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POSITIVE SOLUTIONS FOR SINGULAR BVPs ON THE POSITIVE HALF-LINE ARISING FROM EPIDEMIOLOGY AND COMBUSTION THEORY*

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Abstract In this work, we are concerned with the existence and multiplicity of positive solutions for singular boundary value problems on the half-line. Two problems from epidemiology and combustion theory set on the positive half-line are investigated. We use upper and lower solution techniques combined with fixed point index on cones in appropriate Banach spaces. The results complement recent ones in the literature.

Key words Fixed points index; positive solution; singular problem; cone; lower and upper solution; half-line

2000 MR Subject Classification 34B15; 34B18; 34B40; 47H10

1 Introduction

1.1 The mathematical problems

This article is devoted to the study of the existence of multiple positive solutions to the following boundary value problems set on the positive half-line:

$$\begin{cases} x''(t) - k^2 x(t) + \phi(t) f(t, x(t)) = 0, \quad t \in I, \\ x(0) = 0, \quad \lim\limits_{t \to +\infty} x(t) = 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} -y''(t) + c y'(t) + \lambda y(t) = \phi(t) g(t, y(t)), \quad t \in I, \\ y(0) = 0, \quad \lim\limits_{t \to +\infty} y(t) = 0, \end{cases} \tag{1.2}$$

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where k, c, and λ are real positive constants and $\phi \in C((0,+\infty),(0,+\infty))$. $\mathbb{R}^+$ denotes the set of nonnegative real numbers and $I=(0,+\infty)$. The functions $f,g \in C(\mathbb{R}^+ \times (0,+\infty),\mathbb{R}^+)$ satisfy $\lim\limits_{x\to 0^+} f(t,x)=+\infty$ and $\lim\limits_{y\to 0^+} g(t,y)=+\infty$, that is, $f(t,x)$, $g(t,y)$ may be singular at $x=0$, $y=0$, respectively. In fact, problem (1.2) can be transformed into problem (1.1) if we set $k=\sqrt{\lambda+\frac{c^2}{4}}$, $x(t)=y(t)e^{-\frac{c}{2}t}$, and $f(t,x(t))=e^{-\frac{c}{2}t}g(t,e^{\frac{c}{2}t}x(t))$. In other words, if y is a solution of problem (1.2), then, $x=ye^{-\frac{c}{2}t}$ is a solution of problem (1.1). But if x is a solution of (1.1), then, $y=xe^{\frac{c}{2}t}$ does not necessarily satisfy the boundary condition at positive infinity in (1.2). Problems of types (1.1) and (1.2) arise in many applications in physics, combustion theory, and epidemiology. When $\lambda=0$, we recognize Fisher's Equation in the equation of problem (1.2), where c is the speed of a traveling wave. In epidemiological models, $\lambda=0$ stands for a mortality rate (see [3–5, 9, 15] and the references therein). In [9], problem (1.2) is studied without singularity and the nonlinearity is assumed to satisfy at most linear growth with respect to y. The Schauder fixed point theorem is used to prove the existence of positive solutions. However, various assumptions on g including sub-linear and super-linear conditions were considered in [3]. The existence results of nontrivial solutions obtained in [3] rely on the Krasnozels'kĭi fixed point theorem of cone compression and expansion. A recent fixed point theorem of cone expansion and compression of functional type was used in [4] to prove the existence of positive solutions with some illustrative examples. Existence of two or three solutions when ϕ may be singular at $t=0$ was examined in [5]. In [8, 12], B. Yan et al obtained some existence results of unbounded positive solutions to problem (1.1) with $c=\lambda=0$:

$$\begin{cases} y''(t)+\phi(t)f(t,y(t))=0, \\ y(a)=0,\ \lim\limits_{t\to+\infty} y'(t)=0. \end{cases} \tag{1.3}$$

