



# Finite homomorphisms on rings of continuous functions

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## Abstract

This paper deals with finiteness properties of the homomorphism between the rings of continuous functions  $C(Y) \rightarrow C(X)$  induced by a continuous map  $X \rightarrow Y$ .

The main result says that, for  $X$  and  $Y$  compact Hausdorff spaces, the homomorphism  $C(Y) \rightarrow C(X)$  is finite (i.e.,  $C(X)$  is finitely generated as a  $C(Y)$ -module) if and only if the map  $X \rightarrow Y$  is locally injective.

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## 1. Introduction

Every continuous map  $X \rightarrow Y$  defines, by composition, a homomorphism  $C(Y) \rightarrow C(X)$  between the corresponding algebras of real-valued continuous functions. This paper is devoted to the study of the finiteness properties of this homomorphism.

Our starting point is the well-known result which states that every realcompact space  $X$  is determined by the algebra  $C(X)$  of all real-valued continuous functions defined on it, and that continuous maps between such spaces are in one-to-one correspondence with homomorphisms between their algebras of continuous functions. From this equivalence

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it follows that statements about topological properties of spaces and maps should have a natural translation in terms of algebraic properties of the corresponding algebras and homomorphisms.

Examples of results showing these relationships are the characterization of local homeomorphisms by means of rings of germs of continuous functions [8], the characterization of the dimension of a topological space  $X$  in terms of dense subalgebras of  $C(X)$  [13], the going-up and going-down theorems for the homomorphism  $C(Y) \rightarrow C(X)$  defined by an open and closed map [10], the correspondence between finite (branched) coverings  $X \rightarrow Y$  and integral and flat homomorphisms  $C(Y) \rightarrow C(X)$  [11], the characterization of the branch set of a finite covering by means of modules of Kahler differentials [12], and the characterization of the subalgebras  $A$  of  $C(X)$  containing  $C^*(X)$  that are isomorphic to some  $C(Y)$  and the one-to-one correspondence between the family of these intermediate algebras and the family of realcompact subspaces  $Y$  of  $\beta X$  containing  $X$  [3].

Our main purpose in this paper is to determine the finiteness properties of the homomorphism  $C(Y) \rightarrow C(X)$  (i.e., whether it is finite, integral, singly generated or finitely generated) in terms of the properties of the map  $X \rightarrow Y$ .

A particular case of this problem consists of considering the projection map  $\pi_{\gamma\alpha} : \gamma T \rightarrow \alpha T$  between two Hausdorff compactifications  $\alpha T \leq \gamma T$  of a locally compact, non-compact Hausdorff space  $T$ . In [6] Faulkner studies whether the corresponding homomorphism  $C(\alpha T) \rightarrow C(\gamma T)$  is singly generated. This homomorphism is also studied by Dwornik-Orzechowska and Wajch in [4]. In this latter work the authors investigate the smallest cardinal number of a set  $F \subset C(\gamma T)$  such that  $C(\gamma T)$  is the uniform closure of the algebra generated by  $C(\alpha T) \cup F$ .

We shall see that, in order to study finiteness properties between the associated rings of continuous functions, the problem for compactifications is equivalent to the problem for arbitrary compact Hausdorff spaces.

We obtain Faulkner's theorems (3, 4, and 5 of [6]) as particular cases of our results for finite spaces.

The main result of the paper says that, for  $X$  and  $Y$  compact Hausdorff spaces, the homomorphism  $C(Y) \rightarrow C(X)$  is finite ( $C(X)$  is finitely generated as  $C(Y)$ -module) if and only if the map  $X \rightarrow Y$  is locally injective.

We show examples of finite but not singly generated, as well as integral but not finite, homomorphisms  $C(Y) \rightarrow C(X)$ .

## 2. Preliminaries

For rings of continuous functions and compactifications, we use the same terminology and notation as in [2,7]. For algebraic concepts and results, the reader may consult [1]. Nevertheless we shall review some definitions that will be used throughout the paper.

**Definitions 2.1.** Let  $h : A \rightarrow B$  be a ring homomorphism, and consider  $B$  with the induced structure of  $A$ -module ( $a \cdot b := h(a) \cdot b$ ). Then  $B$  is said to be an  $A$ -algebra.

