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Intersections of maximal ideals in algebras between $C^*(X)$ and $C(X)$

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Abstract

Let $C(X)$ be the algebra of all real-valued continuous functions on a completely regular space X, and $C^*(X)$ the subalgebra of bounded functions. There is a known correspondence between a certain class of z-filters on X and proper ideals in $C^*(X)$ that leads to theorems quite analogous to those for $C(X)$. This correspondence has been generalized by Redlin and Watson to any algebra between $C^*(X)$ and $C(X)$. In the process they have singled out a class of ideals that play a similar geometric role to that of z-ideals in the setting of $C(X)$. We show that these ideals are just the intersections of maximal ideals. It is also known that any algebra A between $C^*(X)$ and $C(X)$ is the ring of fractions of $C^*(X)$ with respect to a multiplicatively closed subset. We make use of this representation to characterize the functions that belong to all the free maximal ideals in A. We conclude by applying our characterization to a subalgebra H of $C(\mathbb{N})$ previously studied by Brooks and Plank.

Keywords: Rings of continuous functions; Rings of fractions; Prime ideal; Maximal ideal; z-ideal; z-filter

AMS classification: 54C40; 13B30

Introduction

Let $C(X)$ be the algebra of all real-valued continuous functions on a nonempty completely regular space X, and $C^*(X)$ the subalgebra of bounded functions. We study those subalgebras of $C(X)$ that contain $C^*(X)$. We shall refer to them as *intermediate*

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algebras on X or, simply, *intermediate algebras*. They are sublattices of $C(X)$ and so they are Φ-algebras in the sense of Henriksen and Johnson. As Φ-algebras these intermediate algebras have been studied by Hager and Henriksen in [7–9], as algebras of functions by Isbell in [10], as β-subalgebras of $C(X)$ by Plank in [12], just as intermediate algebras by Byun in [4,13], and by Acharyya in [1], and finally, as rings of fractions of $C^*(X)$ by Domínguez et al. in [5].

The correspondences between z-filters on X and proper ideals in $C(X)$ provide a geometric (or topological) description of some algebraic properties of $C(X)$. The algebraic structure is usually richer than the geometric one, but the geometric descriptions are powerful tools in the study of $C(X)$.

A function is invertible in $C(X)$ if it is never zero, and in $C^*(X)$ if it is bounded away from zero. In an arbitrary intermediate algebra A, of course, there is no such geometric description of invertibility which is independent of the structure of the algebra A. In [13] Redlin and Watson associate to each $f \in A$ a family $\mathcal{Z}_A(f)$ of zero-sets in X,

$$\mathcal{Z}_A(f) = \left\{E \in Z(X)\colon\ fg|_{E^c} = 1 \text{ for some } g \in A\right\},$$

and show that this family is a z-filter on X just when f is not invertible in A. This correspondence $\mathcal{Z}_A$ extends to a mapping from the set of proper ideals of A into the set of z-filters on X. For I a proper ideal of A, and $\mathcal{F}$ a z-filter on X, one writes

$$\mathcal{Z}_A(I) = \bigcup \left\{\mathcal{Z}_A(f)\colon\ f \in I\right\},$$

$$\mathcal{Z}_A^{-1}(\mathcal{F}) = \left\{f \in A\colon\ \mathcal{Z}_A(f) \subseteq \mathcal{F}\right\}.$$

$\mathcal{Z}_A(I)$ is a z-filter on X, and $\mathcal{Z}_A^{-1}(\mathcal{F})$ is a proper ideal of A. The proper ideals I of A that satisfy $\mathcal{Z}_A^{-1}(\mathcal{Z}_A(I)) = I$ are called $\mathcal{B}$-ideals. They play a similar geometric role to z-ideals in the classical setting of $C(X)$, but the class of $\mathcal{B}$-ideals in $C(X)$ does not agree with that of z-ideals. As a matter of fact, we shall prove in Section 3 that the $\mathcal{B}$-ideals in any intermediate algebra are exactly the intersections of maximal ideals (while the z-ideals in $C(X)$ are basically the sums of intersections of maximal ideals).

The theory developed in [13,4] generalizes to any intermediate algebra the development for $C^*(X)$ outlined in [6, 2L], and it provides unified geometric proofs for many results known separately for $C(X)$ and $C^*(X)$. The same goal is achieved in [5] with an algebraic treatment, by showing that any intermediate algebra A on X is the ring of fractions of $C^*(X)$ with respect to a multiplicatively closed subset. We make use of this representation to characterize the functions that belong to all the free maximal ideals in A. Our characterization is similar to the one given in [1].

Section 1 contains the preliminaries. In Section 2 we describe $\mathcal{Z}_A(f)$ in some particular cases, and summarize the basic facts about the maps $\mathcal{Z}_A$ and $\mathcal{Z}_A^{-1}$. The main purpose of Section 3 is to prove that the $\mathcal{B}$-ideals are just the intersections of maximal ideals. In Section 4 we give a new description of the members of $\mathcal{Z}_A(f)$, and characterize the functions that belong to all the free maximal ideals. We conclude by applying our characterization to a subalgebra H of $C(\mathbb{N})$ previously studied by Brooks and Plank.

An effort has been made to keep the exposition reasonably self-contained.

1. Preliminaries

Concerning rings of continuous functions we shall basically adhere to the notation and terminology in [6]. With respect to algebraic concepts the reader may consult [2]. Nevertheless we shall review some notation and preliminary results that will be used throughout the paper.

We assume that all rings are commutative with identity and that every ring morphism preserves the identities.

Let Y be an intermediate space between X and βX. The restriction morphism from $C(Y)$ to $C(X)$, which sends $g \in C(Y)$ to $g|_X$, is clearly injective. We shall always see $C(Y)$ as an intermediate algebra on X.

1.1. The maximal ideal space of a ring

The *prime ideal space* of a ring R is the set $\operatorname{Spec} R$ of all prime ideals of R endowed with the Zariski topology. The closed subsets in $\operatorname{Spec} R$ are those of the form

$$V(C) = \{P \in \operatorname{Spec} R\colon\ C \subseteq P\},$$

where C is any subset of R. The *maximal ideal space* of R is the subspace $\operatorname{Max} R$ consisting of all maximal ideals in R. Both the prime and maximal ideal spaces are compact spaces but, in general, they are not Hausdorff spaces.

1.2. Rings of fractions

A *multiplicatively closed* subset of a ring R is a subset S of R containing the identity and such that $st \in S$ whenever $s, t \in S$. The set of "fractions" $S^{-1}R = \{f/s\colon\ f \in R,\ s \in S\}$, where $f/s = g/t$ if $rtf = rsg$ for some $r \in S$, is endowed with a ring structure by defining addition and multiplication of "fractions" in the usual way. The ring $S^{-1}R$ is said to be *the ring of fractions* of R with respect to S. The canonical ring morphism $R \to S^{-1}R$, $f \mapsto f/1$, sends each $s \in S$ to an invertible element in $S^{-1}R$.

