}^{12}\\ \mathbf{A}^{21}\ \mathbf{A}^{22}\end{pmatrix} \,\middle|\, \left\{\mathbf{A}^{i\,j}\right\}\subset\mathbb{W}_n \right\}.$$

The structure of a complex linear space in $\mathbb{W}_n$ is inherited by $\mathfrak{W}_n$: it is enough to add the elements and to multiple them by complex scalar in an entry-wise manner.

Given $\mathbf{A}$, $\mathbf{B}$ from $\mathfrak{W}_n$, their "Clifford product" $\mathbf{A}\mathbin{\dot{*}}\mathbf{B}$ is introduced as follows:

$$\mathbf{A}\mathbin{\dot{*}}\mathbf{B}=\begin{pmatrix}\mathbf{A}^{11}\ \mathbf{A}^{12}\\ \mathbf{A}^{21}\ \mathbf{A}^{22}\end{pmatrix}\mathbin{\dot{*}}\begin{pmatrix}\mathbf{B}^{11}\ \mathbf{B}^{12}\\ \mathbf{B}^{21}\ \mathbf{B}^{22}\end{pmatrix}:=\begin{pmatrix}\mathbf{A}^{11}\cdot\mathbf{B}^{11}+\mathbf{A}^{12}\cdot\mathbf{B}^{21};\ \mathbf{A}^{11}\cdot\mathbf{B}^{12}+\mathbf{A}^{12}\cdot\mathbf{B}^{22}\\ \mathbf{A}^{21}\cdot\mathbf{B}^{11}+\mathbf{A}^{22}\cdot\mathbf{B}^{21};\ \mathbf{A}^{21}\cdot\mathbf{B}^{12}+\mathbf{A}^{22}\cdot\mathbf{B}^{22}\end{pmatrix}.$$

This product remains to be associative and distributive:

$$\big(\mathbf{A}\mathbin{\dot{*}}\mathbf{B}\big)\mathbin{\dot{*}}\mathbf{C}=\mathbf{A}\mathbin{\dot{*}}\big(\mathbf{B}\mathbin{\dot{*}}\mathbf{C}\big);$$

$$(\mathbf{A}+\mathbf{B})\mathbin{\dot{*}}\mathbf{C}=\mathbf{A}\mathbin{\dot{*}}\mathbf{C}+\mathbf{B}\mathbin{\dot{*}}\mathbf{C};$$

$$\mathbf{C}\mathbin{\dot{*}}(\mathbf{A}+\mathbf{B})=\mathbf{C}\mathbin{\dot{*}}\mathbf{A}+\mathbf{C}\mathbin{\dot{*}}\mathbf{B}.$$

Thus, we shall consider $\mathfrak{W}_n$ as a complex algebra that is associative, distributive, non-commutative, and with zero divisors and identity.

In the sequel, we need introduce the matrix Dirac operators. We shall consider $\mathfrak{W}_n$-valued functions defined in Ω:

$$\mathbf{F}:\Omega\longrightarrow\mathfrak{W}_n.$$

Now, we need certain matrix operators composed of scalar operators (2.8), (2.9), and acting on $\mathfrak{W}_n$-valued functions of class C^1. We put

$$\overline{\mathbf{D}}:=\begin{pmatrix}\bar\partial & \bar\partial^{\wedge}\\ \bar\partial^{\wedge} & \bar\partial\end{pmatrix},\qquad \overline{\mathbf{D}}^{*}:=\begin{pmatrix}\bar\partial^{\wedge} & \bar\partial\\ \bar\partial & \bar\partial^{\wedge}\end{pmatrix}. \tag{2.14}$$

We can interpret the matrix $\overline{\mathbf{D}}[\mathbf{F}]$ as a result of the "matrix wedge multiplication" of $\mathbf{F}$ by $\overline{\mathbf{D}}$ on the left-hand-side:

$$\overline{\mathbf{D}}\mathbin{\dot{*}}\mathbf{F}:=\begin{pmatrix}\bar\partial & \bar\partial^{\wedge}\\ \bar\partial^{\wedge} & \bar\partial\end{pmatrix}\mathbin{\dot{*}}\begin{pmatrix}\mathbf{F}^{11}\ \mathbf{F}^{12}\\ \mathbf{F}^{21}\ \mathbf{F}^{22}\end{pmatrix}, \tag{2.15}$$

and similarly, $\overline{\mathbf{D}}_r[\mathbf{F}]$ as

$$\mathbf{F}\mathbin{\dot{*}}\overline{\mathbf{D}}:=\begin{pmatrix}\mathbf{F}^{11}\ \mathbf{F}^{12}\\ \mathbf{F}^{21}\ \mathbf{F}^{22}\end{pmatrix}\mathbin{\dot{*}}\begin{pmatrix}\bar\partial & \bar\partial^{\wedge}\\ \bar\partial^{\wedge} & \bar\partial\end{pmatrix}. \tag{2.16}$$

For all above, $\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}^*$, as well as their right–hand side counterparts $\overline{\mathbf{D}}_r$ and $\overline{\mathbf{D}}_r^*$, are also called the matrix Dirac operators.

Let I be the identity operator acting on some linear space of differential forms, then we shall denote by $\mathbb{E}_{2\times 2}$ and $\breve{\mathbb{E}}_{2\times 2}$, respectively, the operators of the (left-hand-side) multiplication by $E_{2\times 2}$ (2×2 unite matrices) and $\breve{E}_{2\times 2}$ on the corresponding linear space of **m.v.d.f.**, that is,

$$E_{2\times 2} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \breve{E}_{2\times 2} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \mathbb{E}_{2\times 2} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \breve{\mathbb{E}}_{2\times 2} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.17}$$

Theorem 2.3 The following operator equalities hold on $C^2(\Omega;\mathfrak{W}_n)$:

$$\overline{\mathbf{D}}\circ\overline{\mathbf{D}}^* = \overline{\mathbf{D}}^*\circ\overline{\mathbf{D}} = \frac{1}{2}\Delta\mathbb{E}_{2\times 2}, \tag{2.18}$$

$$\overline{\mathbf{D}}_r\circ\overline{\mathbf{D}}_r^* = \overline{\mathbf{D}}_r^*\circ\overline{\mathbf{D}}_r = \frac{1}{2}\Delta\mathbb{E}_{2\times 2}. \tag{2.19}$$

Proof We have

$$\overline{\mathbf{D}}\circ\overline{\mathbf{D}}^* = \begin{pmatrix} \bar\partial & \bar\partial^\wedge \\ \bar\partial^\wedge & \bar\partial \end{pmatrix} \dot{*} \begin{pmatrix} \bar\partial^\wedge & \bar\partial \\ \bar\partial & \bar\partial^\wedge \end{pmatrix} = \begin{pmatrix} \bar\partial\bar\partial^\wedge + \bar\partial^\wedge\bar\partial; \ \bar\partial\bar\partial + \bar\partial^\wedge\bar\partial^\wedge \\ \bar\partial^\wedge\bar\partial^\wedge + \bar\partial\bar\partial; \ \bar\partial^\wedge\bar\partial + \bar\partial\bar\partial^\wedge \end{pmatrix}.$$

Now using Theorem 2.2, we get

$$\overline{\mathbf{D}}\circ\overline{\mathbf{D}}^* = \frac{1}{2}\begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} = \frac{1}{2}\Delta\mathbb{E}_{2\times 2}.$$

Also, we have

$$\overline{\mathbf{D}}_r\circ\overline{\mathbf{D}}_r^* = \begin{pmatrix} \bar\partial & \bar\partial^\wedge \\ \bar\partial^\wedge & \bar\partial \end{pmatrix} \dot{*} \begin{pmatrix} \bar\partial^\wedge & \bar\partial \\ \bar\partial & \bar\partial^\wedge \end{pmatrix} = \begin{pmatrix} \bar\partial\bar\partial^\wedge + \bar\partial^\wedge\bar\partial; \ \bar\partial\bar\partial + \bar\partial^\wedge\bar\partial^\wedge \\ \bar\partial^\wedge\bar\partial^\wedge + \bar\partial\bar\partial; \ \bar\partial^\wedge\bar\partial + \bar\partial\bar\partial^\wedge \end{pmatrix} = \frac{1}{2}\Delta\mathbb{E}_{2\times 2}.$$

The rest of the proof is similar to the above.