A similar problem with a Sturm-Liouville operator and mixed boundary conditions was examined in [10]; the existence of a positive solution was obtained when f is positive. In these three recent works, the fixed point index in cones was employed together with the method of upper and lower solution. Compactness arguments were obtained via Corduneanu's criterion on unbounded intervals [6]. In [11], the authors investigated some questions of existence and uniqueness for problem (1.1). Motivated by [8, 11], we prove here the existence of solutions for problems (1.1) and (1.2). The upper and lower solutions techniques are employed. New existence results of multiple positive solutions are also obtained. The proofs rely heavily on detailed properties of the Green's function, on suitable choice of a cone in a Bielecki type Banach space together with the fixed point index on cones [2, 6, 13] and Zima's compactness criterion [14]. This is developed in Section 2 (Theorems 2.1, 2.2) and in Section 3 (Theorem 3.1) for problem (1.1) and in Sections 4 (Theorems 4.1, 4.2), and in Section 5 (Theorem 5.1) for problem (1.2). For each problem, the existence results are illustrated by means of examples of application. First, some auxiliary results are provided hereafter.

1.2 Preliminaries

Let $p: I \longrightarrow I$ be a continuous function. Denote by Y the Banach space consisting of all weighted functions x, continuous on I, satisfying

$$\sup_{t\in I}\{|x(t)|p(t)\}<\infty,$$

equipped with a Bielecki's type norm $\|x\|_p = \sup\limits_{t\in I}\{|x(t)|p(t)\}$.

Definition 1.1 A set of functions $x \in \Omega \subseteq Y$ is said to be almost equicontinuous if it is equicontinuous on each interval $[0, T]$.

Lemma 1.1 [14] If the functions $x \in \Omega$ are almost equi-continuous on I and uniformly bounded in the sense of the norm $\|x\|_q = \sup\limits_{t\in I}\{|x(t)|q(t)\}$, where the function q is positive, continuous on I, and satisfies

$$\lim_{t\to+\infty} \frac{p(t)}{q(t)} = 0,$$

then, Ω is relatively compact in Y.

The following two results will be needed in the sequel [6, 13].

Lemma 1.2 [6, 13] Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E and $A : \overline{\Omega}\cap\mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose $\lambda Ax \neq x, \forall\, x \in \partial\Omega\cap\mathcal{P}, \lambda \in (0, 1]$. Then, $i(A, \Omega\cap\mathcal{P}, \mathcal{P}) = 1$.

Lemma 1.3 [6, 13] Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E, and $A : \overline{\Omega}\cap\mathcal{P} \to \mathcal{P}$ a completely continuous map. Suppose $Ax \not\leq x, \forall\, x \in \partial\Omega\cap\mathcal{P}$. Then, $i(A, \Omega\cap\mathcal{P}, \mathcal{P}) = 0$.

2 A Positive Solution for Problem (1.1)

2.1 General setting and assumptions

Given a real parameter $\theta > k$, consider the weighted Banach space

$$E = \{x \in C(\mathbb{R}^+, \mathbb{R}) : \sup_{t\in\mathbb{R}^+} |x(t)|e^{-\theta t} < \infty\}$$

endowed with the weighted Bielecki's sup-norm

$$\|x\|_\theta = \sup_{t\in\mathbb{R}^+}\{|x(t)|e^{-\theta t}\}.$$

Lemma 2.1 Let z be a function, such that $\phi z \in C(\mathbb{R}^+) \cap L^1(0, +\infty)$. Then, x is a solution of

$$\begin{cases} x''(t) - k^2 x(t) + \phi(t)z(t) = 0, \quad t \in I, \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} x(t) = 0, \end{cases} \tag{2.1}$$

if and only if x is a solution of

$$x(t) = \int_0^{+\infty} G(t, s)\phi(s)z(s)\mathrm{d}s, \tag{2.2}$$

where $G(t, s)$ is given by

$$G(t, s) = \frac{1}{2k}\begin{cases} e^{-kt}(e^{ks} - e^{-ks}), \quad 0 \leq s \leq t < +\infty, \\ e^{-ks}(e^{kt} - e^{-kt}), \quad 0 \leq t \leq s < +\infty. \end{cases} \tag{2.3}$$

Proof (a) The homogeneous problem has only the trivial solution and $\{e^{kt}, e^{-kt}\}$ is a fundamental system of solutions. By the method of variation of constants, we deduce that the unique solution of (2.1) is given by (2.2).