Throughout the paper X will denote a completely regular topological space. As usual $Z(f)$ and $\operatorname{coz} f$ will denote, respectively, the *zero-set* and the *cozero-set* of $f \in C(X)$, i.e., $Z(f) = \{x \in X\colon\ f(x) = 0\}$ and $\operatorname{coz} f = X - Z(f)$. We shall denote by $Z(X)$ the family of all zero-sets in X. For $I \subseteq C(X)$, we shall write $Z(I) = \{Z(f)\colon\ f \in I\}$. Thus, $Z(X) = Z(C(X)) = Z(C^*(X))$. Finally, for $\mathcal{F} \subseteq Z(X)$, we shall write $Z^{-1}(\mathcal{F}) = \{f \in C(X)\colon\ Z(f) \in \mathcal{F}\}$. A proper ideal I of $C(X)$ will be called a *z-ideal* if $Z^{-1}(Z(I)) = I$.

Let S be a multiplicatively closed subset of $C^*(X)$ whose members have void zero-sets. Each formal quotient $f/g \in S^{-1}C^*(X)$ can be identified with the continuous function it defines on X:

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}.$$

In this way the ring of fractions $S^{-1}C^*(X)$ is an intermediate algebra on X. Even more, all intermediate algebras on X are of this type. More precisely, let A be an intermediate algebra between $C^*(X)$ and $C(X)$, and let $U(A)$ be the set of units, or invertible elements, in A. It is clear that $S_A = U(A) \cap C^*(X)$ is a multiplicatively closed subset of $C^*(X)$. It has been proved in [5, 3.1] that A is the ring of fractions of $C^*(X)$ with respect to S_A, i.e., $A = S_A^{-1}C^*(X)$. The key point is that any function $f \in C(X)$ may be written as a fraction with both numerator and denominator in $C^*(X)$:

$$f = \frac{f(1+f^2)^{-1}}{(1+f^2)^{-1}}.$$

A subalgebra A of $C(X)$ is said to be *absolutely convex* if, whenever $|f| \leqslant |g|$, with $f \in C(X)$ and $g \in A$, then $f \in A$. Any intermediate algebra on X is an absolutely convex subalgebra of $C(X)$ and so a sublattice of $C(X)$ (see [5, 3.3]).

1.3. Maximal ideal spaces as models for βX

The Stone–Čech compactification of a completely regular space X is a compact Hausdorff space βX containing X as a dense subspace, and characterized by a universal property: "Every continuous mapping of X into a compact Hausdorff space can be continuously extended to βX".

Both $\operatorname{Max} C(X)$ and $\operatorname{Max} C^*(X)$ are compact Hausdorff spaces, each one containing a dense copy of X. The point $x \in X$ is identified with the maximal ideal $M^x = \{f \in C(X)\colon\ f(x) = 0\}$ in $\operatorname{Max} C(X)$, and with $M^{*x} = M^x \cap C^*(X)$ in $\operatorname{Max} C^*(X)$. These spaces are models for βX. Each point p in βX is identified with the maximal ideal $M^p = \{f \in C(X)\colon\ p \in \operatorname{cl}_{\beta X} Z(f)\}$ in $\operatorname{Max} C(X)$, and with $M^{*p} = \{f \in C^*(X)\colon\ f^\beta(p) = 0\}$ in $\operatorname{Max} C^*(X)$, where f^β denotes the continuous extension of f to βX. It is well known that M^{*p} is the unique maximal ideal in $C^*(X)$ containing $M^p \cap C^*(X)$.

As we have preserved the letter A to denote an intermediate algebra, we shall denote by $\mathcal{U}^p$ the z-ultrafilter on X corresponding to the point $p \in \beta X$, i.e.,

$$\mathcal{U}^p = \left\{Z \in Z(X)\colon\ p \in \operatorname{cl}_{\beta X} Z\right\} = Z(M^p).$$

Let A be an intermediate algebra on X. Every prime ideal of A is contained in a unique maximal ideal. It is a well-known result that in this situation $\operatorname{Max} A$ is a Hausdorff space. For any $p \in \beta X$, let M_A^p denote the unique maximal ideal of A containing the prime ideal $M^p \cap A$. With this notation, $M_C^p = M^p$ and $M_{C^*}^p = M^{*p}$. For $x \in X$, $M_A^x = \{f \in A\colon\ f(x) = 0\}$. The space $\operatorname{Max} A$ is also a model for βX. Each point p in βX is identified with the maximal ideal M_A^p in $\operatorname{Max} A$.

1.4. The extension f^* and the space $\upsilon_f X$

If the function $f \in C(X)$ is regarded as a continuous mapping of X into the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$, it has an extension $f^*\colon \beta X \to \mathbb{R}^*$. The set of points in βX where f^* takes real values is denoted by $\upsilon_f X$, i.e.,

$$\upsilon_f X = \left\{p \in \beta X\colon\ f^*(p) \neq \infty\right\}.$$

The space $\upsilon_f X$ is locally compact and σ-compact, and it is the largest subspace of βX to which f can be continuously extended.

1.5. Singly generated intermediate algebras

Let $f \in C(X)$. We shall denote by $C^*(X)[f]$ the smallest intermediate algebra containing f, that is,

$$C^*(X)[f] = \left\{ \sum_{i=0}^{n} g_i f^i \colon g_i \in C^*(X),\ n = 0, 1, 2, \ldots \right\}.$$

We shall say that $C^*(X)[f]$ is a *singly generated* intermediate algebra.

Certainly $f \in C(\upsilon_f X)$, and so $C^*(X)[f] \subseteq C(\upsilon_f X)$. In [5, 3.4] it has been shown:

(a) Let c be a real number, $c > 1$. Every singly generated intermediate algebra on X is $C^*(X)[f]$ for some $f \geqslant c$.

(b) If $f \geqslant c > 1$ for some $c \in \mathbb{R}$, then

$$C^*(X)[f] = \left\{ g \in C(X) \colon |g| \leqslant f^n \text{ for some } n \in \mathbb{N} \right\}.$$

One can easily see that if $A = C^*(X)[f]$ for some $f \geqslant c > 1$, then the multiplicatively closed subset $S_A = U(A) \cap C^*(X)$ is the set

$$S_A = \left\{ g \in C^*(X) \colon |g| \geqslant \frac{1}{f^n} \text{ for some } n \in \mathbb{N} \right\}.$$

2. The z-filters $\mathcal{Z}_A(I)$ and the ideals $\mathcal{Z}_A^{-1}(\mathcal{F})$

We summarize the basic facts about the maps $\mathcal{Z}_A$ and $\mathcal{Z}_A^{-1}$. We include some definitions and results selected from papers of Byun, Redlin and Watson (see [13,4]).

Definition 2.1. Let A be an intermediate algebra on X, and E a cozero-set in X. We shall say that a function $f \in A$ is *regular* on E (with respect to A) if there exists a function g in A such that $fg|_E = 1$.