According to the proof, we can find that the compositions of two operators act on $\mathfrak{W}_n$-functions.

Let X be $M\times M$. $T^{1,0}(M\times M)$ is the holomorphic vector bundle of rank n over X and η is a holomorphic section to $T^{1,0}(M\times M)$ such that $\{\eta=0\}=\Delta=\{(z,\zeta)\in M\times M:\zeta=z\}$.

Let g be a C^∞ Hermitian metric on $T^{1,0}M$. It induces a Hermitian metric on $T^{1,0}(M\times M)$ (also denoted by g), and induces an antilinear map $\sigma: T^{1,0}(M\times M)\longrightarrow T^{*1,0}(M\times M)$, $\eta\longmapsto \langle\cdot,\eta\rangle_g$. Let D be the Chern connection of $T^{1,0}(M\times M)$ with respect to g, ∇ be the Chern connection of $T^{*1,0}(M\times M)$ with respect to g^*, g^* is induced by g on $T^{*1,0}(M\times M)$. Let $\widehat{\eta}$ be a C^∞ section, $M\times M\longrightarrow T^{*1,0}(M\times M)$, defined by $\widehat{\eta}=\sigma\circ\eta$ such that it satisfies the following condition: for all $\eta\in T^{1,0}(M\times M)$, $\langle\sigma\eta,\eta\rangle_g\geq 0$ and the map $\|\eta(z,\zeta)\|_\sigma := (\langle\sigma\eta,\eta\rangle_g)^{\frac{1}{2}}$ defines a norm on every fibre over $T^{1,0}(M\times M)$. Denote $\langle\sigma\eta,\eta\rangle_g=\big|\eta(z,\zeta)\big|_g^2$.

Assume that U is an open set of the coordinate atlas of complex manifold M, and $(e_j)_{j=1}^n$ is the trivial holomorphic frame of $T^{1,0}(M\times M)$. The metric g is given by a Hermitian matrix H in the frame which only depends on the variable z, while the induced metric of g in $T^{*1,0}(M\times M)$ is given by the matrix $\overline{H}^{-1}$ in the dual frame.

Let u, $\widehat{u}$ be the representations of η, $\widehat{\eta}$ in the selected frame, respectively. The representations of $D\eta$, $\nabla\widehat{\eta}$ in this frame are $du+(H^{-1}\partial H)\wedge u$, $d\widehat{u}+(\overline{H}\partial\overline{H}^{-1})\wedge\widehat{u}$, respectively, while by definition of $\widehat{\eta}$, we have $\widehat{u}=\overline{H}\bar{u}$. Then, we define

$$\bar{\sigma}:=\bar{\sigma}_\zeta=\sum_{j=1}^n c_j\ \bar{\partial}\widehat{u}_{[j]}\wedge Du\cdot i^j, \tag{2.20}$$

$$\bar{\sigma}^*:=\bar{\sigma}^*_\zeta=(-1)^n\sum_{j=1}^n c_j\ \nabla\widehat{u}\wedge\partial u_{[j]}\cdot\widehat{i^j}, \tag{2.21}$$

where

$$Du=Du_1\wedge\cdots\wedge Du_n,\qquad \nabla\widehat{u}=\nabla\widehat{u}_1\wedge\cdots\wedge\nabla\widehat{u}_n,\qquad\text{and}\qquad c_j=\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\,i)^n}\,(-1)^{j-1}.$$

They will serve as entries of the following matrices:

$$\overline{\boldsymbol{\sigma}}:=\overline{\boldsymbol{\sigma}}_\zeta=\begin{pmatrix}\bar{\sigma} & \bar{\sigma}^*\\ \bar{\sigma}^* & \bar{\sigma}\end{pmatrix}, \tag{2.22}$$

$$\overline{\boldsymbol{\sigma}}^*:=\overline{\boldsymbol{\sigma}}^*_\zeta=\begin{pmatrix}\bar{\sigma}^* & \bar{\sigma}\\ \bar{\sigma} & \bar{\sigma}^*\end{pmatrix}. \tag{2.23}$$

Let ([5])

$$\theta(u):=\begin{cases}-2\dfrac{(n-2)!}{\pi^n}\dfrac{1}{|u|^{2(n-1)}}, & n>1,\\[2mm] \dfrac{2}{\pi}\ln(|u|), & n=1;\end{cases}$$

where $u=u(\zeta,z)$, if $\zeta\neq z$, $u(\zeta,z)\neq 0$; if $\zeta=z$, $u(z,z)=0$. If $u=z$, then, $\theta(z)$ is a fundamental solution of the complex Laplace operator on $\mathbb{C}^n$, that is, $\Delta_{\mathbb{C}^n}\mathbb{E}_{2\times2}\left[\theta_{\mathbb{C}^n}E_{2\times2}\right]=\delta_{\mathbb{C}^n}E_{2\times2}$, where $\Delta_{\mathbb{C}^n}=\sum\limits_{k=1}^n\frac{\partial^2}{\partial z_k\partial\bar{z}_k}$ (Diracś delta on $\mathbb{C}^n$) is in the distributional sense.

Hence, we have the fundamental solution of the operators $\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}^*$:

$$\begin{aligned}\mathbf{K}_{\overline{\mathbf{D}}}(u):&=\overline{\mathbf{D}}^*\left[\theta(u)E_{2\times2}\right](u)\\ &=\begin{pmatrix}\bar{\partial}^\wedge[\theta(u)];\ \bar{\partial}[\theta(u)]\\ \bar{\partial}[\theta(u)];\ \bar{\partial}^\wedge[\theta(u)]\end{pmatrix}\\ &=\sum_{\alpha=1}^n\begin{pmatrix}\sum\limits_{\beta=1}^n g^{\bar{\beta}\alpha}\dfrac{\partial\theta(u)}{\partial u_\alpha}\,\widehat{i^\beta}; & \dfrac{\partial\theta(u)}{\partial\widehat{u}_\alpha}\,i^\alpha\\ \dfrac{\partial\theta(u)}{\partial\widehat{u}_\alpha}\,i^\alpha; & \sum\limits_{\beta=1}^n g^{\bar{\beta}\alpha}\dfrac{\partial\theta(u)}{\partial u_\alpha}\,\widehat{i^\beta}\end{pmatrix}\\ &=2\frac{(n-1)!}{\pi^n}\sum_{\alpha=1}^n\begin{pmatrix}\sum\limits_{\beta=1}^n g^{\bar{\beta}\alpha}\dfrac{\widehat{u}_\alpha}{|u|^{2n}}\,\widehat{i^\beta}; & \dfrac{u_\alpha}{|u|^{2n}}\,i^\alpha\\ \dfrac{u_\alpha}{|u|^{2n}}\,i^\alpha; & \sum\limits_{\beta=1}^n g^{\bar{\beta}\alpha}\dfrac{\widehat{u}_\alpha}{|u|^{2n}}\,\widehat{i^\beta}\end{pmatrix},\end{aligned} \tag{2.24}$$

