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THERE DO NOT EXIST MINIMAL ALGEBRAS BETWEEN $C^*(X)$ AND $C(X)$ WITH PRESCRIBED REAL MAXIMAL IDEAL SPACES

J. M. DOMÍNGUEZ[1] (Valladolid) and J. GÓMEZ-PÉREZ[2] (León)

Abstract. Let $C(X)$ be the algebra of all real-valued continuous functions on a completely regular Hausdorff space X, and $C^*(X)$ the subalgebra of bounded functions. We prove that for any intermediate algebra A between $C^*(X)$ and $C(X)$, other than $C^*(X)$, there exists a smaller intermediate algebra with the same real maximal ideals as in A. The space X is called A-compact if any real maximal ideal in A corresponds to a point in X. It follows that, for a noncompact space X, there does not exist any minimal intermediate algebra A for which X is A-compact. This completes the answer to a question raised by Redlin and Watson in 1987.

Introduction

Let $C(X)$ be the algebra of all real-valued continuous functions on a nonempty completely regular Hausdorff space X, and $C^*(X)$ the subalgebra of bounded functions. We are concerned with subalgebras of $C(X)$ containing $C^*(X)$. We shall refer to them as *intermediate algebras on* X.

A maximal ideal M of an intermediate algebra A is said to be real if the quotient field A/M is isomorphic to $\mathbf{R}$. The space $\upsilon_A X$ of all real maximal ideals in A is identified with the space of those points in βX to which all the functions in A can be continuously extended. The space X is called A-compact if $X = \upsilon_A X$. In view of this definition, C-compact spaces are the realcompact spaces while C^*-compact spaces are the compact ones.

In 1987 Redlin and Watson [5] raised the following question: Does there exist in some sense a minimal intermediate algebra A_m for which X is A_m-compact? In 1997 Acharyya, Chattopadhyay and Ghosh [1] gave an affirmative answer to Redlin and Watson's question by defining a suitable ordering among the intermediate algebras on X. At the same time, they made the following conjecture: There does not exist any minimal intermediate algebra

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A on $\mathbf{N}$, in the usual inclusion sense, for which $\mathbf{N}$ becomes A-compact. We show that the conjecture holds for a general noncompact X. Even more: For any intermediate algebra A on X other than $C^*(X)$, there exists a smaller intermediate algebra $B \subset A$ such that $\upsilon_B X = \upsilon_A X$.

1. Preliminaries

We shall basically adhere to the notation and terminology in [4]. Throughout the paper X will be a nonempty completely regular Hausdorff topological space, and βX will denote the Stone–Čech compactification of X. As usual $Z(f)$ will denote the *zero-set* of $f \in C(X)$, i.e., $Z(f) = \{x \in X : f(x) = 0\}$.

Let A be an intermediate algebra on X. We shall denote by $\operatorname{Max} A$ the set of all maximal ideals of A endowed with the Zariski (or Stone) topology. Both $\operatorname{Max} C(X)$ and $\operatorname{Max} C^*(X)$ are models for βX. Each point p in βX is identified with the maximal ideal $M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$ in $\operatorname{Max} C(X)$, and with $M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}$ in $\operatorname{Max} C^*(X)$, where f^β denotes the continuous extension of f to βX. It is well-known that every prime ideal of A is contained in a unique maximal ideal. For any $p \in \beta X$, let M_A^p denote the unique maximal ideal of A containing the prime ideal $M^p \cap A$. The space $\operatorname{Max} A$ is also a model for βX. Each point p in βX is identified with the maximal ideal M_A^p in $\operatorname{Max} A$.

1.1. The space of real maximal ideals. If the function $f \in C(X)$ is regarded as a continuous mapping of X into the one-point compactification $\mathbf{R}^* = \mathbf{R} \cup \{\infty\}$ of $\mathbf{R}$, it has an extension $f^* : \beta X \to \mathbf{R}^*$. The set of points in βX where f^* takes real values is denoted by $\upsilon_f X$, i.e.,

$$\upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty\}.$$

Notice that $\upsilon_f X = \upsilon_{e^f} X$ if $f \geqq 0$, and that $\upsilon_f X = \upsilon_{\ln f} X$ if $f \geqq c > 0$ for some $c \in \mathbf{R}$.