Lemma 2.2. *Let A be an intermediate algebra on X. Let f, g be two functions in A, and E, F two cozero-sets in X.*

(a) *If f is regular on E and $F \subseteq E$, then f is regular on F.*

(b) *If f is regular on both E and F, then f is regular on $E \cup F$.*

(c) *If $f(x) \geqslant \varepsilon > 0$ for all $x \in E$, then f is regular on E.*

(d) *If f is regular on E and $0 < f(x) \leqslant g(x)$ for all $x \in E$, then g is regular on E.*

(e) *If f is regular on E and g is regular on F, then fg is regular on $E \cap F$, and $f^2 + g^2$ is regular on $E \cup F$.*

Proof. See [13, Lemma 1]. □

Let A be an intermediate algebra on X, and $f \in A$. Following Redlin and Watson we define

$$\mathcal{Z}_A(f) = \left\{ E \in Z(X) \colon f \text{ is regular on} E^c \right\}.$$

For $I \subseteq A$, and $\mathcal{F} \subseteq Z(X)$, we write

$$\mathcal{Z}_A(I) = \bigcup \{\mathcal{Z}_A(f)\colon\ f \in I\},$$

$$\mathcal{Z}_A^{-1}(\mathcal{F}) = \{f \in A\colon\ \mathcal{Z}_A(f) \subseteq \mathcal{F}\}.$$

When the intermediate algebra is $C(X)$ or $C^*(X)$, we shall write $\mathcal{Z}_C$ or $\mathcal{Z}_{C^*}$, to simplify the notation.

The next lemma is an easy consequence of the previous one.

Lemma 2.3. *Let A be an intermediate algebra on X, and let f, g be two functions in A.*

(a) $\mathcal{Z}_A(f) = \mathcal{Z}_A(f^n)$ *for all* $n \in \mathbb{N}$.

(b) $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f) \cap \mathcal{Z}_A(g)$.

(c) $\mathcal{Z}_A(f^2 + g^2) \supseteq \mathcal{Z}_A(f) \cup \mathcal{Z}_A(g)$.

(d) *If* $|f| \leqslant |g|$, *then* $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(g)$. *In particular,* $\mathcal{Z}_A(f) = \mathcal{Z}_A(|f|)$.

Proposition 2.4. *Let A be an intermediate algebra on X, f a function in A, I a proper ideal of A, and $\mathcal{F}$ a z-filter on X.*

(a) $\mathcal{Z}_A(f)$ *is a z-filter on X if and only if f is not a unit of A.*

(b) $\mathcal{Z}_A(I)$ *is a z-filter on X.*

(c) $\mathcal{Z}_A^{-1}(\mathcal{F})$ *is a proper ideal of A.*

Proof. See [13, Theorems 1–3]. Although a proof of (c) can be found in [13, Theorem 3], let us give a shorter one.

If $f \in \mathcal{Z}_A^{-1}(\mathcal{F})$ and $g \in A$, then $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f) \subseteq \mathcal{F}$, whence fg is in $\mathcal{Z}_A^{-1}(\mathcal{F})$. Now let f and g be two functions in $\mathcal{Z}_A^{-1}(\mathcal{F})$. To see that $f+g$ is in $\mathcal{Z}_A^{-1}(\mathcal{F})$ we must show that $\mathcal{Z}_A(f+g) \subseteq \mathcal{F}$. Let $E \in \mathcal{Z}_A(f+g)$. There exists $h \in A$ such that $(f+g)h = 1$ on E^c. Let us consider the zero-sets:

$$E_1 = \{x \in X\colon\ |f(x)h(x)| \leqslant \tfrac{1}{3}\},$$

$$E_2 = \{x \in X\colon\ |g(x)h(x)| \leqslant \tfrac{1}{3}\}.$$

If $x \in E^c$, then $x \in (E_1 \cap E_2)^c$. Hence $E \supseteq E_1 \cap E_2$. By Lemma 2.2(c), $E_1 \in \mathcal{Z}_A(fh) \subseteq \mathcal{Z}_A(f) \subseteq \mathcal{F}$, and similarly $E_2 \in \mathcal{F}$. Therefore, $E_1 \cap E_2 \in \mathcal{F}$, and so $E \in \mathcal{F}$. Finally, as $\emptyset \notin \mathcal{F}$, there are no units in $\mathcal{Z}_A^{-1}(\mathcal{F})$, so that $\mathcal{Z}_A^{-1}(\mathcal{F})$ is a proper ideal. □

A two-way correspondence between the z-filters on X and the proper ideals in A has been established. Let us point out that we are generalizing the theory for $C^*(X)$ (as outlined in [6, 2L]), which is far more complicated than the one for $C(X)$.

Let $f \in C(X)$ and $\varepsilon > 0$. Following Gillman and Jerison we define

$$E_\varepsilon(f) = \{x \in X\colon\ |f(x)| \leqslant \varepsilon\}.$$

For $I \subseteq C^*(X)$, and $\mathcal{F} \subseteq Z(X)$, we write

$$E(I) = \{E_\varepsilon(f)\colon\ f \in I,\ \varepsilon > 0\},$$

$$E^{-1}(\mathcal{F}) = \{f \in C^*(X)\colon\ E_\varepsilon(f) \in \mathcal{F} \text{ for all } \varepsilon > 0\}.$$

Let A be an intermediate algebra on X, and $f \in A$. It follows from Lemma 2.2(c) that, for any $\varepsilon > 0$, $E_\varepsilon(f) \in \mathcal{Z}_A(|f|) = \mathcal{Z}_A(f)$. Suppose now that $f \in C^*(X)$ and $E \in Z(X)$. It is almost evident that $E \in Z_{C^*}(f)$ if and only if $E \supseteq E_\varepsilon(f)$ for some $\varepsilon > 0$. To see that this cannot be generalized to a general intermediate algebra, take $X = \mathbb{N}$ and define $f(n) = 1/n$. The function f is invertible on X and so $\emptyset \in Z_C(f)$, but no $E_\varepsilon(f)$ is the empty set. We shall get the right generalization in Proposition 4.2.

Let I be any proper ideal of $C^*(X)$. Taking into account that $E(I)$ is a z-filter (see [6, 2L.5]), it follows from the previous statements that $\mathcal{Z}_{C^*}(I) = E(I)$.

Let $f \in C(X)$, without any additional assumption on f. We are going to describe the members of $\mathcal{Z}_A(f)$ when A is the smallest intermediate algebra containing f.

Proposition 2.5. *Let $f \in C(X)$, $A = C^*(X)[f]$ and $E \in Z(X)$. The following conditions are equivalent*:

(1) $E \in \mathcal{Z}_A(f)$.

(2) $E \supseteq E_\varepsilon(f)$, *for some* $\varepsilon > 0$.