$$
\begin{aligned}
\mathbf{K}_{\overline{\mathbf{D}}^*}(u) :&= \overline{\mathbf{D}}\left[\theta(u)E_{2\times 2}\right](u) \\
&= \begin{pmatrix} \bar{\partial}[\theta(u)]; & \bar{\partial}^\wedge[\theta(u)] \\ \bar{\partial}^\wedge[\theta(u)]; & \bar{\partial}[\theta(u)] \end{pmatrix} \\
&= \sum_{\alpha=1}^{n} \begin{pmatrix} \dfrac{\partial\theta(u)}{\partial\widehat{u}_\alpha}\, i^\alpha; & \displaystyle\sum_{\beta=1}^{n} g^{\bar{\beta}\alpha}\frac{\partial\theta(u)}{\partial u_\alpha}\,\widehat{i^\beta} \\ \displaystyle\sum_{\beta=1}^{n} g^{\bar{\beta}\alpha}\frac{\partial\theta(u)}{\partial u_\alpha}\,\widehat{i^\beta}; & \dfrac{\partial\theta(u)}{\partial\widehat{u}_\alpha}\, i^\alpha \end{pmatrix} \\
&= 2\frac{(n-1)!}{\pi^n}\sum_{\alpha=1}^{n} \begin{pmatrix} \dfrac{u_\alpha}{|u|^{2n}}\, i^\alpha; & \displaystyle\sum_{\beta=1}^{n} g^{\bar{\beta}\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\,\widehat{i^\beta} \\ \displaystyle\sum_{\beta=1}^{n} g^{\bar{\beta}\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\,\widehat{i^\beta}; & \dfrac{u_\alpha}{|u|^{2n}}\, i^\alpha \end{pmatrix}.
\end{aligned} \tag{2.25}
$$

Formally, for (2.19), one can set

$$\mathbf{K}_{\overline{\mathbf{D}}_r} := \overline{\mathbf{D}}_r^*\left[\theta(u)E_{2\times 2}\right], \qquad \mathbf{K}_{\overline{\mathbf{D}}_r}^* := \overline{\mathbf{D}}_r\left[\theta(u)E_{2\times 2}\right].$$

As the matrix $\theta(u)E_{2\times 2}$ is scalar–valued, one has

$$\mathbf{K}_{\overline{\mathbf{D}}} = \mathbf{K}_{\overline{\mathbf{D}}_r}, \qquad \mathbf{K}_{\overline{\mathbf{D}}^*} = \mathbf{K}_{\overline{\mathbf{D}}_r}^*.$$

Using the factorization (2.18), we have

$$\overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}\right] = \overline{\mathbf{D}}\circ\overline{\mathbf{D}}^*\left[\theta(u)E_{2\times 2}\right] = \frac{1}{2}\Delta\mathbb{E}_{2\times 2}\left[\theta(u)E_{2\times 2}\right].$$

The same for (2.25). By that reason, we shall call each of $\mathfrak{W}_n$-valued functions (2.24) and (2.25) the matrix Cauchy-Dirac kernel for the theory of $\mathfrak{W}_n$-valued functions from the null-set of the corresponding operator.

Now, we derive the Borel-Pompeiu formulas for $\mathfrak{W}_n$-valued functions.

Theorem 2.4 (Borel-Pompeiu) Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$ on a Kaehler manifold M, and let $\mathbf{F}\in C^1(\Omega;\ \mathfrak{W}_n)\cap C^0(\Omega\cup\partial\Omega;\ \mathfrak{W}_n)$. Then, the following equalities hold in Ω:

$$
\begin{aligned}
2\mathbf{F}(z) = &\int_{\zeta\in\partial\Omega}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}_\zeta\,\dot{*}\,\mathbf{F}(\zeta) - \int_{\zeta\in\Omega}\overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}_\zeta\right]\dot{*}\,\mathbf{F}(\zeta) \\
&-\int_{\zeta\in\Omega}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}_\zeta\,\dot{*}\,\overline{\mathbf{D}}[\mathbf{F}](\zeta),
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
2\mathbf{F}(z) = &\int_{\zeta\in\partial\Omega}\mathbf{K}_{\overline{\mathbf{D}}^*}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}^*_\zeta\,\dot{*}\,\mathbf{F}(\zeta) - \int_{\zeta\in\Omega}\overline{\mathbf{D}}^*\left[\mathbf{K}_{\overline{\mathbf{D}}^*}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}^*_\zeta\right]\dot{*}\,\mathbf{F}(\zeta) \\
&-\int_{\zeta\in\Omega}\mathbf{K}_{\overline{\mathbf{D}}^*}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}^*_\zeta\,\dot{*}\,\overline{\mathbf{D}}^*[\mathbf{F}](\zeta).
\end{aligned} \tag{2.27}
$$

Remark 2.5 If the curvature form of the $T^{1,0}(M\times M)$ with respect to the connection D is zero, that is, $C(T^{1,0}(M\times M)) = D^2 = 0$, then, $\overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}_\zeta\right] = 0$, $\overline{\mathbf{D}}^*\left[\mathbf{K}_{\overline{\mathbf{D}}^*}(\zeta,z)\,\dot{*}\,\boldsymbol{\overline{\sigma}}^*_\zeta\right] = 0$, and the second integrals on the right-hand side of (31) and (2.27) disappear [2, 5, 6].

Essentially, formula (2.26) and (2.27) are Bochner-Martinelli formula for matrix-valued continuous functions with Witt basis, so this theorem can be proved by the method of complex analysis on the manifolds (see [2] Chapter 4) combined with the method of Clifford analysis [8, 9].

Proof The proof will be given for the first case only. Take $z \in \Omega$ fixed and choose $\varepsilon > 0$ such that the closed ball $\overline{B}(z,\varepsilon) \subset \Omega$. Where the closed ball $\overline{B}(z,\varepsilon)$ has center at $z \in \Omega$, with radius in Kaehler metric sense less than $\varepsilon > 0$.