(b) Conversely, if x satisfies (2.2), then clearly that $x(0)=0$ and $x''(t)-k^2x(t)+\phi(t)z(t)=0$. To show that $x(+\infty)=0$, we split the integral in (2.2). As $\phi z\in L^1(0,+\infty)$, then for any $\varepsilon>0$, there exists $a_1>0$ such that, for all $t\geq a_1$, $\int_t^{+\infty}\phi(s)z(s)\mathrm{d}s\leq\frac{\varepsilon}{3}$, in particular, $\int_{a_1}^{+\infty}\phi(s)z(s)\mathrm{d}s\leq\frac{\varepsilon}{3}$. As $\lim\limits_{t\to+\infty}e^{-kt}=0$, then, there exists $a_2>0$, such that

$$\forall\, t\geq a_2,\quad e^{-kt}\leq\frac{\varepsilon}{3(e^{ka_1}-e^{-ka_1})\int_0^{a_1}\phi(s)z(s)\mathrm{d}s}.$$

Hence, for any $t\geq\max\{a_1,a_2\}$, we have

$$\begin{aligned}2kx(t)&=e^{-kt}\int_0^t(e^{ks}-e^{-ks})\phi(s)z(s)\mathrm{d}s+(e^{kt}-e^{-kt})\int_t^{+\infty}e^{-ks}\phi(s)z(s)\mathrm{d}s\\&=e^{-kt}\int_0^{a_1}(e^{ks}-e^{-ks})\phi(s)z(s)\mathrm{d}s+e^{-kt}\int_{a_1}^t(e^{ks}-e^{-ks})\phi(s)z(s)\mathrm{d}s\\&\quad+(e^{kt}-e^{-kt})\int_t^{+\infty}e^{-ks}\phi(s)z(s)\mathrm{d}s\\&\leq e^{-kt}(e^{ka_1}-e^{-ka_1})\int_0^{a_1}\phi(s)z(s)\mathrm{d}s+\int_{a_1}^t\phi(s)z(s)\mathrm{d}s+\int_t^{+\infty}\phi(s)z(s)\mathrm{d}s\\&\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Some properties of the Green's function G are given hereafter. For $t\geq0$, let

$$\gamma(t):=(e^{2kt}-1)e^{-(\theta+3k)t}$$

and

$$\widetilde{\gamma}(t):=\gamma(t)e^{-\theta t}.$$

Lemma 2.2 The Green's function G satisfies:
(a) $G(t,s)\geq0,\quad\forall\, t,s\in\mathbb{R}^+$.
(b) $G(t,s)\leq G(s,s)\leq\frac{1}{2k},\quad\forall\, t,s\in\mathbb{R}^+$.
(c) $G(t,s)e^{-\mu t}\leq G(s,s)e^{-ks},\quad\forall\, t,s\in I,\ \forall\,\mu\geq k$.
(d) $G(t,s)e^{-\mu t}\geq\gamma(t)G(\tau,s)e^{-\mu\tau},\quad\forall\, t,s,\tau\in\mathbb{R}^+,\ \forall\,\mu\geq k$.

Proof It is easy to prove the properties (a), (b), and (c). So, we only prove (d):

$$\frac{G(t,s)e^{-\mu t}}{G(\tau,s)e^{-\mu\tau}}=\begin{cases}\dfrac{e^{-ks}(e^{kt}-e^{-kt})}{e^{-k\tau}(e^{ks}-e^{-ks})}e^{\mu\tau}e^{-\mu t}, & t\leq s\leq\tau,\\[2mm] \dfrac{e^{-kt}(e^{ks}-e^{-ks})}{e^{-ks}(e^{k\tau}-e^{-k\tau})}e^{\mu\tau}e^{-\mu t}, & \tau\leq s\leq t,\\[2mm] \dfrac{e^{-kt}}{e^{-k\tau}}e^{\mu\tau}e^{-\mu t}, & s\leq t\leq\tau,\\[2mm] \dfrac{e^{kt}-e^{-kt}}{e^{k\tau}-e^{-k\tau}}e^{\mu\tau}e^{-\mu t}, & \tau\leq t\leq s,\\[2mm] \dfrac{e^{-kt}}{e^{-k\tau}}e^{\mu\tau}e^{-\mu t}, & s\leq\tau\leq t,\\[2mm] \dfrac{e^{kt}-e^{-kt}}{e^{k\tau}-e^{-k\tau}}e^{\mu\tau}e^{-\mu t}, & t\leq\tau\leq s.\end{cases}$$