Proof. It follows from Lemma 2.2(c) that (2) implies (1). Now we shall prove the converse. Let $E \in \mathcal{Z}_A(f)$, and assume that (2) does not hold. Consequently, there is a sequence (x_n) in E^c such that $\lim_{n\to\infty} f(x_n) = 0$. Set $D = \{x_1, x_2, \ldots\}$, and $D_k = \{x_k, x_{k+1}, \ldots\}$ for $k \in \mathbb{N}$. As $Z(f) \subseteq E$, f is never zero on D, and so $f(D)$ cannot be compact. Therefore D cannot be compact either, that is, there exists $p \in \mathrm{cl}_{\beta X} D - D$. Hence $p \in \mathrm{cl}_{\beta X} D_k$, and so $f^*(p) \in \mathrm{cl}_{\mathbb{R}^*} f(D_k)$. As this happens for any $k \in \mathbb{N}$, $f^*(p) = 0$. Since $E \in \mathcal{Z}_A(f)$, there exists $g \in A = C^*(X)[f]$ such that $fg = 1$ on E^c. Both functions f and g are in $C(\upsilon_f X)$, as $C^*(X)[f] \subseteq C(\upsilon_f X)$, and certainly $p \in \upsilon_f X$; therefore $(fg)^*(p) = f^*(p)g^*(p)$. Finally, on the one hand, $(fg)^*(p) = 0$, since $f^*(p) = 0$, but, on the other hand, $(fg)^*(p) = 1$, as $p \in \mathrm{cl}_{\beta X} D$ and $fg = 1$ on $D \subseteq E^c$. Of course, this is a contradiction. □

Let A be an intermediate algebra on X, and $f \in A$. If $E \in \mathcal{Z}_A(f)$, then $fg = 1$ in E^c for some $g \in A$, and so

$$Z(f) \subseteq \bigl\{x \in X\colon\ f(x)g(x) \neq 1\bigr\} \subseteq E.$$

We conclude that any member of $\mathcal{Z}_A(f)$ is a zero-set-neighborhood of $Z(f)$. To see that, in general, the converse does not hold take $X = \mathbb{N}$ and define $f(1) = 0$, $f(n) = 1/n$ for $n \neq 1$. Since X is a discrete space, $Z(f)$ is a zero-set-neighborhood of itself. Nevertheless $Z(f) \notin \mathcal{Z}_{C^*}(f)$.

Theorem 2.6. *For X a normal space and $f \in C(X)$, the members of $\mathcal{Z}_C(f)$ are the zero-set-neighborhoods of $Z(f)$.*

Proof. Let E be a zero-set-neighborhood of $Z(f)$. Let U be an open set such that $Z(f) \subseteq U \subseteq E$. The function $1/f : U^c \to \mathbb{R}$ has a continuous extension $h \in C(X)$. Since $fh = 1$ on E^c, it follows that $E \in \mathcal{Z}_C(f)$. □

3. $\mathcal{B}$-ideals as intersections of maximal ideals

The next lemma is immediate from the definitions of $\mathcal{Z}_A$ and $\mathcal{Z}_A^{-1}$.

Lemma 3.1 [4, 1.5]. *Let A be an intermediate algebra on X, I a proper ideal of A, and $\mathcal{F}$ a z-filter on X.*
(a) $\mathcal{Z}_A^{-1}(\mathcal{Z}_A(I)) \supseteq I$.
(b) $\mathcal{Z}_A(\mathcal{Z}_A^{-1}(\mathcal{Z}_A(I))) = \mathcal{Z}_A(I)$.
(c) $\mathcal{Z}_A(\mathcal{Z}_A^{-1}(\mathcal{F})) \subseteq \mathcal{F}$.
(d) $\mathcal{Z}_A^{-1}(\mathcal{Z}_A(\mathcal{Z}_A^{-1}(\mathcal{F}))) = \mathcal{Z}_A^{-1}(\mathcal{F})$.

Byun and Watson note in [4] that the inclusions in (a) and (c) may be proper, although in the classical setting of $C(X)$ one always has the equality $Z(Z^{-1}(\mathcal{F})) = \mathcal{F}$. Let us insist that this development is not a generalization of that carried out in the study of $C(X)$.

Definition 3.2. Let A be an intermediate algebra on X, and I a proper ideal of A. It is said that I is a $\mathcal{B}$-*ideal* if $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(I)$ implies $f \in I$. This condition is equivalent to $\mathcal{Z}_A^{-1}(\mathcal{Z}_A(I)) = I$.

Let us recall that an ideal I of a ring R is said to be a *radical* ideal if f is in I whenever f^n is in I for some $n \in \mathbb{N}$. Any radical ideal is an intersection of prime ideals (see [2, 1.14] or [6, 0.18]). It is evident that each maximal ideal is a $\mathcal{B}$-ideal. It follows from Lemma 2.3(a) that any $\mathcal{B}$-ideal is a radical ideal, and so it is an intersection of prime ideals. It is also immediate that any intersection of $\mathcal{B}$-ideals is a $\mathcal{B}$-ideal. From all the above-mentioned, one deduces that any intersection of maximal ideals is a $\mathcal{B}$-ideal.

Notice that the $\mathcal{B}$-ideals of $C^*(X)$ are the e-ideals studied in [6], where they are characterized as the intersections of maximal ideals. In Theorem 3.13 we shall generalize this characterization to any intermediate algebra.

For $p \in \beta X$, let O^p be the intersection of the prime ideals of $C(X)$ contained in M^p, and $\mathcal{E}^p$ the intersection of the prime z-filters on X contained in $\mathcal{U}^p$. With this notation, $\mathcal{E}^p = Z(O^p)$. For A an intermediate algebra on X, we shall denote by O_A^p the intersection of the prime ideals of A contained in M_A^p. It is known that, in general, $M_A^p \neq M^p \cap A$. Next we shall see that O_A^p has a better behavior in this respect. First we need an elementary lemma taken from [11, 1.4].

Lemma 3.3. *Let C be a ring, and A a subring of C. For every prime ideal Q of A, there exists a prime ideal P of C such that $P \cap A \subseteq Q$.*

Proof. For the sake of completeness let us repeat the short argument in [11]. The set $S = A - Q$ is a multiplicatively closed subset of C, and $0 \notin S$. According to [6, 0.16] there exists a prime ideal P in C such that $P \cap S = \emptyset$. Then $P \cap A \subseteq Q$. □

Proposition 3.4. *Let A be an intermediate algebra on X. For any $p \in \beta X$,*

$$O_A^p = O^p \cap A.$$

Proof. If P is a prime ideal in $C(X)$ contained in M^p, then $P \cap A$ is a prime ideal in A, and $P \cap A \subseteq M^p \cap A \subseteq M_A^p$. So $O_A^p \subseteq O^p \cap A$. Let us prove the other inclusion. Let Q be a prime ideal of A contained in M_A^p. We should show that $O^p \cap A \subseteq Q$. By the previous lemma, there exists a prime ideal P in $C(X)$ such that $P \cap A \subseteq Q$. Then $P \subseteq M^p$, since $P \cap A$ is a prime ideal of A contained in M_A^p. Therefore $O^p \cap A \subseteq P \cap A \subseteq Q$. □

Remark 3.5. The previous result, for $A = C^*(X)$, can be seen in [6, 7J].

The following lemma will be useful later.

Lemma 3.6 [4, 3.1]. *Let A be an intermediate algebra on X, and I a proper ideal of A. Then*

$$\mathcal{Z}_A(I) = \mathcal{Z}_{C^*}\big(I \cap C^*(X)\big).$$

Proof. The argument in [4, 3.1] needs some correction. It is enough to see that $\mathcal{Z}_A(I) \subseteq \mathcal{Z}_{C^*}(I \cap C^*(X))$, as the other inclusion is obvious. For any $f \in I$ and any $E \in \mathcal{Z}_A(f)$, there exists $g \in A$ such that $fg = 1$ on E^c. Let $h = fg$ and $u = 1/(1+h^2)$. Both functions u and hu are in $C^*(X)$, hence $hu = fgu \in I \cap C^*(X)$. Finally, by Lemma 2.2(c), $E \in \mathcal{Z}_{C^*}(hu)$, since $hu \geqslant 1/2$ on E^c. □

We observe next that the mapping $\mathcal{Z}_A$ does not distinguish between different prime ideals of A contained in the same maximal ideal. First we consider the case $A = C^*(X)$.