By Stokes formula, we have

$$\begin{aligned}&\int_{\zeta\in\partial\left(\Omega\setminus\overline{B}(z,\varepsilon)\right)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \mathbf{F}(\zeta)\\ &= \int_{\zeta\in\left(\Omega\setminus\overline{B}(z,\varepsilon)\right)} \overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta\right] \dot{*} \mathbf{F}(\zeta)\\ &\quad + \int_{\zeta\in\left(\Omega\setminus\overline{B}(z,\varepsilon)\right)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \overline{\mathbf{D}}[\mathbf{F}](\zeta). \end{aligned} \tag{2.28}$$

As $\overline{\mathbf{D}}[\mathbf{F}](\zeta)$ is continuous in Ω and Ω is a bounded set, and $\overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta\right] \dot{*} \mathbf{F}(\zeta)$ and $\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \overline{\mathbf{D}}[\mathbf{F}](\zeta)$ are Lebesque absolutely integrable on Ω, consequently, by taking the limit as $\varepsilon \longrightarrow 0^+$, we obtain in the right-hand side of (2.28):

$$\int_{\zeta\in\Omega} \overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta\right] \dot{*} \mathbf{F}(\zeta) + \int_{\zeta\in\Omega} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \overline{\mathbf{D}}[\mathbf{F}](\zeta).$$

As to the left-hand side of (33), it can be put into the form

$$\int_{\zeta\in\partial\Omega} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \mathbf{F}(\zeta) \quad - \int_{\zeta\in S(z,\,\varepsilon)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \mathbf{F}(\zeta).$$

Now, we begin to prove this equality

$$\lim_{\varepsilon\longrightarrow 0^+} \int_{\zeta\in S(z,\,\varepsilon)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \mathbf{F}(\zeta) = 2\mathbf{F}(z).$$

Assume that $\mathbf{F}(z) := \begin{pmatrix} \mathbf{F}^{11}(z) \ \mathbf{F}^{12}(z) \\ \mathbf{F}^{21}(z) \ \mathbf{F}^{22}(z) \end{pmatrix}$, where $\mathbf{F}^{ij} \in \mathbb{W}_n(\Omega)$ and $\mathbf{F}(z) \in \mathfrak{W}_n(\Omega)$.

First of all,

$$\begin{aligned}&\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta\\ &= 2\frac{(n-1)!}{\pi^n} \sum_{\alpha=1}^n \begin{pmatrix} \sum_{\beta=1}^n g^{\bar{\beta}\alpha} \dfrac{\widehat{u}_\alpha}{|u|^{2n}} \widehat{i^\beta}; & \dfrac{u_\alpha}{|u|^{2n}} i^\alpha \\ \dfrac{u_\alpha}{|u|^{2n}} i^\alpha; & \sum_{\beta=1}^n g^{\bar{\beta}\alpha} \dfrac{\widehat{u}_\alpha}{|u|^{2n}} \widehat{i^\beta} \end{pmatrix}\\ &\quad \dot{*} \frac{(-1)^{\frac{n(n-1)}{2}}}{(2i)^n} \sum_{j=1}^n \begin{pmatrix} (-1)^{j-1} \bar{\partial}\widehat{u}_{[j]} \wedge Du \cdot i^j; & (-1)^{n+j-1} \nabla\widehat{u} \wedge \partial u_{[j]} \cdot \widehat{i^j} \\ (-1)^{n+j-1} \nabla\widehat{u} \wedge \partial u_{[j]} \cdot \widehat{i^j}; & (-1)^{j-1} \bar{\partial}\widehat{u}_{[j]} \wedge Du \cdot i^j \end{pmatrix}.\end{aligned}$$

The integral $\int_{\zeta\in S(z,\,\varepsilon)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z) \dot{*} \overline{\boldsymbol{\sigma}}_\zeta \dot{*} \mathbf{F}(\zeta)$ can be divided into two cases of $\alpha = j$ and $\alpha \neq j$.

When $\alpha = j$, from $1, \cdots, n$, we have

$$
\begin{aligned}
&2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\sum_{\alpha=1}^{n}\\
&\begin{pmatrix}
\begin{array}{l}(-1)^{\alpha-1}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot\widehat{i^\beta}\cdot i^\alpha\\ \quad+(-1)^{n+\alpha-1}\frac{u_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot i^\alpha\cdot\widehat{i^\alpha};\end{array} &
\begin{array}{l}(-1)^{n+\alpha-1}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot\widehat{i^\beta}\cdot\widehat{i^\alpha}\\ \quad+(-1)^{\alpha-1}\frac{u_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot i^\alpha\cdot i^\alpha\end{array}\\
\begin{array}{l}(-1)^{n+\alpha-1}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot\widehat{i^\beta}\cdot\widehat{i^\alpha}\\ \quad+(-1)^{\alpha-1}\frac{u_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot i^\alpha\cdot i^\alpha;\end{array} &
\begin{array}{l}(-1)^{\alpha-1}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot\widehat{i^\beta}\cdot i^\alpha\\ \quad+(-1)^{n+\alpha-1}\frac{u_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot i^\alpha\cdot\widehat{i^\alpha}\end{array}
\end{pmatrix}\\
=\;&2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\sum_{\alpha=1}^{n}(-1)^{\alpha-1}\\
&\begin{pmatrix}
\begin{array}{l}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot\widehat{i^\beta}\cdot i^\alpha\\ \quad+(-1)^{n}\frac{u_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot i^\alpha\cdot\widehat{i^\alpha};\end{array} &
(-1)^{n}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot\widehat{i^\beta}\cdot\widehat{i^\alpha}\\
(-1)^{n}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\nabla\hat{u}\wedge\partial u_{[\alpha]}\cdot\widehat{i^\beta}\cdot\widehat{i^\alpha}; &
\begin{array}{l}\sum\limits_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot\widehat{i^\beta}\cdot i^\alpha\\ \quad+(-1)^{n}\frac{u_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot i^\alpha\cdot\widehat{i^\alpha}\end{array}
\end{pmatrix}\\
=\;&2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\sum_{\alpha=1}^{n}(-1)^{\alpha-1}\Bigg\{\Bigg[\sum_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot\widehat{i^\beta}\cdot i^\alpha+(-1)^n\\
&\frac{u_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot i^\alpha\cdot\widehat{i^\alpha}\Bigg]\cdot E_{2\times 2}+\Bigg[(-1)^n\sum_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\nabla\hat{u}\wedge\partial u_{[\alpha]}\cdot\widehat{i^\beta}\cdot\widehat{i^\alpha}\Bigg]\cdot\breve{E}_{2\times 2}\Bigg\}.
\end{aligned}
$$

So, we follow that the case of $\alpha = j$ in $\int_{\zeta\in S(z,\,\varepsilon)}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta, z)\,\dot{*}\,\overline{\boldsymbol{\sigma}}_\zeta\,\dot{*}\,\mathbf{F}(\zeta)$ is

$$
\begin{aligned}
&2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\int_{\zeta\in S(z,\,\varepsilon)}\sum_{\alpha=1}^{n}(-1)^{\alpha-1}\\
&\Bigg\{\Bigg[\sum_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot\widehat{i^\beta}\cdot i^\alpha+(-1)^n\frac{u_\alpha}{|u|^{2n}}\nabla\widehat{u}\wedge\partial u_{[\alpha]}\cdot i^\alpha\cdot\widehat{i^\alpha}\Bigg]\cdot E_{2\times 2}\\
&+\Bigg[(-1)^n\sum_{\beta=1}^{n} g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\nabla\hat{u}\wedge\partial u_{[\alpha]}\cdot\widehat{i^\beta}\cdot\widehat{i^\alpha}\Bigg]\cdot\breve{E}_{2\times 2}\Bigg\}\,\dot{*}\,\mathbf{F}(\zeta)\\
=\;&2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\int_{\zeta\in S(z,\,\varepsilon)}\sum_{\alpha,\beta=1}^{n}(-1)^{\alpha-1}g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\\
&\wedge Du\cdot\widehat{i^\beta}\cdot\ i^\alpha\ \cdot E_{2\times 2}\,\dot{*}\,\mathbf{F}(\zeta).
\end{aligned}
$$

From [10], we give a point ζ on an Kaehler manifold, then there exists a complex coordinate system normal at ζ. That is to say, $g^{\bar\beta\alpha}(\zeta) = \delta_{\bar\beta\alpha}$.