Following [2], we shall denote by $\upsilon_A X$ the space of all real maximal ideals of an intermediate algebra A on X. It is a known result that M_A^p is a real maximal ideal of A if and only if $f^*(p) \neq \infty$ for any $f \in A$. Thus,

$$\upsilon_A X = \bigcap \{\upsilon_f X : f \in A\} \subseteqq \beta X.$$

A subalgebra A of $C(X)$ is said to be *absolutely convex* if, whenever $|f| \leqq |g|$, with $f \in C(X)$ and $g \in A$, then $f \in A$.

1.2. Proposition. *If A is an intermediate algebra on X, then A is an absolutely convex subalgebra of $C(X)$ and so a sublattice of $C(X)$.*

PROOF. Let us repeat the short argument in [3, 3.3]. If $|f| \leqq |g|$, with $f \in C(X)$ and $g \in A$, then $f(1+g^2)^{-1}$ is in $C^*(X) \subseteqq A$, and so $f = (1+g^2)f(1+g^2)^{-1}$ is in A. □

2. The theorem

Let A be an intermediate algebra on X containing unbounded functions. We shall prove that there exists a smaller intermediate algebra $B \subset A$ such that $\upsilon_B X = \upsilon_A X$.

Although formally unnecessary, we think that it will be quite illustrative to examine first the case when A is the smallest intermediate algebra containing a given unbounded function.

Let $f \in C(X)$. We shall denote by $A[f]$ the smallest intermediate algebra containing both A and f, that is,

$$A[f] = \left\{ \sum_{i=0}^{n} g_i f^i : g_i \in A,\ n = 0,1,2,\dots \right\}.$$

If $f \notin A$ we shall say that $A[f]$ is *singly generated* over A.

Notice that

$$\upsilon_{A[f]} X = \upsilon_A X \cap \upsilon_f X.$$

If $Z(f) = \emptyset$ and $1/f \in C^*(X)$, then

$$A[f] = \{ g f^n : g \in A,\ n = 0,1,2,\dots \},$$

since $\sum_{i=0}^{n} g_i f^i = (g_0/f^n + g_1/f^{n-1} + \dots + g_n) f^n$.

If we, further, assume that $f \geqq c > 1$ for some $c \in \mathbf{R}$, then

$$C^*(X)[f] = \left\{ h \in C(X) : |h| \leqq f^n \text{ for some } n \in \mathbf{N} \right\}.$$

Let c be any real number. By Proposition 1.2, an intermediate algebra contains a given function $g \in C(X)$ if and only if it contains $|g| + c$. So that, $A[g] = A\big[|g| + c\big]$. This shows that every intermediate algebra that is singly generated over A is $A[f]$ for some $f \geqq c$.

Now, taking into account that $\lim_{r\to\infty} e^r/r^n = \infty$, one may easily conclude that, for any nonnegative unbounded function $f \in C(X)$, the composition $\exp \circ f = e^f$ does not belong to $C^*(X)[f]$ (see [3, 2.3 and 3.4]).

2.1. EXAMPLE. (See [1, 4.1].) Let A be an intermediate algebra that is singly generated over $C^*(X)$. We may assume that $A = C^*(X)[f]$ with $f \geqq 1$. Then $0 \leqq \ln f \leqq f$, and so $\ln f \in A$. According to the above, the function f is not in the intermediate algebra $B = C^*(X)[\ln f]$. Therefore, B is properly contained in A. Nevertheless, $\upsilon_A X = \upsilon_f X = \upsilon_{\ln f} X = \upsilon_B X$. $\square$

For any intermediate space Y between X and βX, the restriction morphism from $C(Y)$ to $C(X)$, which sends $g \in C(Y)$ to $g|_X$, is clearly injective. We shall always consider $C(Y)$ as an intermediate algebra on X.