Lemma 3.7 [6, 7R]. *For any $p \in \beta X$,*

$$\mathcal{Z}_{C^*}\big(O^p \cap C^*(X)\big) = \mathcal{Z}_{C^*}(M^{*p}) = \mathcal{E}^p.$$

Theorem 3.8. *Let A be an intermediate algebra on X. For any $p \in \beta X$,*

$$\mathcal{Z}_A(O_A^p) = \mathcal{Z}_A(M_A^p) = \mathcal{E}^p.$$

Proof. We shall see that $\mathcal{E}^p = \mathcal{Z}_A(O_A^p) \subseteq \mathcal{Z}_A(M_A^p) \subseteq \mathcal{E}^p$.

(1) $\mathcal{Z}_A(O_A^p) = \mathcal{Z}_{C^*}(O_A^p \cap C^*(X)) = \mathcal{Z}_{C^*}(O^p \cap C^*(X)) = \mathcal{E}^p$.

(2) $\mathcal{Z}_A(M_A^p) = \mathcal{Z}_{C^*}(M_A^p \cap C^*(X)) \subseteq \mathcal{Z}_{C^*}(M^{*p}) = \mathcal{E}^p$. □

Corollary 3.9. *Let A be an intermediate algebra on X. For any $p \in \beta X$,*

$$\mathcal{Z}_A^{-1}(\mathcal{E}^p) = \mathcal{Z}_A^{-1}(\mathcal{U}^p) = M_A^p.$$

Proof. It is enough to observe that $M_A^p \subseteq \mathcal{Z}_A^{-1}(\mathcal{E}^p) \subseteq \mathcal{Z}_A^{-1}(\mathcal{U}^p)$, and then take into account the maximality of M_A^p. □

Corollary 3.10. *Let A be an intermediate algebra on X, and p a point of βX. If $\mathcal{F}$ is a prime z-filter on X contained in $\mathcal{U}^p$, then*

$$\mathcal{Z}_A^{-1}(\mathcal{F}) = M_A^p.$$

Proof. Since $\mathcal{F} \subseteq \mathcal{U}^p$ and $\mathcal{F}$ is a prime z-filter, $\mathcal{E}^p \subseteq \mathcal{F}$. So $M_A^p = \mathcal{Z}_A^{-1}(\mathcal{E}^p) \subseteq \mathcal{Z}_A^{-1}(\mathcal{F})$. By the maximality of M_A^p, we conclude that $M_A^p = \mathcal{Z}_A^{-1}(\mathcal{F})$. □

Now we may obtain Plank's geometric description of the functions in M_A^p.

Let A be an intermediate algebra on X, $p \in \beta X$, and $f \in M_A^p$. For any $\varepsilon > 0$,

$$E_\varepsilon(f) \in \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(M_A^p) \subseteq \mathcal{U}^p.$$

Hence $p \in \mathrm{cl}_{\beta X} E_\varepsilon(f)$, and so $f^*(p) \leqslant \varepsilon$. Therefore $f^*(p) = 0$.

Proposition 3.11 (see [12,6, 7D]). *Let A be an intermediate algebra on X. For any $p \in \beta X$,*

$$M_A^p = \{ f \in A\colon\ (fg)^*(p) = 0 \ \text{for every } g \in A \}.$$

Proof. It is easy to see that the set on the right is a proper ideal of A, and we have just shown that the ideal M_A^p is contained in that set. Now take into account the maximality of M_A^p. □

Next we shall prove that the $\mathcal{B}$-ideals are precisely the intersections of maximal ideals.

Lemma 3.12. *Any z-filter on X is an intersection of prime z-filters.*

Proof. For $\mathcal{F}$ a z-filter on X, the z-ideal $Z^{-1}(\mathcal{F})$ is an intersection of prime ideals, and so it is the intersection of the prime ideals that are minimal between those containing it. Each one of these minimal prime ideals is a z-ideal, by [6, 14.7]. Therefore, $Z^{-1}(\mathcal{F})$ is an intersection of prime z-ideals, and so $\mathcal{F}$ is an intersection of prime z-filters. □

Theorem 3.13. *Let A be an intermediate algebra on X. The $\mathcal{B}$-ideals of A are just the intersections of maximal ideals.*

Proof. It is known that any intersection of maximal ideals of A is a $\mathcal{B}$-ideal of A. We prove the converse. According to Lemma 3.1, a $\mathcal{B}$-ideal of A is an ideal of the form $\mathcal{Z}_A^{-1}(\mathcal{F})$, $\mathcal{F}$ being a z-filter on X. Since any z-filter on X is an intersection of prime z-filters (by the previous lemma), the result follows from Corollary 3.10. □

Remark 3.14. It is well known that, for $p \in \beta X$,

$$O^p = \{ f \in C(X)\colon\ p \in \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, Z(f) \}.$$

We shall see that, for A an intermediate algebra on X and $p \in \beta X$,

$$O_A^p = \{ f \in A\colon\ p \in \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, \mathcal{Z}_A(f) \},$$

where $\mathrm{cl}_{\beta X}\, \mathcal{Z}_A(f)$ is the set of cluster points of $\mathcal{Z}_A(f)$ in βX, that is,

$$\mathrm{cl}_{\beta X}\, \mathcal{Z}_A(f) = \bigcap \{ \mathrm{cl}_{\beta X}\, E\colon\ E \in \mathcal{Z}_A(f) \}.$$

This geometric description of the functions in O_A^p will provide an alternative proof to Proposition 3.4. Our argument will involve some knowledge of the Zariski topology on the prime ideal space of a ring.

Recall that, for R a ring and $f \in R$,

$$V(f) = \{P \in \operatorname{Spec} R\colon\ f \in P\}.$$

Let M be a maximal ideal of a ring R, and denote by $O(M)$ the intersection of the prime ideals contained in M. It is not difficult to check (see [11, 1.1]) that

$$O(M) = \bigl\{f \in R\colon\ V(f) \text{ is a neighborhood of } M \text{ in } \operatorname{Spec} R\bigr\}.$$

Moreover, if $\bigcap \operatorname{Spec} R = \bigcap \operatorname{Max} R$, then

$$O(M) = \bigl\{f \in R\colon\ V(f) \cap \operatorname{Max} R \text{ is a neighborhood of } M \text{ in } \operatorname{Max} R\bigr\}.$$

Next we apply this result to the case of an intermediate algebra on X. Let A be such an algebra, and $f \in A$. If we identify $\operatorname{Max} A$ with βX, then

$$V(f) \cap \operatorname{Max} A = \bigl\{p \in \beta X\colon\ f \in M_A^p\bigr\}.$$