Hence, the case of $\alpha = j$ in $\int_{\zeta\in S(z,\,\varepsilon)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta, z) \dot{*}\, \overline{\boldsymbol{\sigma}}_\zeta \,\dot{*}\, \mathbf{F}(\zeta)$ is

$$2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}} \int_{\zeta\in S(z,\,\varepsilon)} \sum_{\alpha=1}^{n}(-1)^{\alpha-1}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar{\partial}\widehat{u}_{[\alpha]} \wedge Du\cdot E_{2\times 2} \,\dot{*}\, \mathbf{F}(\zeta).$$

In succession, we begin to consider another case of $\alpha \neq j$ in $\int_{\zeta\in S(z,\,\varepsilon)} \mathbf{K}_{\overline{\mathbf{D}}}(\zeta, z) \dot{*}\, \overline{\boldsymbol{\sigma}}_\zeta \,\dot{*}\, \mathbf{F}(\zeta)$. We first investigate the relations between the above-introduced matrices $\overline{\boldsymbol{\sigma}}$, $\overline{\boldsymbol{\sigma}}^*$, and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^*$, and the normal vector to a surface in M. Let Γ be a real $(2n-1)$-surface in M of class C^1. Denote by $\mathbf{n}_\zeta = (n_{1\zeta}, \cdots, n_{n\zeta})$ the outward pointing normal unit vector to Γ at $\zeta \in \Gamma$ and dS be a surface differential form on Γ. Now, consider on the surface Γ

$$\begin{aligned}
n_j dS = \Big\{&(-1)^{(2j-1)-1}\frac{1}{2}(\nabla\widehat{u}_1 + Du_1)\wedge(-\frac{1}{2i})(\nabla\widehat{u}_1 - Du_1)\wedge\cdots\cdots\\
&\wedge\frac{1}{2}(\nabla\widehat{u}_{j-1} + Du_{j-1})\wedge(-\frac{1}{2i})(\nabla\widehat{u}_{j-1} - Du_{j-1})\wedge(-\frac{1}{2i})(\nabla\widehat{u}_j - Du_j)\\
&\wedge\frac{1}{2}(\nabla\widehat{u}_{j+1} + Du_{j+1})\wedge(-\frac{1}{2i})(\nabla\widehat{u}_{j+1} - Du_{j+1})\wedge\cdots\wedge\frac{1}{2}(\nabla\widehat{u}_n + Du_n)\\
&\wedge(-\frac{1}{2i})(\nabla\widehat{u}_n - Du_n) + (-1)^{2j-1}i\frac{1}{2}(\nabla\widehat{u}_1 + Du_1)\wedge(-\frac{1}{2i})(\nabla\widehat{u}_1 - Du_1)\\
&\wedge\cdots\wedge\frac{1}{2}(\nabla\widehat{u}_{j-1} + Du_{j-1})\wedge(-\frac{1}{2i})(\nabla\widehat{u}_{j-1} - Du_{j-1})\wedge\frac{1}{2}(\nabla\widehat{u}_j + Du_j)\\
&\wedge\frac{1}{2}(\nabla\widehat{u}_{j+1} + Du_{j+1})\wedge(-\frac{1}{2i})(\nabla\widehat{u}_{j+1} - Du_{j+1})\wedge\cdots\cdots\\
&\wedge\frac{1}{2}(\nabla\widehat{u}_n + Du_n)\wedge(-\frac{1}{2i})(\nabla\widehat{u}_n - Du_n)\Big\}\Big|_\Gamma\\
=\Big\{&\frac{(-1)^n}{2^{2n-1}i^n}(\nabla\widehat{u}_j - Du_j)\wedge\bigwedge_{k=1,k\neq j}^{n}(-2)\nabla\widehat{u}_k\wedge Du_k\\
&.-\frac{(-1)^n}{2^{2n-1}i^n}(\nabla\widehat{u}_j + Du_j)\wedge\bigwedge_{k=1,k\neq j}^{n}(-2)\nabla\widehat{u}_k\wedge Du_k\Big\}\Big|_\Gamma\\
=2&\frac{1}{(2i)^n}\Big\{Du_j\wedge\bigwedge_{k=1,k\neq j}^{n}\nabla\widehat{u}_k\wedge Du_k\Big\}\Big|_\Gamma\\
=2&\frac{(-1)^{\frac{(n-2)(n-1)}{2}}}{(2i)^n}\left\{Du_j\wedge\bar{\partial}\widehat{u}_{[j]}\wedge\partial u_{[j]}\right\}\Big|_\Gamma\\
=2&\frac{(-1)^{\frac{n(n-1)}{2}}}{(2i)^n}(-1)^{j-1}\left\{\bar{\partial}\widehat{u}_{[j]}\wedge Du\right\}\Big|_\Gamma.
\end{aligned}$$

Analogously, we have

$$\bar{n}_j dS = 2(-1)^n\frac{(-1)^{\frac{n(n-1)}{2}}}{(2i)^n}(-1)^{j-1}\left\{\nabla\widehat{u}\wedge\partial u_{[j]}\right\}\Big|_\Gamma,$$

and hence,

$$\overline{\sigma}|_\Gamma = \frac{1}{2}\sum_{j=1}^{n}\begin{pmatrix} n_j\ i^j;\ \bar{n}_j\ \widehat{i^j}\\ \bar{n}_j\ \widehat{i^j};\ n_j\ i^j\end{pmatrix}dS_\zeta = \frac{1}{2}\sum_{j=1}^{n}(n_j\ i^j\ E_{2\times 2} + \bar{n}_j\ \widehat{i^j}\ \breve{E}_{2\times 2})dS_\zeta.$$

Other cases are similar.

So, we obtain

$$\begin{aligned}&\int_{\zeta\in S(z,\,\varepsilon)}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\dot{*}\,\overline{\boldsymbol{\sigma}}_\zeta\,\dot{*}\,\mathbf{F}(\zeta)\\&=\frac{(n-1)!}{\pi^n}\int_{\zeta\in S(z,\,\varepsilon)}\sum_{\alpha=1}^n\begin{pmatrix}\sum\limits_{\beta=1}^n g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\widehat{i^\beta}; & \frac{u_\alpha}{|u|^{2n}}i^\alpha\\ \frac{u_\alpha}{|u|^{2n}}i^\alpha; & \sum\limits_{\beta=1}^n g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\widehat{i^\beta}\end{pmatrix}\\&\quad\dot{*}\sum_{j=1}^n\begin{pmatrix}n_j\ i^j;\ \bar n_j\ \widehat{i^j}\\ \bar n_j\ \widehat{i^j};\ n_j\ i^j\end{pmatrix}\dot{*}\,\mathbf{F}(\zeta)dS_\zeta.\end{aligned}\tag{2.29}$$