The homomorphism  $h$  is finite, and  $B$  is a finite  $A$ -algebra, if  $B$  is a finitely generated  $A$ -module, i.e., if there exists a finite set of elements  $b_1, \dots, b_n \in B$  such that every element of  $B$  is a linear combination of  $b_1, \dots, b_n$  with coefficients in  $A$ , i.e.,  $B = Ab_1 + \dots + Ab_n$ .

An element  $b \in B$  is integral over  $A$  if there exists a monic polynomial  $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in A[x]$  such that  $P(b) = b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0$ .

The homomorphism  $h$  is integral, and  $B$  is an integral  $A$ -algebra, if every element of  $B$  is integral over  $A$ .

The homomorphism  $h$  is of finite type, and  $B$  is a finitely generated  $A$ -algebra, if there exists a finite set of elements  $b_1, \dots, b_n \in B$  such that every element of  $B$  can be written as a polynomial in  $b_1, \dots, b_n$  with coefficients in  $A$ , i.e.,  $B = A[b_1, \dots, b_n]$ .

The homomorphism  $h$  is singly generated, and  $B$  is a singly generated  $A$ -algebra, if there exists an element  $b \in B$  such that  $B = A[b]$ .

Recall that a ring homomorphism is finite if and only if it is of finite type and integral.

### 3. Finite spaces

In this section we study the simplest case of the problem: when the space  $Y$  is finite. First, we investigate the finiteness properties of  $C(X)$  as an  $\mathbb{R}$ -algebra, i.e., of the homomorphism  $\mathbb{R} = C(\{p\}) \rightarrow C(X)$  given by a constant map  $X \rightarrow Y = \{p\}$ .

**Proposition 3.1.** *The following conditions are equivalent for a completely regular (Hausdorff) space  $X$ :*

- (1)  $X$  is a finite space.
- (2)  $C(X)$  is a finite  $\mathbb{R}$ -algebra.
- (3)  $C(X)$  is an integral  $\mathbb{R}$ -algebra.
- (4)  $C(X)$  is a singly generated  $\mathbb{R}$ -algebra.
- (5)  $C(X)$  is a finitely generated  $\mathbb{R}$ -algebra.

**Proof.** (1)  $\Rightarrow$  (2) If  $X = \{x_1, \dots, x_n\}$ , then  $C(X) = C(\{x_1\}) \oplus \dots \oplus C(\{x_n\}) = \mathbb{R} \oplus \dots \oplus \mathbb{R} = \mathbb{R}^n$ .

(2)  $\Rightarrow$  (3) Every finite ring homomorphism is integral.

(3)  $\Rightarrow$  (1) If every function  $f \in C(X)$  is a root of a polynomial with real coefficients, then  $f(X)$  is finite, and then so is  $X$ , because in an infinite completely regular space we can always define a real continuous function with infinite range.

(1)  $\Rightarrow$  (4) Let us adapt the argument in [6, Theorem 3]. We shall see that  $C(X) = \mathbb{R}[f]$ , for any function  $f$  that separates points in  $X = \{x_1, x_2, \dots, x_n\}$ . Set  $\lambda_i = f(x_i)$ , and take

$$f_i = \frac{\prod_{i \neq j} (f - \lambda_j)}{\prod_{i \neq j} (\lambda_i - \lambda_j)}, \quad \text{for } 1 \leq i \leq n.$$

Certainly  $f_i \in \mathbb{R}[f]$ . Moreover,  $f_i(x_i) = 1$  and  $f_i(x_j) = 0$  for  $i \neq j$ . Finally observe that, for any  $g \in C(X)$ ,  $g = \sum_{i=1}^n g(x_i) f_i \in \mathbb{R}[f]$ .

(4)  $\Rightarrow$  (5) This follows from the definitions.

(5)  $\Rightarrow$  (1) The number of minimal prime ideals of a finitely generated  $\mathbb{R}$ -algebra is finite (because it is a noetherian ring). In  $C(X)$  each prime ideal is contained in a unique maximal ideal. Hence, if  $C(X)$  is a finitely generated  $\mathbb{R}$ -algebra, then it has only finitely many maximal ideals, and so the space  $X$  is finite.  $\square$

**Corollary 3.2.** *Let  $\pi : X \rightarrow Y$  be a continuous map between compact Hausdorff spaces. If the induced homomorphism  $C(Y) \rightarrow C(X)$  is finite (integral, singly generated or finitely generated), then each fibre  $\pi^{-1}(y)$  is a finite set.*