According to Corollary 3.9, $f \in M_A^p$ if and only if $\mathcal{Z}_A(f) \subseteq \mathcal{U}^p$ or, equivalently, $p \in \operatorname{cl}_{\beta X} \mathcal{Z}_A(f)$. Hence,

$$V(f) \cap \operatorname{Max} A = \operatorname{cl}_{\beta X} \mathcal{Z}_A(f).$$

Also, $\bigcap \operatorname{Spec} A = \bigcap \operatorname{Max} A = \{0\}$. Therefore,

$$\begin{aligned} O_A^p &= O(M_A^p) \\ &= \bigl\{f \in A\colon\ \operatorname{cl}_{\beta X} \mathcal{Z}_A(f) \text{ is a neighborhood of } p \text{ in } \beta X\bigr\} \\ &= \bigl\{f \in A\colon\ p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} \mathcal{Z}_A(f)\bigr\}. \end{aligned}$$

Of course, $\operatorname{cl}_{\beta X} \mathcal{Z}_C(f) = \operatorname{cl}_{\beta X} Z(f)$. But, for f in an arbitrary intermediate algebra A, if $Z(f) = \emptyset$ and f is not a unit of A, then $\operatorname{cl}_{\beta X} \mathcal{Z}_A(f) \neq \operatorname{cl}_{\beta X} Z(f)$. So that, in general, the inclusion $\operatorname{cl}_{\beta X} Z(f) \subseteq \operatorname{cl}_{\beta X} \mathcal{Z}_A(f)$ may be proper. Nevertheless, according to the above geometric description of O_A^p, the algebraic equality

$$O_A^p = O^p \cap A, \quad \text{for every } p \in \beta X,$$

is equivalent to the following geometric one:

$$\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} \mathcal{Z}_A(f) = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f), \quad \text{for every } f \in A.$$

Let us give a direct proof of the latter equality. If V is an open neighborhood of p in βX that is contained in $\operatorname{cl}_{\beta X} \mathcal{Z}_A(f)$, then $V \cap X \subseteq \bigcap \mathcal{Z}_A(f)$. We already know that, for any $\varepsilon > 0$, $E_\varepsilon(f) \in \mathcal{Z}_A(|f|) = \mathcal{Z}_A(f)$. So that

$$\bigcap \mathcal{Z}_A(f) \subseteq \bigcap \bigl\{E_\varepsilon(f)\colon\ \varepsilon > 0\bigr\} \subseteq Z(f).$$

Therefore, $V \cap X \subseteq Z(f)$, and so $p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)$. This shows that

$$\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} \mathcal{Z}_A(f) \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f).$$

The other inclusion is immediate.

4. Intersection of all the free maximal ideals

Let A be an intermediate algebra on X. We shall temporarily denote by A_F the intersection of all the free maximal ideals in A, i.e.,

$$A_F = \bigcap \{ M_A^p \colon \ p \in \beta X - X \}.$$

Our main purpose in this section is to achieve a topological description of the functions in A_F. In the extreme cases $A = C^*(X)$ or $A = C(X)$, the functions that belong to all the free maximal ideals have already been described, though imposing some conditions on X. It is an easy exercise [6, 7F] to see that the intersection of all the free maximal ideals in $C^*(X)$ is the set $C_\infty(X)$ of all functions in $C(X)$ that vanish at infinity (i.e., all f in $C(X)$ such that $\{x \in X \colon \ |f(x)| \geqslant 1/n\}$ is compact for every $n \in \mathbb{N}$). On the other hand, it is not a trivial result that, for X realcompact, the intersection of all the free maximal ideals in $C(X)$ is the set $C_K(X)$ of all continuous functions with compact support (see [6, 8.19]).

Let us show how the extreme cases delimit our object. We shall see that

$$C_K(X) \subseteq C(X)_F \subseteq A_F \subseteq C^*(X)_F = C_\infty(X).$$

Let $f \in C_K(X)$, and $p \in \beta X - X$. Clearly, $p \in \mathrm{cl}_{\beta X} Z(f)$ or, equivalently, $f \in M^p$. This shows that $C_K(X) \subseteq C(X)_F$. Assume now that B is an intermediate algebra on X containing A. It only rest to prove that $B_F \subseteq A_F$. We have already seen that $f^*(p) = 0$ for $f \in M_A^p$. So that, for $f \in A_F$, f^* vanish on $\beta X - X$, and so $f \in C_\infty(X) \subseteq C^*(X)$. Thus, $A_F \subseteq C^*(X)$. Finally, if $f \in B_F$ and $p \in \beta X - X$, then $f \in M_B^p$ and also $f \in C^*(X) \subseteq A$, so that $f \in M_B^p \cap A \subseteq M_A^p$. This shows that $B_F \subseteq A_F$.

For our purpose, it will be useful to take into account the representation of the intermediate algebra A as a ring of fractions of $C^*(X)$. Let us recall that if $U(A)$ is the set of units of A and $S_A = U(A) \cap C^*(X)$, then $A = S_A^{-1} C^*(X)$.

Lemma 4.1. *Let A be an intermediate algebra on X, and $S_A = U(A) \cap C^*(X)$. Then*

$$A = \left\{ f \in C(X) \colon \ |f| \leqslant \left| \frac{1}{g} \right| \ \text{for some } g \in S_A \right\}.$$

Proof. If $|f| \leqslant |1/g|$ for some $g \in S_A$, then $f \in A$, since A is an absolutely convex subalgebra of $C(X)$ and $1/g \in A$. Conversely, if $f \in A$, then $(1 + f^2)^{-1} \in S_A$, and $|f| \leqslant 1 + f^2 = 1/(1 + f^2)^{-1}$. □

Next we shall give a new description of the zero-sets in $\mathcal{Z}_A(f)$.

For $f, g \in C(X)$, it will be useful to write

$$E_g(f) = \{ x \in X \colon \ |f(x)| \leqslant |g(x)| \},$$

$$E^g(f) = \{ x \in X \colon \ |f(x)| \geqslant |g(x)| \}.$$

Proposition 4.2. *Let A be an intermediate algebra on X, $E \in Z(X)$, and $f \in A$. The following statements are equivalent*:

(1) $E \in \mathcal{Z}_A(f)$.

(2) $|f| \geqslant |g|$ *on* E^c*, for some* $g \in S_A$.
(3) $E \supseteq E_g(f)$*, for some* $g \in S_A$.

Proof. Assume that $E \in \mathcal{Z}_A(f)$. Then there exists $h \in A$ such that $fh = 1$ on E^c. By the previous lemma, $|h| \leqslant |1/g|$, for some $g \in S_A$. Hence $|f| = |1/h| \geqslant |g|$ on E^c. This shows that (1) implies (2). Next we shall prove that (2) implies both (1) and (3). Suppose that $|f| \geqslant |g|$ on E^c, for some $g \in S_A$. On the one hand, since g is a unit of A, g is regular on E^c and, by Lemma 2.2(d), $E \in \mathcal{Z}_A(f)$. On the other hand, $E \supseteq \{x \in X\colon |f(x)| < |g(x)|\}$ and, as g is never zero, the last set contains $\{x \in X\colon |f(x)| \leqslant |\frac{1}{2}g(x)|\} = E_{g/2}(f)$. Thus $E \supseteq E_{g/2}(f)$ and, indeed, $\frac{1}{2}g \in S_A$. Finally, it is clear that (3) implies (2). □

We shall use the concept of a *small* set.