For the spheres $n_j=\frac{1}{|u|^{2n}}u_j$ and $\bar n_j=\frac{1}{|u|^{2n}}\widehat{u}_j$, when we consider $\alpha\neq j$, we see that it tends to zero if and only if the following integral tends to zero when ε tends to zero:

$$\int_{\zeta\in S(z,\,\varepsilon)}\sum_{\alpha\neq j}\begin{pmatrix}\sum\limits_{\beta=1}^n g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\widehat{i^\beta}; & \frac{u_\alpha}{|u|^{2n}}i^\alpha\\ \frac{u_\alpha}{|u|^{2n}}i^\alpha; & \sum\limits_{\beta=1}^n g^{\bar\beta\alpha}\frac{\widehat{u}_\alpha}{|u|^{2n}}\widehat{i^\beta}\end{pmatrix}\dot{*}\begin{pmatrix}\frac{u_j}{|u|^{2n}}i^j;\ \frac{\widehat{u}_j}{|u|^{2n}}\widehat{i^j}\\ \frac{\widehat{u}_j}{|u|^{2n}}\widehat{i^j};\ \frac{u_j}{|u|^{2n}}i^j\end{pmatrix}\dot{*}\,E_{2\times2}dS_\zeta.$$

To prove the last identity, it is sufficient to prove that, for any $\alpha\neq j$,

$$\int_{\zeta\in S(z,\,\varepsilon)}\widehat{u}_\alpha\cdot u_j\ dS_\zeta=0,\qquad\int_{\zeta\in S(z,\,\varepsilon)}\widehat{u}_\alpha\cdot\widehat{u}_j\ dS_\zeta=0.\tag{2.30}$$

As $Du=du+(H^{-1}\partial H)\wedge u$, so dS_ζ equals the Euclidian area element of $S(z,\,\varepsilon)$ plus a lower dimensional area element on the sphere $S(z,\,\varepsilon)$, therefore, to prove the last identity, it is sufficient to prove that equalities (2.30) is true in the Euclidian case, but this result was proved in detail in [3], pp 42–43.

Therefore,

$$\begin{aligned}&\lim_{\varepsilon\longrightarrow0^+}\int_{\zeta\in S(z,\,\varepsilon)}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\dot{*}\,\overline{\boldsymbol{\sigma}}_\zeta\,\dot{*}\,\mathbf{F}(\zeta)\\&=2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\lim_{\varepsilon\longrightarrow0^+}\int_{\zeta\in S(z,\,\varepsilon)}\sum_{\alpha=1}^n(-1)^{\alpha-1}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du\cdot E_{2\times2}\,\dot{*}\,\mathbf{F}(\zeta).\end{aligned}$$

Set

$$U(\zeta,z):=2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\sum_{\alpha=1}^n(-1)^{\alpha-1}\frac{\widehat{u}_\alpha}{|u|^{2n}}\bar\partial\widehat{u}_{[\alpha]}\wedge Du$$

then,

$$\lim_{\varepsilon\longrightarrow0^+}\int_{\zeta\in S(z,\,\varepsilon)}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\dot{*}\,\overline{\boldsymbol{\sigma}}_\zeta\,\dot{*}\,\mathbf{F}(\zeta)=\lim_{\varepsilon\longrightarrow0^+}\int_{\zeta\in S(z,\,\varepsilon)}U(\zeta,z)\cdot E_{2\times2}\,\dot{*}\,\mathbf{F}(\zeta).$$

We only consider $\mathbf{F}^{11}(\zeta)\in\mathbb{W}_n(\Omega)$, and the others are similar. Hence, we need prove ([1–2]).

$$\lim_{\varepsilon\longrightarrow0^+}\int_{\zeta\in S(z,\,\varepsilon)}U(\zeta,z)\cdot\mathbf{F}^{11}(\zeta)=2\mathbf{F}^{11}(z).\tag{2.31}$$

Consider a neighborhood $U_\Delta \subseteq M\times M$ of the diagonal $\Delta := \{(z,z) : z\in M\}$ such that, for every fixed $z\in M$, the map $\eta(z,\zeta)$ is biholomorphic for all $\zeta\in M$ with $(z,\zeta)\in U_\Delta$. Consider the open set

$$U_\varepsilon := \{(z,\zeta)\in U_\Delta : |\eta(z,\zeta)|_g < \varepsilon\}, \qquad \varepsilon>0,$$

and let $B(z,\varepsilon) := U_\varepsilon$, $\partial B(z,\varepsilon) = S(z,\varepsilon) = \partial U_\varepsilon$. Let $\{(U_j, e_j)\}$ be a fixed holomorphic atlas of M, such that $U_j \subset\subset M$ for all j, then, for all $(z,\zeta)\in U_\varepsilon$ with $z\in U_j$, we have $\zeta\in U_j$. Therefore, (2.31) can be written as

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{(z,\zeta)\in(U_j\times U_j)\cap S(z,\,\varepsilon)} U(\zeta,z)\cdot \mathbf{F}^{11}(\zeta) = 2\,\mathbf{F}^{11}(z). \tag{2.32}$$

By [2] §4.15.2 and [6], when $(z,\zeta)\in(U_j\times U_j)$, in local coordinates the kernel $U(\zeta,z)$ can be expressed by

$$U(\zeta,z) = 2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\frac{\overline{\omega}'_{z,\zeta}(\widehat{u})\wedge\omega_{z,\zeta}(u)}{|u|^{2n}} + O(|u|^{-2n+2}),$$

specially, in $S(z,\varepsilon)\cap(U_j\times(U_j\cap\overline{D}))$, we have

$$U(\zeta,z) = 2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\frac{\overline{\omega}'_{z,\zeta}(\widehat{u})\wedge\omega_{z,\zeta}(u)}{\varepsilon^{2n}} + O(\varepsilon^{-2n+2}).$$

As the measure of the point set $(U_j\times U_j)\cap S(z,\varepsilon)$ is $O(\varepsilon^{2n-1})$, so (2.32) becomes

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{(z,\zeta)\in(U_j\times U_j)\cap S(z,\,\varepsilon)} K(\zeta,z)\cdot \mathbf{F}^{11}(\zeta) = 2\,\mathbf{F}^{11}(z),$$

where

$$K(\zeta,z) = 2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\frac{\overline{\omega}'_{z,\zeta}(\widehat{u})\wedge\omega_{z,\zeta}(u)}{\varepsilon^{2n}}.$$

We can find $\varepsilon_0>0$ such that the map $T : (U_j\times U_j)\cap B(z,\varepsilon_0)\longrightarrow \mathbb{C}^n\times\mathbb{C}^n$ defined by $T(z,\zeta) := (e_j(z), u(z,\zeta))$ is biholomorphic from $(U_j\times U_j)\cap B(z,\varepsilon_0)$ onto some open set $W_{\varepsilon_0}\subseteq\mathbb{C}^n\times\mathbb{C}^n$. Let $T^{-1} : W_{\varepsilon_0}\longrightarrow (U_j\times U_j)\cap B(z,\varepsilon_0)$ be the inverse of T. Let $\alpha(\tau,\lambda) : W_{\varepsilon_0}\longrightarrow U_j$ be the holomorphic map defined by $\alpha(e_j(z), u(z,\zeta)) = \zeta$ for all $(z,\zeta)\in(U_j\times U_j)\cap B(z,\varepsilon_0)$. Then,

$$T^{-1}(\tau,\lambda) = (e_j^{-1}(\tau), \alpha(\tau,\lambda)) \quad \text{for all} \quad (\tau,\lambda)\in W_{\varepsilon_0}. \tag{2.33}$$

Further, it is clear that

$$u(T^{-1}(\tau,\lambda)) = \lambda \quad \text{for all} \quad (\tau,\lambda)\in W_{\varepsilon_0} \tag{2.34}$$

and

$$T^{-1}(\tau,0) = (e_j^{-1}(\tau), e_j^{-1}(\tau)) \quad \text{for all} \quad \tau\in e_j(U_j). \tag{2.35}$$