As a consequence,

$$\frac{G(t,s)e^{-\mu t}}{G(\tau,s)e^{-\mu\tau}}=\begin{cases}\dfrac{1}{(e^{2ks}-1)}e^{(\mu+k)\tau}(e^{2kt}-1)e^{-(\mu+k)t}, & t\le s\le\tau,\\ \dfrac{(e^{2ks}-1)}{(e^{2k\tau}-1)}e^{(\mu+k)\tau}e^{-(\mu+k)t}, & \tau\le s\le t,\\ e^{(\mu+k)\tau}e^{-(\mu+k)t}, & s\le t\le\tau,\\ \dfrac{e^{(\mu+k)\tau}}{(e^{2k\tau}-1)}(e^{2kt}-1)e^{-(\mu+k)t}, & \tau\le t\le s,\\ e^{(\mu+k)\tau}e^{-(\mu+k)t} & s\le\tau\le t,\\ \dfrac{e^{(\mu+k)\tau}}{(e^{2k\tau}-1)}(e^{2kt}-1)e^{-(\mu+k)t}, & t\le\tau\le s.\end{cases}$$

As $\mu\ge k$, we have

$$\frac{G(t,s)e^{-\mu t}}{G(\tau,s)e^{-\mu\tau}}\ge\begin{cases}(e^{2kt}-1)e^{-(\mu+k)t}, & t\le s\le\tau,\\ e^{-(\mu+k)t}, & \tau\le s\le t,\\ e^{-(\mu+k)t}, & s\le t\le\tau,\\ (e^{2kt}-1)e^{-(\mu+k)t}, & \tau\le t\le s,\\ e^{-(\mu+k)t}, & s\le\tau\le t,\\ (e^{2kt}-1)e^{-(\mu+k)t}, & t\le\tau\le s.\end{cases}$$

Consequently,

$$\frac{G(t,s)e^{-\mu t}}{G(\tau,s)e^{-\mu\tau}}\ge\begin{cases}(e^{2kt}-1)e^{-(\mu+3k)t}=\gamma(t), & t\le s\le\tau,\\ (e^{2kt}-1)e^{-(\mu+3k)t}=\gamma(t), & \tau\le s\le t,\\ (e^{2kt}-1)e^{-(\mu+3k)t}=\gamma(t), & s\le t\le\tau,\\ (e^{2kt}-1)e^{-(\mu+3k)t}=\gamma(t), & \tau\le t\le s,\\ (e^{2kt}-1)e^{-(\mu+3k)t}=\gamma(t), & s\le\tau\le t,\\ (e^{2kt}-1)e^{-(\mu+3k)t}=\gamma(t), & t\le\tau\le s.\end{cases}$$

2.2 An auxiliary problem

For some positive real number h, consider the auxiliary boundary value problem

$$\begin{cases}x''(t)-k^2x(t)+\phi(t)f(t,x(t))=0,\ t>0,\\ x(0)=h,\quad \lim\limits_{t\to+\infty}x(t)=0,\end{cases}\tag{2.4}$$

where $f\in\mathcal{C}(I\times I,\mathbb{R})$.