2.2. LEMMA. *Let A be an intermediate algebra on X, and $p \in \beta X - \upsilon_A X$. If $D = C\big(X \cup \{p\}\big) \cap A$, then $\upsilon_D X = \upsilon_A X \cup \{p\}$.*

PROOF. Clearly $\upsilon_A X \cup \{p\} \subseteqq \upsilon_D X$. Let us prove the reverse inclusion. Let $q \in \beta X - \big(\upsilon_A X \cup \{p\}\big)$. We shall see that $q \notin \upsilon_D X$. On the one hand, as $q \notin \upsilon_A X$, there exists $f \in A$ such that $f^*(q) = \infty$. On the other hand, as $q \neq p$, and M^{*p} is the unique maximal ideal in $C^*(X)$ containing $M^p \cap C^*(X)$, then M^{*q} does not contain $M^p \cap C^*(X)$, and so there exists $g \in M^p \cap C^*(X)$ such that $g^\beta(q) \neq 0$. The function fg is in $M^p \cap A$ and, certainly, $M^p \subseteqq C\big(X \cup \{p\}\big)$. Therefore, $fg \in C\big(X \cup \{p\}\big) \cap A = D$. Finally, $(fg)^*(q) = \infty$. Thus, $q \notin \upsilon_D X$. $\square$

2.3. LEMMA. *Let A be an intermediate algebra on X, and let $f \in C(X)$ be a nonnegative function such that $f \notin C(\upsilon_A X)$. Then $e^f \notin A[f]$.*

PROOF. Suppose, on the contrary, that $e^f \in A[f]$. We may assume, without lost of generality, that $f \geqq 1$. Therefore, there exists $g \in A$ such that $e^f = gf^n$, for some $n \in \mathbf{N}$. Since $f \notin C(\upsilon_A X)$, there must be a point p in $\upsilon_A X$ such that $f^*(p) = \infty$. Although $p \in \upsilon_A X \subseteqq \upsilon_g X$, from the equality $g = e^f / f^n$, one deduces that $g^*(p) = \infty$. This is a contradiction. $\square$

2.4. THEOREM. *Let A be an intermediate algebra on X containing unbounded functions. There exists another intermediate algebra B, properly contained in A, such that $\upsilon_B X = \upsilon_A X$.*

PROOF. Let f be an unbounded function in A such that $f \geqq 1$. Take $p \in \beta X - \upsilon_f X$, and set $D = C\big(X \cup \{p\}\big) \cap A$. Finally, set $B = D[\ln f]$. Clearly, $B \subseteqq A$. To see that it is a proper inclusion we shall prove that $f \notin B$. Since $p \notin \upsilon_A X \subseteqq \upsilon_f X$, we may apply Lemma 2.2 to conclude that $\upsilon_D X = \upsilon_A X \cup \{p\}$. Now, $\ln f \notin C(\upsilon_D X)$, since $p \notin \upsilon_{\ln f} X = \upsilon_f X$. By Lemma 2.3, $f \notin D[\ln f] = B$. Finally,

$$\upsilon_B X = \upsilon_D X \cap \upsilon_{\ln f} X = \big(\upsilon_A X \cup \{p\}\big) \cap \upsilon_f X = \upsilon_A X. \qquad \square$$

Let A be an intermediate algebra on X. Following Redlin and Watson, we say that the space X is *A-compact* if $X = \upsilon_A X$.

2.5. COROLLARY. (See [1] and [5].) *For a noncompact space X, there does not exist any minimal intermediate algebra A on X for which X is A-compact.*

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DEPARTAMENTO DE ÁLGEBRA GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE VALLADOLID
47005 VALLADOLID
SPAIN
E-MAIL: JMD@AGT.UVA.ES

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE LEÓN
24071 LEÓN
SPAIN
E-MAIL: DEMJGP@ISIDORO.UNILEON.ES