We set $Z_\varepsilon := T((U_j\times U_j)\cap S(z,\varepsilon))$ for $0<\varepsilon\le\varepsilon_0$. We can find $0<\varepsilon_1\le\varepsilon_0$ such that

$$Z_\varepsilon \subset\subset W_{\varepsilon_0} \quad \text{for} \quad 0<\varepsilon\le\varepsilon_1, \tag{2.36}$$

and, for some constant $C<\infty$, $|u(z,\zeta)| \le C \parallel \eta(z,\zeta)\parallel_\sigma \le C\varepsilon$ for $0<\varepsilon\le\varepsilon_1$ and $(z,\zeta)\in(U_j\times U_j)\cap S(z,\varepsilon)$, that is,

$$|\lambda|\le C\varepsilon \quad \text{for} \quad 0<\varepsilon\le\varepsilon_1 \quad \text{and} \quad (\tau,\lambda)\in Z_\varepsilon. \tag{2.37}$$

From (2.35) we obtain $\widehat{u}(T^{-1}(\tau,0))=0$ for all $\tau\in e_j(U_j)$. As $\widehat{u}\circ T^{-1}$ is C^∞ on W_{ε_0}, together with (2.36) and (2.37), this implies that there is a constant $C<\infty$ such that

$$\left|\widehat{u}(T^{-1}(\tau,\lambda))\right|\le C\varepsilon \quad \text{and} \quad \| \, d_\tau\widehat{u}(T^{-1}(\tau,\lambda)) \, \|\le C\varepsilon$$

for all $0<\varepsilon\le\varepsilon_1$ and $(\tau,\lambda)\in Z_\varepsilon$. Therefore, we can find a constant $C<\infty$, such that

$$\| \, \overline{\omega}_{\tau,\lambda}'(\widehat{u}(T^{-1}(\tau,\lambda)))-\overline{\omega}_{\lambda}'(\widehat{u}(T^{-1}(\tau,\lambda))) \, \|\le C\varepsilon^2, \tag{2.38}$$

for all $0<\varepsilon\le\varepsilon_1$ and $(\tau,\lambda)\in Z_\varepsilon$, where $\overline{\omega}_{\tau,\lambda}'(\widehat{u}):=\frac{1}{(n-1)!}\det_{1,n-1}\left(\widehat{u},(\bar\partial_\tau+\bar\partial_\lambda)\widehat{u}\right)$ and $\overline{\omega}_{\lambda}'(\widehat{u}):=\frac{1}{(n-1)!}\det_{1,n-1}(\widehat{u},\bar\partial_\lambda\widehat{u})$.

Let $(e_j^{-1})^*\mathbf{F}^{11}$ and $\alpha^*\mathbf{F}^{11}$ be the pull-backs defined by e_j^{-1} and α, respectively, and let

$$(e_j^{-1})^*\mathbf{F}^{11}(\tau)=\sum_{1\le i_1<\cdots<i_q\le n}\mathbf{F}^{11}_{i_1\cdots i_q}(\tau)\, d\overline{\tau}_{i_1}\wedge\cdots\wedge d\overline{\tau}_{i_q} \quad \text{for} \quad \tau\in e_j\,(U_j). \tag{2.39}$$

If $e_{j_1},\cdots,e_{j_n}$ are the components of e_j, then, it follows that

$$\mathbf{F}^{11}(\zeta)=\sum_{1\le i_1<\cdots<i_q\le n}\mathbf{F}^{11}_{i_1\cdots i_q}(e_j(\zeta))\, d\overline{e}_{ji_1}(\zeta)\wedge\cdots\wedge d\overline{e}_{ji_q}(\zeta) \quad \text{for} \quad \zeta\in U_j\,, \tag{2.40}$$

and

$$\alpha^*\mathbf{F}^{11}(\tau,\lambda)=\sum_{1\le i_1<\cdots<i_q\le n}\mathbf{F}^{11}_{i_1\cdots i_q}(e_j(\alpha(\tau,\lambda)))\, d_{\tau,\lambda}\overline{e}_{ji_1}(\alpha(\tau,\lambda))\wedge\cdots\wedge\, d_{\tau,\lambda}\overline{e}_{ji_q}(\alpha(\tau,\lambda)) \quad \text{for} \quad (\tau,\lambda)\in W_{\varepsilon_0}. \tag{2.41}$$

Define

$$\alpha_\tau^*\mathbf{F}^{11}(\tau,\lambda):=\sum_{1\le i_1<\cdots<i_q\le n}\mathbf{F}^{11}_{i_1\cdots i_q}(e_j(\alpha(\tau,\lambda)))\, d_{\tau}\overline{e}_{ji_1}(\alpha(\tau,\lambda))\wedge\cdots\wedge\, d_{\tau}\overline{e}_{ji_q}(\alpha(\tau,\lambda)) \quad \text{for} \quad (\tau,\lambda)\in W_{\varepsilon_0}. \tag{2.42}$$

From (2.35), (2.36), and (2.37), we obtain a constant C, such that

$$|\, e_j(\alpha(\tau,\lambda))-\tau|\le C\varepsilon \quad \text{and} \quad \| \, d_\tau e_j(\alpha(\tau,\lambda))-d\tau \, \|\le C\varepsilon$$

for all $0<\varepsilon\le\varepsilon_1$, and $(\tau,\lambda)\in Z_\varepsilon$. According to (2.39) and (2.42), this implies that

$$\lim_{\varepsilon\longrightarrow 0^+}\sup_{(\tau,\,\lambda)\in Z_\varepsilon}\| \, \alpha_\tau^*\mathbf{F}^{11}(\tau,\lambda)-(e_j^{-1})^*\mathbf{F}^{11}(\tau) \, \|=0. \tag{2.43}$$

Now, we are ready to prove (2.32). We denote by $(T^{-1})^*(K\wedge\mathbf{F}^{11})$ the pull-back of $K\wedge\mathbf{F}^{11}$ with respect to T^{-1}. Then,

$$\int_{(z,\,\zeta)\in(U_j\times U_j)\cap S(z,\,\varepsilon)}K(\zeta,\,z)\wedge\mathbf{F}^{11}(\zeta)=\int_{(\tau,\,\lambda)\in Z_\varepsilon}(T^{-1})^*(K\wedge\mathbf{F}^{11})(\tau,\lambda). \tag{2.44}$$

Because

$$K(\zeta,z)=2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\frac{\overline{\omega}_{z,\zeta}'(\widehat{u})\wedge\omega_{z,\zeta}(u)}{\varepsilon^{2n}}$$

for $(z,\zeta)\in(U_j\times U_j)\cap S(z,\varepsilon)$, it follows from (2.33) and (2.34) that for $(\tau,\lambda)\in Z_\varepsilon$

$$(T^{-1})^*(K\wedge\mathbf{F}^{11})(\tau,\lambda)=2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\alpha^*\mathbf{F}^{11}(\tau,\lambda)\wedge(T^{-1}(\tau,\lambda))\wedge\frac{\overline{\omega}'_{\tau,\lambda}(\widehat{u}(T^{-1}(\tau,\lambda)))\wedge\omega_\lambda(\lambda)}{\varepsilon^{2n}}. \tag{2.45}$$