**Proof.** By Tietze's extension theorem,  $C(\pi^{-1}(y))$  is a quotient ring of  $C(X)$ . This implies that if  $C(Y) \rightarrow C(X)$  is finite (integral, singly generated, finitely generated), then so is  $\mathbb{R} = C(y) \rightarrow C(\pi^{-1}(y))$ .  $\square$

**Corollary 3.3.** *Let  $Y$  be a finite space, and  $\pi : X \rightarrow Y$  be a continuous map. The following conditions are equivalent:*

- (1)  $X$  is a finite space.
- (2)  $C(Y) \rightarrow C(X)$  is finite.
- (3)  $C(Y) \rightarrow C(X)$  is integral.
- (4)  $C(Y) \rightarrow C(X)$  is singly generated.
- (5)  $C(Y) \rightarrow C(X)$  is finitely generated.

**Proof.** (1)  $\Rightarrow$  (2), (1)  $\Rightarrow$  (3), (1)  $\Rightarrow$  (4), and (1)  $\Rightarrow$  (5) If  $C(X)$  is a finite (integral, singly generated or finitely generated)  $\mathbb{R}$ -algebra, then it has the same property as  $C(Y)$ -algebra.

The converse follows directly from 3.2.  $\square$

#### 4. Extensions associated to compactifications

We start with a simplification of the problem.

**Proposition 4.1.** *Let  $\pi : X \rightarrow Y$  be a continuous map between compact Hausdorff spaces. If  $F \subset Y$  is a closed subset such that  $\pi : X - \pi^{-1}(F) \rightarrow Y - F$  is injective, then  $C(X)$  is a finite (respectively, integral, singly generated, finitely generated)  $C(Y)$ -algebra if and only if  $C(\pi^{-1}(F))$  is a finite (respectively, integral, singly generated, finitely generated)  $C(F)$ -algebra.*

**Proof.** By Tietze's extension theorem,  $C(F) = C(Y)/I_F$ , where  $I_F = \{g \in C(Y) : g(F) = \{0\}\}$ , and  $C(\pi^{-1}(F)) = C(X)/I_{\pi^{-1}(F)}$ . Since  $\pi$  is injective in  $X - \pi^{-1}(F)$ , every function  $f \in I_{\pi^{-1}(F)}$  is constant on the fibres so that,  $I_{\pi^{-1}(F)} = I_F \subset C(Y)$ .

Finally, observe that the homomorphisms  $C(Y) \rightarrow C(X)$  and  $C(F) = C(Y)/I_F \rightarrow C(\pi^{-1}(F)) = C(X)/I_F$  have the same finiteness properties.  $\square$

**Corollary 4.2.** *Let  $\pi : X \rightarrow Y$  be a continuous map between compact Hausdorff spaces. If the set  $F = \{y \in Y : |\pi^{-1}(y)| > 1\}$  is finite, then the following conditions are equivalent:*

- (1)  $\pi^{-1}(F)$  is finite.
- (2)  $C(Y) \rightarrow C(X)$  is finite.
- (3)  $C(Y) \rightarrow C(X)$  is integral.
- (4)  $C(Y) \rightarrow C(X)$  is singly generated.
- (5)  $C(Y) \rightarrow C(X)$  is finitely generated.

**Proof.** The proof follows from 3.3 since the homomorphisms  $C(Y) \rightarrow C(X)$  and  $C(F) = C(Y)/I_F \rightarrow C(\pi^{-1}(F)) = C(X)/I_F$  have the same finiteness properties.  $\square$

The following two results show that, in order to study finiteness properties between the associated rings of continuous functions, the problem for compactifications of a given locally compact Hausdorff space  $T$  is equivalent to the problem for arbitrary compact Hausdorff spaces.

It is well known (see [2, 1.30]) that, if  $\pi_{\gamma\alpha} : \gamma T \rightarrow \alpha T$  is the projection map between two Hausdorff compactifications  $\alpha T \leq \gamma T$  of  $T$ , then  $\pi_{\gamma\alpha}$  carries  $\gamma T - T$  onto  $\alpha T - T$ , and so it induces an injective ring homomorphism  $C(\alpha T - T) \rightarrow C(\gamma T - T)$ .