Definition 2.1 A function $\alpha\in C(\mathbb{R}^+,I)\cap C^2(I,\mathbb{R})$ is called a lower solution of (2.4) if α satisfies

$$\begin{cases}\alpha''(t)-k^2\alpha(t)+\phi(t)f(t,\alpha(t))\ge 0,\\ \alpha(0)\le h,\quad \lim\limits_{t\to+\infty}\alpha(t)\le 0.\end{cases}$$

A function $\beta\in C(\mathbb{R}^+,I)\cap C^2(I,\mathbb{R})$ is called an upper solution of (2.4) if β satisfies

$$\begin{cases}\beta''(t)-k^2\beta(t)+\phi(t)f(t,\beta(t))\le 0,\\ \beta(0)\ge h,\quad \lim\limits_{t\to+\infty}\beta(t)\ge 0.\end{cases}$$

If there exists an upper solution β and a lower solution α with $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{R}^+$, then, we can define the set

$$D_\alpha^\beta(t) = \{x \in \mathbb{R} : \ \alpha(t) \leq x \leq \beta(t)\}, \quad \text{for each} \quad t \geq 0.$$

Theorem 2.1 Assume that α, β are lower and upper solutions of problem (2.4) respectively with $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{R}^+$. Moreover, suppose that there exists some $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\sup_{x \in D_\alpha^\beta(t)} e^{-\theta t}|f(t,x)| \leq \delta(t), \quad \forall\, t \in \mathbb{R}^+,$$

and

$$\int_0^{+\infty} e^{\theta s}\phi(s)\delta(s)\mathrm{d}s < +\infty. \tag{2.5}$$

Then, problem (2.4) has at least one solution $x^* \in E$ with

$$\alpha(t) \leq x^*(t) \leq \beta(t), \ \ t \in \mathbb{R}^+.$$

Proof We follow the same lines as in [11]. Consider the truncation function

$$f^*(t,x) = \begin{cases} f(t,\alpha(t)), & x < \alpha(t) \\ f(t,x), & \alpha(t) \leq x \leq \beta(t) \\ f(t,\beta(t)), & x > \beta(t). \end{cases}$$

As the function $f^* \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ has no singularity, define the regular problem

$$\begin{cases} x''(t) - k^2 x(t) + \phi(t) f^*(t, x(t)) = 0, \ t > 0 \\ x(0) = h, \quad \lim\limits_{t \to +\infty} x(t) = 0. \end{cases} \tag{2.6}$$

To show that problem (2.6) has at least one solution x^*, let the operator A_1 be defined on E by

$$A_1 x(t) = h e^{-kt} + \int_0^{+\infty} G(t,s)\phi(s) f^*(s, x(s))\mathrm{d}s.$$

(2.5) implies that A_1 is well defined. By Lemma 2.1, x is a solution of (2.6) if and only if x is a fixed point of A_1.

Step 1 $A_1(E) \subset E$. For $x \in E$ and $t \in \mathbb{R}^+$, we have the estimates

$$\begin{aligned} |A_1 x(t)| e^{-\theta t} &= h e^{-(k+\theta)t} + \int_0^{+\infty} e^{-\theta t} G(t,s)\phi(s)|f^*(s,x(s))|\mathrm{d}s \\ &\leq h + \frac{1}{2k}\int_0^{+\infty} e^{-\theta s}\phi(s)|f^*(s,x(s))|\mathrm{d}s \\ &\leq h + \frac{1}{2k}\int_0^{+\infty} \phi(s) e^{-\theta s}|f^*(s,x(s))|\mathrm{d}s \\ &\leq h + \frac{1}{2k}\int_0^{+\infty} \phi(s)\delta(s)\mathrm{d}s < +\infty. \end{aligned}$$

Then, $A_1 x \in E$.

Step 2 A_1 is continuous. Let some sequence $\{x_n\}_{n\geq 1} \subseteq E$ be such that $\lim\limits_{n\to+\infty} x_n = x_0 \in E$. By the continuity of the functions f, α, and β, we deduce the continuity of f^* too. Then, for each fixed $s \in \mathbb{R}^+$,

$$|f^*(s, x_n(s)) - f^*(s, x_0(s))| \longrightarrow 0,\ n \to +\infty.$$

In fact, we even have the convergence of $f^*(\cdot, x_n(\cdot))$ to $f^*(\cdot, x_0(\cdot))$ on every compact interval of $\mathbb{R}^+$. Moreover, we have

$$\begin{aligned}\|A_1x_n - A_1x_0\|_\theta &= \sup_{t\in\mathbb{R}^+} |Ax_n(t) - Ax_0(t)|e^{-\theta t}\\ &\leq \sup_{t\in\mathbb{R}^+} \int_0^{+\infty} e^{-\theta t} G(t,s)\phi(s)|f^*(s,x_n(s)) - f^*(s,x_0(s))|\mathrm{d}s\\ &\leq \frac{1}{2k}\int_0^{+\infty} \phi(s)e^{-\theta s}|f^*(s,x_n(s)) - f^*(s,x_0(s))|\mathrm{d}s.\end{aligned}$$