Definition 4.3. Let E be a subset of X. It is said that E is a *small* set if every zero-set contained in E is compact.

Following Redlin and Watson, we write

$$\mathcal{K} = \left\{E \in Z(X)\colon\ E^c \text{ is small}\right\}.$$

One can easily see that

$$\mathcal{K} = \bigcap \left\{\mathcal{U}^p\colon\ p \in \beta X - X\right\},$$

and so

$$\mathcal{Z}_A^{-1}(\mathcal{K}) = \bigcap \left\{M_A^p\colon\ p \in \beta X - X\right\}.$$

Theorem 4.4 (see also [1, Theorem 2.2]). *Let A be an intermediate algebra on X, and let $S_A = U(A) \cap C^*(X)$. A function $f \in A$ is in $\bigcap\{M_A^p\colon\ p \in \beta X - X\}$ if and only if $E^g(f)$ is compact for every $g \in S_A$.*

Proof. Assume that $E^g(f)$ is not compact for some $g \in S_A$. Then there exists $p \in \mathrm{cl}_{\beta X} E^g(f)$ such that $p \notin X$. By Proposition 4.2, $E_{g/2}(f) \in \mathcal{Z}_A(f)$. Since g is never zero, the zero-sets $E^g(f)$ and $E_{g/2}(f)$ are disjoint, and so their closures in βX are disjoint too. As $p \in \mathrm{cl}_{\beta X} E^g(f)$, it follows that $p \notin \mathrm{cl}_{\beta X} E_{g/2}(f)$. Hence $\mathcal{Z}_A(f)$ is not contained in $\mathcal{U}^p$ or, equivalently, $f \notin M_A^p$.

Suppose now that $E^g(f)$ is compact for every $g \in S_A$. We shall see that $\mathcal{Z}_A(f) \subseteq \mathcal{K}$. Let E be a zero-set in $\mathcal{Z}_A(f)$. By Proposition 4.2, there exists $g \in S_A$ such that $|f| \geqslant |g|$ on E^c, and so $E^c \subseteq E^g(f)$. Since $E^g(f)$ is compact, any zero-set contained in E^c is compact too. Therefore, E^c is small. □

Remark 4.5. Let A be an intermediate algebra on X, and $f \in A$. If h is a unit of A, and $E^g(f)$ is compact for every $g \in S_A$, then $E^h(f)$ is compact too. Notice that, for h a unit of A, the function $g = h^2/(1 + h^2)$ is a bounded unit, and $|g| \leqslant |h|$. Hence $g \in S_A = U(A) \cap C^*(X)$, and $E^h(f) \subseteq E^g(f)$.

It follows from the previous theorem that

$$\bigcap\{M^{*p}\colon\ p\in\beta X-X\}=C_\infty(X),$$

where $C_\infty(X)$ denotes the set of functions in $C(X)$ that vanish at infinity.

Next we shall pay some attention to the case $A=C(X)$.

Corollary 4.6 [6, 4E.2]. *A function $f\in C(X)$ is in $\bigcap\{M^p\colon\ p\in\beta X-X\}$ if and only if every zero-set disjoint from $Z(f)$ is compact.*

Proof. Assume there exists a noncompact zero-set $Z(h)$ disjoint from $Z(f)$, and take $g=(|h|+|f|)\wedge 1$. If $x\in Z(h)$, then $|g(x)|\leqslant|f(x)|$, whence $Z(h)\subseteq E^g(f)$. Since $Z(h)$ is not compact, $E^g(f)$ cannot be compact either. Conversely, suppose now that $f\notin M^p$ for some $p\in\beta X-X$, that is, $p\notin \mathrm{cl}_{\beta X} Z(f)$. There exists $h\in C^*(X)$ such that $Z(h^\beta)$ is a neighborhood of p in βX disjoint from $Z(f)$. Then $Z(h)$ is a zero-set disjoint from $Z(f)$, and it is not compact because $p\in \mathrm{cl}_{\beta X} Z(h)$. □

Now we shall recover the classical result [6, 8.19]. Recall that $C_K(X)$ is the set of all functions in $C(X)$ with compact support.

Corollary 4.7. *If X is a realcompact space, then*

$$\bigcap\{M^p\colon\ p\in\beta X-X\}=C_K(X).$$

Proof. The inclusion $C_K(X)\subseteq\bigcap\{M^p\colon\ p\in\beta X-X\}$ always holds. Let $f\in C(X)$ such that $f\notin C_K(X)$. Since X is realcompact, there is a noncompact closed subset S of X that is completely separated from $Z(f)$ (as it is shown in [6, 8.19]), and so there is a noncompact zero-set containing S and disjoint from $Z(f)$. Hence $f\notin\bigcap\{M^p\colon\ p\in\beta X-X\}$. □

For a singly generated intermediate algebra, the characterization of the functions in all the free maximal ideals can be simplified a bit more, as it is shown in the next corollary.

Let $f,l\in C(X)$, with $l\geqslant 0$ and $Z(l)=\emptyset$. For $n\in\mathbb{N}$, we shall write

$$F_n(f)=\left\{x\in X\colon\ |f(x)|\geqslant\frac{1}{l^n(x)}\right\}.$$

Corollary 4.8. *Let A be a singly generated intermediate algebra on X, $A=C^*(X)[l]$, with $l\geqslant c>1$. A function $f\in A$ is in $\bigcap\{M_A^p\colon\ p\in\beta X-X\}$ if and only if $F_n(f)$ is compact for all $n\in\mathbb{N}$.*

Proof. Assume that $f\in\bigcap\{M_A^p\colon\ p\in\beta X-X\}$. The function $1/l^n$ is in S_A, and $F_n(f)=E^{1/l^n}(f)$. By Theorem 4.4, $F_n(f)$ is compact. Conversely, suppose now that $F_n(f)$ is compact for all n. Let g be a function in S_A. According to Section 1.5, $|g|\geqslant 1/l^n$ for some n. Then

$$E^g(f)=\{x\in X\colon\ |f(x)|\geqslant|g(x)|\}\subseteq\left\{x\in X\colon\ |f(x)|\geqslant\frac{1}{l^n(x)}\right\}=F_n(f),$$

and the last set is compact. Hence $E^g(f)$ is compact too. It follows from Theorem 4.4 that $f \in \bigcap\{M_A^p\colon\ p \in \beta X - X\}$. $\square$

Remark 4.9 (see [1, 3.4]). Let X be a locally compact and σ-compact, but not compact space, and let l be as in the corollary. If we further assume that l is a *perfect* mapping (i.e., $l^{-1}(K)$ is compact for each compact $K \subseteq \mathbb{R}$), then the function $1/e^l$ belongs to all the free maximal ideals of $C^*(X)[l]$, yet it does not belong to $C_K(X)$.

Next we shall examine the classical intermediate algebra on $\mathbb{N}$ studied by Brooks and Plank in [3,12], respectively.