**Corollary 4.3.** *Let  $T$  be a locally compact, non-compact Hausdorff space, and let  $\alpha T \leq \gamma T$  be two Hausdorff compactifications of  $T$ . Then,  $C(\gamma T)$  is a finite (respectively, integral, singly generated, finitely generated)  $C(\alpha T)$ -algebra if and only if  $C(\gamma T - T)$  is a finite (respectively, integral, singly generated, finitely generated)  $C(\alpha T - T)$ -algebra.*

**Proof.** The proof follows from 4.1.  $\square$

Next we shall adapt a theorem of Magill to our needs (see [9, 2.1] and [2, 7.2]).

**Theorem 4.4 (Magill).** *Let  $\pi : X \rightarrow Y$  be a surjective continuous map between compact Hausdorff spaces. There exist a locally compact Hausdorff space  $T$  and Hausdorff compactifications  $\alpha T \leq \gamma T$  such that  $X = \gamma T - T$ ,  $Y = \alpha T - T$  and  $\pi$  is the projection map between  $\gamma T - T$  and  $\alpha T - T$ .*

**Proof.** We shall see that, in fact, one may choose  $\gamma T = \beta T$ . It is well-known (see [2, 4.17]) that there exist a locally compact Hausdorff space  $T$  and a homeomorphism  $h_1 : \beta T - T \rightarrow X$ . The composition  $\pi h_1$  is a surjective continuous map between  $\beta T - T$  and  $Y$ . According to [2, 7.2], there exist a compactification  $\alpha T$  of  $T$  and a homeomorphism  $h_2 : \alpha T - T \rightarrow Y$ . Moreover, this homeomorphism  $h_2$ , as defined in [2, 7.2], makes the following diagram commutative:

$$\begin{array}{ccc}
 \beta T - T & \xrightarrow{h_1} & X \\
 \pi_{\beta\alpha} \downarrow & & \downarrow \pi \\
 \alpha T - T & \xrightarrow{h_2} & Y \quad \square
 \end{array}$$

Observe that, in order to study the finiteness properties of the homomorphism  $C(Y) \rightarrow C(X)$  given by a continuous map  $\pi : X \rightarrow Y$  between compact Hausdorff spaces, we

can assume that  $\pi$  is a surjective map: the restriction homomorphism  $C(Y) \rightarrow C(\pi(X))$  is surjective, because  $\pi(X)$  is a closed subspace of  $Y$ . Therefore, the homomorphism  $C(Y) \rightarrow C(X)$  is finite (integral, etc.), if and only if  $C(\pi(X)) \rightarrow C(X)$  is finite (integral, etc.).

Faulkner's theorems on extensions of continuous function rings associated to compactifications [6, Theorems 3, 4 and 5] can be easily deduced from our results.

**Corollary 4.5.** *Let  $\alpha T$  have a finite remainder  $\alpha T - T$  and let  $\alpha T \leq \gamma T$ . The following conditions are equivalent:*

- (1)  $\gamma T$  has a finite remainder.
- (2)  $C(\alpha T) \rightarrow C(\gamma T)$  is finite.
- (3)  $C(\alpha T) \rightarrow C(\gamma T)$  is integral.
- (4)  $C(\alpha T) \rightarrow C(\gamma T)$  is singly generated.
- (5)  $C(\alpha T) \rightarrow C(\gamma T)$  is finitely generated.

The equivalence between conditions (1) and (4) was proved by Faulkner [6], as a consequence of the two following results, which we obtain from 3.1.

**Corollary 4.6** (Faulkner). *Let  $\pi_{\gamma\alpha}: \gamma T \rightarrow \alpha T$  be the projection map between two Hausdorff compactifications  $\alpha T \leq \gamma T$  of  $T$ . If the set  $\bigcup\{\pi_{\gamma\alpha}^{-1}(p): |\pi_{\gamma\alpha}^{-1}(p)| > 1\}$  is finite, then there exists an  $f \in C(\gamma T)$  such that  $C(\gamma T) = C(\alpha T)[f]$ .*

**Proof.** In this case,  $F = \{p \in \alpha T: |\pi^{-1}(p)| > 1\}$  is obviously finite.  $\square$

The following result, which is a particular case of 3.2, is a partial converse of this Corollary.