Because the volume of Z_ε is of order ε^{2n-1}, it follows from (2.38), (2.44), and (2.45) that the proof of (2.32) will be completed if we show that

$$\lim_{\varepsilon\longrightarrow0^+}2\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\int_{(\tau,\lambda)\in Z_\varepsilon}\alpha^*\mathbf{F}^{11}(\tau,\lambda)\wedge(T^{-1}(\tau,\lambda))\wedge\frac{\overline{\omega}'_{\tau,\lambda}(\bar{u}(T^{-1}(\tau,\lambda)))\wedge\omega_\lambda(\lambda)}{\varepsilon^{2n}}=2\int_{\tau\in e_j(U_j)}(e_j^{-1})^*\mathbf{F}^{11}(\tau). \tag{2.46}$$

As, for every $\tau\in e_j(U_j)$, the set of all $\lambda\in M$ with $(\tau,\lambda)\in Z_\varepsilon$ is of real dimension $2n-1$, and as the form $\overline{\omega}'_{\tau,\lambda}(\widehat{u}(T^{-1}(\tau,\lambda)))\wedge\omega_\lambda(\lambda)$ is of degree $2n-1$ in λ, in (2.46) the form $\alpha^*\mathbf{F}^{11}$ can be replaced by $\alpha_\tau^*\mathbf{F}^{11}$. In view of (2.43), this implies that in (2.46) the form $\alpha^*\mathbf{F}^{11}$ can be replaced by $(e_j^{-1})^*\mathbf{F}^{11}$. Therefore, it is sufficient to prove that, for every fixed $\tau\in e_j(U_j)$,

$$\lim_{\varepsilon\longrightarrow0^+}\frac{(n-1)!}{(2\pi i)^n}(-1)^{\frac{n(n-1)}{2}}\int_{\{\lambda\in\mathbb{C}^n:(\tau,\lambda)\in Z_\varepsilon\}}\frac{\overline{\omega}'_{\tau,\lambda}(\widehat{u}(T^{-1}(\tau,\lambda)))\wedge\omega_\lambda(\lambda)}{\varepsilon^{2n}}=1.$$

This follows from the Leray formula, because, for $(\tau,\lambda)\in Z_\varepsilon$,

$$\overline{\omega}'_{\tau,\lambda}(\widehat{u}(T^{-1}(\tau,\lambda)))\wedge\omega_\lambda(\lambda)=\omega'(\widehat{u}(T^{-1}(\tau,\lambda)))\wedge\omega_\lambda(\lambda)$$

and, by (2.34),

$$\left\langle\widehat{u}(T^{-1}(\tau,\lambda)),\lambda\right\rangle=\left\langle\widehat{u}(T^{-1}(\tau,\lambda)),u(T^{-1}(\tau,\lambda))\right\rangle=\|\eta(T^{-1}(\tau,\lambda))\|_\sigma^2=\varepsilon^2.$$

Let Ω be an open set in M and $\mathbf{F}\in C^1(\Omega;\mathfrak{W}_n)$. Set

1. $\mathbf{F}$ is called $\overline{\mathbf{D}}$-monogenic if $\overline{\mathbf{D}}[\mathbf{F}]=0$ in Ω; that is, $\overline{\mathfrak{D}}(\Omega):=\ker\overline{\mathbf{D}}$;
2. $\mathbf{F}$ is called $\overline{\mathbf{D}}^*$-monogenic if $\overline{\mathbf{D}}^*[\mathbf{F}]=0$ in Ω; that is, $\overline{\mathfrak{D}}^*(\Omega):=\ker\overline{\mathbf{D}}^*$;
3. $\mathbf{F}$ is called $\overline{\mathbf{D}}_r$-monogenic if $\overline{\mathbf{D}}_r[\mathbf{F}]=0$ in Ω; that is, $\overline{\mathfrak{D}}_r(\Omega):=\ker\overline{\mathbf{D}}_r$;
4. $\mathbf{F}$ is called $\overline{\mathbf{D}}_r^*$-monogenic if $\overline{\mathbf{D}}_r^*[\mathbf{F}]=0$ in Ω; that is, $\overline{\mathfrak{D}}_r^*(\Omega):=\ker\overline{\mathbf{D}}_r^*$.

Hence, we immediately get Cauchy's integral representations for monogenic $\mathfrak{W}_n$-valued functions.

Theorem 2.6 Let Ω be a bounded domain with the topological boundary $\partial\Omega$, which is a piecewise smooth surface on the Kaehler manifold M, and the curvature form of the fibre of $T^{1,0}(M\times M)$ with respect to the connection D is zero, that is, $C(T^{1,0}(M\times M))=D^2=0$, and let $\mathbf{F}\in C^1(\Omega;\ \mathfrak{W}_n)\cap C^0(\Omega\cup\partial\Omega;\ \mathfrak{W}_n)$.

1. If $\mathbf{F}\in\overline{\mathfrak{D}}(\Omega)$, then, the following equality holds in Ω:

$$2\,\mathbf{F}(z)=\int_{\zeta\in\partial\Omega}\mathbf{K}_{\overline{\mathbf{D}}}(\zeta,z)\,\dot{*}\,\overline{\boldsymbol{\sigma}}_\zeta\,\dot{*}\,\mathbf{F}(\zeta).$$

2. If $\mathbf{F} \in \overline{\mathfrak{D}}^*(\Omega)$, then, the following equality holds in Ω:

$$2\,\mathbf{F}(z) = \int_{\zeta\in\,\partial\Omega} \mathbf{K}_{\overline{\mathbf{D}}^*}(\zeta, z) \,\dot{*}\, \overline{\boldsymbol{\sigma}}_\zeta^* \,\dot{*}\, \mathbf{F}(\zeta).$$

It follows trivially from the Borel-Pompeiu formulas in Theorem 2.4.

References

[1] Henkin G M, Leiterer J. Theory of Functions on Complex Manifolds. Berlin: Akademie-Verlag, 1984

[2] Zhong T D, Huang S. Complex Analysis in Several Variables (in Chinese). Shijiazhuang: Hebei Education Press, 1990

[3] Rocha-Chávez R, Shapiro M, Sommen F. Integral theorems for functions and differential forms in C^n. Boca Raton, London, New York, Washington, D C: Chapman & Hall/CRC, 2002

[4] Morrow J, Kodaira K. Complex Manifolds. New York: Holt, Rinehart & Winston, 1971

[5] Zhong T D, Qiu C H, Tang D M. Hyperholomorphic $\overline{D}$-problem in Kaehler manifolds//Tao Qian, Kit Lan Kou, Yan Yang. Proceedings of Sympositium on Hyper-Complex Analysis. Macau: University of Macau, 2008: 109–134

[6] Demailly J P, Laurent-Thiebaut C. Formulaes intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein. Ann Scient Eć Norm Sup, 1987, **20**(4): 579–598

[7] Berndtsson B. Cauchy-Leray forms and vector bundles. Ann Scien Éc Norm Sup, 1991, **24**(4): 319–337

[8] Sommen F. Martinelli-Bochner type formula in complex Clifford analysis. Z Anal Anwendungen, 1987, **6**(1): 75–82

[9] Zhong T D, Qiu C H, Tang D M. Plemelj formula for Bochner–Martinelli type integral in complex Clifford analysis//Lu Q K, Yin W P. Several Complex Variables in China on Its Research and Developments. Beijing: Science Press, 2009: 251–252

[10] Wu H H. The Bochner Technique in Differential Geometry. Harwood Academic publishers, 1988

[11] Yano K, Bochner S. Curvature and Betti Numbers//Ann of Math Studies 32. Princeton University Press, 1953