Let H denote the intermediate algebra on $\mathbb{N}$ consisting of those functions $f \in C(\mathbb{N})$ such that

$$\limsup_{n\to\infty} \sqrt[n]{|f(n)|} \leqslant 1.$$

For $f \in C(\mathbb{N})$, let $\bar{f}(n) = \sqrt[n]{|f(n)|}$. One can see in [12, 7.1] that

$$f \in H \quad \text{if and only if} \quad \bar{f} \in C^*(\mathbb{N}) \text{ and } \bar{f}^\beta \leqslant 1 \text{ on } \beta\mathbb{N} - \mathbb{N},$$

where $\bar{f}^\beta$ is the continuous extension of $\bar{f}$ to $\beta\mathbb{N}$. Plank also shows there that a function $f \in H$ is a unit of H if and only if $Z(f) = \emptyset$ and $\bar{f}^\beta = 1$ on $\beta\mathbb{N} - \mathbb{N}$.

Let us now recover Corollary 2.3.1 in [3], which is obtained there as a consequence of a theorem that was shown to be wrong by Plank in [12, 7.6].

Corollary 4.10 [3, 2.3.1]. *A function $f \in H$ is in $\bigcap\{M_H^p\colon\ p \in \beta\mathbb{N} - \mathbb{N}\}$ if and only if $\bar{f}^\beta(p) < 1$ for every $p \in \beta\mathbb{N} - \mathbb{N}$.*

Proof. Plank shows in [12, 7.2] that f is in $\bigcap\{M_H^p\colon\ p \in \beta\mathbb{N} - \mathbb{N}\}$ if $\bar{f}^\beta(p) < 1$ for every $p \in \beta\mathbb{N} - \mathbb{N}$. Let us prove the converse. Assume there exists $p \in \beta\mathbb{N} - \mathbb{N}$ such that $\bar{f}^\beta(p) = 1$. We shall exhibit a unit g of H such that $E^g(f)$ is not compact. It follows from the assumption that there is a sequence (n_k) in $\mathbb{N}$ such that $|\bar{f}(n_k) - 1| \leqslant 1/2^k$, and $n_k \neq n_j$ for $k \neq j$. Set $D = \{n_k\colon\ k = 1, 2, \ldots\}$. We define the function g as follows: $g(n) = f(n)$ for $n \in D$, and $g(n) = 1$ for $n \notin D$. Notice that $Z(g) = \emptyset$, since f is never zero on D. Also, $g \in H$, since $|g| \leqslant |f| \vee 1 \in H$. Next we shall see that $\bar{g}^\beta = 1$ on $\beta\mathbb{N} - \mathbb{N}$. For $k \in \mathbb{N}$, set $D_k = \{k, k+1, \ldots\}$. Clearly, $\bar{g}^\beta(q) = 1$ if $q \in \mathrm{cl}_{\beta\mathbb{N}}(\mathbb{N} - D)$. On the other hand, if $q \in \mathrm{cl}_{\beta\mathbb{N}} D$, then $q \in \mathrm{cl}_{\beta\mathbb{N}} D_k$, and so

$$\bar{g}^\beta(q) \in \mathrm{cl}_{\mathbb{R}}\, \bar{g}(D_k) = \mathrm{cl}_{\mathbb{R}}\, \bar{f}(D_k) \subseteq \left[1 - \frac{1}{2^k}, 1 + \frac{1}{2^k}\right].$$

As this happens for every $k \in \mathbb{N}$, $\bar{g}^\beta(q) = 1$. This shows, according to Plank's characterization of the units of H, that g is such a unit. Certainly, the set $E^g(f) = \{n \in \mathbb{N}\colon\ |f(n)| \geqslant |g(n)|\}$ is not compact because it contains the infinite set D. Finally, it follows from Theorem 4.4, taking into account Remark 4.5, that f does not belong to all the free maximal ideals of H. $\square$

For any intermediate algebra A on X, and any maximal ideal M of A, the residue class field A/M contains a canonical copy of $\mathbb{R}$: the set of residual classes of the constant functions. When this canonical copy of $\mathbb{R}$ is the entire field A/M, one says that M is a *real* maximal ideal of A. We shall denote by $\upsilon_A X$ the set of all real maximal ideals of A. The space $\upsilon_A X$ can be identified to the space of those points in $\beta X = \operatorname{Max} A$ to which all the functions in A can be continuously extended, i.e.,

$$\upsilon_A X = \bigcap \{\upsilon_f X\colon\ f \in A\}.$$

Hence $A \subseteq C(\upsilon_A X)$. With this notation, $\upsilon_C X = \upsilon X$ (the Hewitt realcompactification of X) and $\upsilon_{C^*} X = \beta X$. It is well known that $C(X) = C(\upsilon X)$ and $C^*(X) = C(\beta X)$, but, in general, the inclusion $A \subseteq C(\upsilon_A X)$ may be proper (see [5, 4.3 and 2.3]).

Definition 4.11. Let A be an intermediate algebra on X. Following Redlin and Watson, we shall say that the space X is *A-compact* if the image of the canonical immersion $X \to \operatorname{Max} A$, $x \mapsto M_A^x$, is the set $\upsilon_A X$ of all real maximal ideals of A.

In view of this definition, the C-compact spaces are the realcompact spaces while the C^*-compact spaces are the compact ones. Clearly, if X is A-compact and B is an intermediate algebra on X containing A, then X is B-compact.

Proposition 4.12. *Let A be a singly generated intermediate algebra on X, $A = C^*(X)[l]$, with $Z(l) = \emptyset$ and $1/l \in C^*(X)$. The space X is A-compact if and only if $1/l \in C_\infty(X)$.*

Proof. Observe that $\upsilon_A X = \upsilon_l X = \operatorname{coz}(1/l)^\beta$. □

Corollary 4.13. *The following statements are equivalent*:
(1) *X is locally compact and σ-compact.*
(2) *X supports a continuous function that has no zeros, but vanishes at infinity.*
(3) *X is A-compact, for some singly generated intermediate algebra A.*

Proof. The equivalence of (1) with (2) is well known. The rest follows from the previous proposition. □

Remark 4.14. Let A be an intermediate algebra on X. It was stated in [4, 5.7] that if X is an A-compact space, then the intersection of all the free maximal ideals in A is $C_K(X)$. This is false. Let H be as in Corollary 4.10. On the one hand, the space $\mathbb{N}$ is H-compact, since the function $g(n) = n$ is in H and $\mathbb{N} = \upsilon_g \mathbb{N}$. On the other hand, the function $f(n) = n/2^n$ belongs to all the free maximal ideals of H, since $\lim_{n\to\infty} \bar{f}(n) = 1/2 > 1$. Nevertheless, $f \notin C_K(\mathbb{N})$. The error was also pointed out by Acharyya, Chattopadhyay and Ghosh in [1, 3.4]. They considered $A = C^*(\mathbb{N})[g]$ and showed that the function $1/e^g$ belongs to all the free maximal ideals in A.

Note. This work was presented in Morelia, Michoacan, Mexico in the Second Ibero-American Conference on Topology and its Applications in March 1997. In May 1997 we found the paper [1] by Acharyya, Chattopadhyay and Ghosh.

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