**Corollary 4.7** (Faulkner). *Let  $\pi_{\gamma\alpha}: \gamma T \rightarrow \alpha T$  be the projection map between two Hausdorff compactifications  $\alpha T \leq \gamma T$  of  $T$ . If  $C(\gamma T) = C(\alpha T)[f]$ , then the fibres of  $\pi_{\gamma\alpha}$  are finite.*

In [6], Faulkner asks whether the complete converse of 4.6 also holds. In fact, it does not: Dwornik-Orzechowska and Wajch show in [4] that, for every cardinal number  $\kappa$  there exist a locally compact Hausdorff space  $T$  and compactifications  $\alpha T \leq \gamma T$  such that  $C(\gamma T) = C(\alpha T)[f]$  for some  $f \in C^*(T)$ , but the collection of all those fibres of the map  $\pi_{\gamma\alpha}$  which have more than one point is of cardinality  $\kappa$ . For the sake of completeness, we include the following example.

**Example 4.8.** The converse of 4.6 is not true: Let  $T = \mathbb{R} \times I$ , where  $I = [0, 1]$ . Let  $\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$  and  $\gamma\mathbb{R} = \mathbb{R} \cup \{+\infty, -\infty\}$  be, respectively, the one-point and the two-point compactifications of  $\mathbb{R}$ . Then,  $\alpha T = I \times \omega\mathbb{R}$  and  $\gamma T = I \times \gamma\mathbb{R}$  are Hausdorff compactifications of  $T$  and  $\alpha T \leq \gamma T$ . The remainders of these compactifications are  $\alpha T - T = I$  and  $\gamma T - T = I \times \{+\infty, -\infty\}$ , so that  $C(\gamma T - T) = C(\alpha T - T) \oplus C(\alpha T - T)$ . If  $f(\{+\infty\} \times I) = \{0\}$  and  $f(\{-\infty\} \times I) = \{1\}$  then,  $C(\gamma T - T) = C(\alpha T - T)[f]$  and

$C(\gamma T) = C(\alpha T)[f]$ . But the set  $\bigcup\{\pi_{\gamma\alpha}^{-1}(p) : |\pi_{\gamma\alpha}^{-1}(p)| > 1\} = I \times \{+\infty, -\infty\}$  is not finite.

### 5. Finite homomorphisms

**Proposition 5.1** [11, Proposition 5.6]. *Let  $\pi : X \rightarrow Y$  be a continuous map between realcompact spaces. If the homomorphism  $C(Y) \rightarrow C(X)$  is finite, then the continuous extension of  $\pi$  to the Stone–Čech compactifications,  $\beta\pi : \beta X \rightarrow \beta Y$ , is a locally injective map.*

**Theorem 5.2.** *Let  $\pi : X \rightarrow Y$  be a continuous map between compact Hausdorff spaces. The homomorphism  $C(Y) \rightarrow C(X)$  is finite if and only if the map  $\pi : X \rightarrow Y$  is locally injective.*

**Proof.** Suppose that  $\pi : X \rightarrow Y$  is locally injective. Every point  $x \in X$  has a cozero neighbourhood  $U$  such that  $\pi$  is injective on  $\bar{U}$ , the closure of  $U$ . Then  $C(\bar{U}) \simeq C(\pi(\bar{U}))$  and, by Tietze’s extension theorem, the homomorphism  $C(Y) \rightarrow C(\pi(\bar{U})) \simeq C(\bar{U})$  is surjective.

The space  $X$  can be covered by a finite number of these cozero sets,  $X = \text{coz}(g_1) \cup \dots \cup \text{coz}(g_n)$ . Since  $\text{coz}(g_i) = \text{coz}(g_i^2)$ , we can take  $g_i \geq 0, \forall i$ . The functions  $h_i = g_i / (g_1 + \dots + g_n), i = 1, \dots, n$ , generate  $C(X)$  as a  $C(Y)$ -module: for every  $f \in C(X)$ , there exist functions  $f_1, \dots, f_n \in C(Y)$  such that  $f = f_i$  in  $\text{coz}(g_i)$ , i.e.,  $g_i \cdot f = f_i \cdot g_i$ , so  $(\sum g_i) f = f_1 \cdot g_1 + \dots + f_n \cdot g_n$  and  $f = f_1 \cdot h_1 + \dots + f_n \cdot h_n$ .

The converse follows from 5.1.  $\square$

**Proposition 5.3.** *Let  $\pi : X \rightarrow Y$  be a continuous map between realcompact spaces. If the homomorphism  $C(Y) \rightarrow C(X)$  is finite, then  $\pi$  is a closed map and the space  $X$  can be covered by a finite number of cozero sets,  $X = \text{coz}(g_1) \cup \dots \cup \text{coz}(g_n)$ , such that  $\pi$  is injective on each closure  $\overline{\text{coz}(g_i)}$ . Consequently,  $|\pi^{-1}(y)| \leq n$  for every  $y \in Y$ .*

**Proof.** Assume that the homomorphism  $\phi : C(Y) \rightarrow C(X)$  ( $\phi(f) = f \circ \pi$ ) is finite. According to 5.1, the map  $\beta\pi : \beta X \rightarrow \beta Y$  is locally injective, so that  $\beta X$  may be covered by a finite number of cozero sets such that  $\beta\pi$  is injective on each closure, and obviously the same happens for  $X$  and  $\pi$ .

Next we shall prove that  $\pi$  is a closed map. First of all, take into account that  $\beta\pi : \beta X \rightarrow \beta Y$  is a closed map. If we show that  $\beta\pi$  transforms  $\beta X - X$  into  $\beta Y - Y$ , then  $X = \beta\pi^{-1}(Y)$ , and so  $\pi = \beta\pi|_X : X \rightarrow Y$  is also a closed map (see [5, 2.1.4]).

In order to prove that  $\beta\pi$  carries  $\beta X - X$  into  $\beta Y - Y$ , we are going to describe the space  $\beta X$  and the map  $\beta\pi$  in terms of prime ideals in  $C(X)$ .

Let  $\text{Spec } C(X)$  be the prime spectrum of the ring  $C(X)$ , that is, the set of prime ideals in  $C(X)$  endowed with the Zariski (or hull-kernel) topology. Recall that the subspace  $\text{Max } C(X)$  of  $\text{Spec } C(X)$  consisting of all maximal ideals in  $C(X)$  is just  $\beta X$ .

Each prime ideal in  $C(X)$  is contained in an unique maximal ideal (see [7, 2.11]) and the map  $r_X : \text{Spec } C(X) \rightarrow \text{Max } C(X)$  that sends each prime ideal in  $C(X)$  to the unique maximal ideal in  $C(X)$  containing it, is a continuous retraction.

The map between the prime spectra  $\text{Spec } C(X) \rightarrow \text{Spec } C(Y)$  that sends each prime ideal  $P$  in  $C(X)$  to the prime ideal  $\phi^{-1}(P) = \{f \in C(Y) : f \circ \pi \in P\}$  is also a continuous map. The restriction of this map to  $\beta X$ , composed with the continuous retraction  $r_Y : \text{Spec } C(Y) \rightarrow \text{Max } C(Y) = \beta Y$ , is just  $\beta\pi : \beta X \rightarrow \beta Y$ .

Given a point  $p \in \beta X$ , let  $M_p$  be the corresponding maximal ideal in  $C(X)$ . According to the above description of  $\beta\pi$ , the maximal ideal in  $C(Y)$  corresponding to the point  $q = \beta\pi(p)$  is just  $r_Y(\phi^{-1}(M_p))$ , that is,  $M_q = r_Y(\phi^{-1}(M_p))$ . As the homomorphism  $\phi : C(Y) \rightarrow C(X)$  is finite,  $\phi^{-1}(M_p)$  is a maximal ideal in  $C(Y)$  (see [1, 5.8]), and so  $M_q = r_Y(\phi^{-1}(M_p)) = \phi^{-1}(M_p)$ . The homomorphism induced by  $\phi$  between the quotient fields  $C(Y)/M_q \rightarrow C(X)/M_p$  is injective and also finite. Therefore,  $C(X)/M_p$  is an algebraic extension of  $C(Y)/M_q$ . Moreover, the field  $C(X)/M_p$  is totally ordered [7, Theorem 5.5] and  $C(Y)/M_q$  is real-closed, that is to say, it has no proper algebraic extensions to an ordered field [7, Theorem 13.4]. Hence, the homomorphism  $C(Y)/M_q \rightarrow C(X)/M_p$  is an isomorphism. This implies that  $C(Y)/M_q = \mathbb{R}$  or, equivalently,  $q \in Y$  if and only if  $C(X)/M_p = \mathbb{R}$ , i.e.,  $p \in X$ .  $\square$

The converse of this result is true for normal spaces  $Y$ .

**Theorem 5.4.** *Let  $\pi : X \rightarrow Y$  be a continuous map between realcompact spaces and suppose that  $Y$  is a normal space. The homomorphism  $C(Y) \rightarrow C(X)$  is finite if and only if  $\pi$  is a closed map and the space  $X$  can be covered by a finite number of cozero sets,  $X = \text{coz}(g_1) \cup \dots \cup \text{coz}(g_n)$ , such that  $\pi$  is injective on each closure  $\overline{\text{coz}(g_i)}$ .*

**Proof.** The proof is entirely analogous to 5.2.  $\square$

The following example shows that the converse of 5.1 is not true, and that Theorem 5.2 does not hold for non compact spaces.

**Example 5.5.** A continuous injective map  $\pi : X \rightarrow Y$  such that  $\beta\pi : \beta X \rightarrow \beta Y$  is an homeomorphism and the homomorphism  $C(Y) \rightarrow C(X)$  is not finite.

Let  $\Sigma = \mathbb{N} \cup \{p\}$ , where  $p \in \beta\mathbb{N} - \mathbb{N}$ .  $\Sigma$  is a realcompact and normal space, and  $\mathbb{N}$  is dense and  $C^*$ -embedded in  $\Sigma$ . Therefore,  $\beta\mathbb{N} = \beta\Sigma$ . The homomorphism  $C(\Sigma) \rightarrow C(\mathbb{N})$  induced by the inclusion map  $\mathbb{N} \rightarrow \Sigma$  is not finite, because  $\mathbb{N}$  is not closed in  $\Sigma$ .

A class of locally injective continuous maps of especial interest consists of unbranched coverings, classically studied in connection with the fundamental group. Unbranched finite coverings (locally injective, open and closed continuous maps with finite fibres) and branched finite coverings (open and closed continuous maps with finite fibres) are characterized in terms of rings of continuous functions in [11]: A continuous map between topological manifolds  $\pi : X \rightarrow Y$  is an unbranched finite covering (respectively, a branched finite covering) if and only if the induced homomorphism  $C(Y) \rightarrow C(X)$  is finite and flat (respectively, integral and flat).



We have as yet obtained no analogues of Theorem 5.2 or 5.4 for integral, singly generated or finitely generated homomorphisms. However, we do have some partial results. In a paper in preparation, we prove that if the homomorphism  $C(Y) \rightarrow C(X)$  induced by a continuous map  $\pi : X \rightarrow Y$ , between compact Hausdorff spaces, is singly generated, then it is finite, and consequently the map  $\pi$  is locally injective. The converse of this result is not true.

**Example 5.6.** A finite homomorphism  $C(Y) \rightarrow C(X)$  that is not singly generated.

Let  $\pi : S^2 \rightarrow \mathbb{P}^2$  be the natural projection map from the unit sphere onto the real projective plane,  $\pi(x_1, x_2, x_3) = \pi(-x_1, -x_2, -x_3)$ .

Since this map is locally injective, the corresponding homomorphism  $C(\mathbb{P}^2) \rightarrow C(S^2)$  is finite. Let us prove that it is not singly generated.

If there exists  $f \in C(S^2)$  such that  $C(S^2) = C(\mathbb{P}^2)[f]$ , then  $f$  must be injective on each fibre of  $\pi$ , because  $C(S^2)$  separates points in  $S^2$ . But, by Borsuk–Ulam’s theorem, for every  $f \in C(S^2)$  there exists  $p \in S^2$  such that  $f(p) = f(-p)$ . Therefore,  $C(S^2)$  is not singly generated over  $C(\mathbb{P}^2)$ .

We do not know if every finitely generated homomorphism  $C(Y) \rightarrow C(X)$  is finite. With respect to the integral case, we know that not every integral homomorphism  $C(Y) \rightarrow C(X)$  is finite. To give an example of this, we use that the homomorphism  $C(Y) \rightarrow C(X)$ , induced by a finite branched covering between topological manifolds  $\pi : X \rightarrow Y$ , is integral (proved in [11]).

**Example 5.7.** An integral homomorphism  $C(Y) \rightarrow C(X)$  that is not finite.

Let  $X = \mathbb{C}$  (the complex plane),  $Y = \mathbb{C}$  and  $\pi : X \rightarrow Y$ , defined by  $\pi(z) = z^n$ . This map is a finite branched covering, so that the homomorphism  $C(Y) \rightarrow C(X)$  is integral. This homomorphism is not finite, since  $\pi$  is not locally injective at  $z = 0$ .

For an example with compact spaces, one might consider  $X = Y = \{z = x + iy \in \mathbb{C} : |z|^2 = x^2 + y^2 = 1\}$ .

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