As $e^{-\theta s}|f^*(s,x_n(s)) - f^*(s,x_0(s))| \leq 2\delta(s),\ \forall\, s \in \mathbb{R}^+$, the Lebesgue dominated convergence theorem implies that the right-hand term tends to zero as $n \to +\infty$. Our claim follows.

Step 3 $A_1(E)$ is compact. Indeed,

(a) $A_1(E)$ is uniformly bounded. To prove this, let $\mu \in (k, \theta)$. Then, for $x \in E$, we have

$$\begin{aligned}|A_1x(t)|e^{-\mu t} &= he^{-(k+\mu)t} + \int_0^{+\infty} e^{-\mu t}G(t,s)\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\leq h + \int_0^{+\infty} e^{-ks}G(s,s)\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\leq h + \frac{1}{2k}\int_0^{+\infty} e^{-ks}\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\leq h + \frac{1}{2k}\int_0^{+\infty} e^{(\theta-k)s}\phi(s)\delta(s)\mathrm{d}s < +\infty.\end{aligned}$$

Hence, $A_1(E)$ is uniformly bounded with respect to the norm $\|.\|_\mu$.

(b) $A_1(E)$ is equicontinuous. For a given $T > 0$, $x \in E$, and $t, t' \in [0, T]$, we have

$$\begin{aligned}&|A_1x(t) - A_1x(t')| \leq h|e^{-kt} - e^{-kt'}|\\ &\quad + \int_0^{+\infty} |G(t,s) - G(t',s)|\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\leq h|e^{-kt} - e^{-kt'}| + \int_0^{T} |G(t,s) - G(t',s)|\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\quad + \int_T^{+\infty} |G(t,s) - G(t',s)|\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &= h|e^{-kt} - e^{-kt'}| + \int_0^{T} |G(t,s) - G(t',s)|\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\quad + \frac{1}{2k}[(e^{kt} - e^{-kt}) - (e^{kt'} - e^{-kt'})]\int_T^{+\infty} e^{-ks}\phi(s)|f^*(s,x(s))|\mathrm{d}s\\ &\leq h|e^{-kt} - e^{-kt'}| + \int_0^{T} |G(t,s) - G(t',s)|\phi(s)e^{\theta s}\delta(s)\mathrm{d}s\\ &\quad + \frac{1}{2k}[(e^{kt} - e^{-kt}) - (e^{kt'} - e^{-kt'})]\int_T^{+\infty} e^{(\theta-k)s}\phi(s)\delta(s)\mathrm{d}s.\end{aligned}$$

We have proved that, for any $\varepsilon > 0$ and $T > 0$, there exists $\eta > 0$ such that $|A_1x(t) - A_1x(t')| < \varepsilon$ for all $t, t' \in [0, T]$ and $|t - t'| < \eta$. Then, Zima's compactness criterion (Lemma 1.1 with $q(t) = e^{-\mu t}$) guarantees that $A_1(E)$ is relatively compact. Finally, by the Schauder fixed point theorem, A_1 has at least one fixed point $x^* \in E$, solution of problem (2.6).

Step 4 It only remains to show that $\alpha(t) \le x^*(t) \le \beta(t), \forall\, t \in \mathbb{R}^+$, in which case x^* is also a solution of (2.4). On the contrary, suppose that some point $t^* \in \mathbb{R}^+$ exists and satisfies $x^*(t^*) > \beta(t^*)$ and let $z(t) = x^*(t) - \beta(t)$. As $z(0) \le 0$, $z(\infty) \le 0$, and $z(t^*) > 0$, then, z must have a positive maximum at $t_0 \in I$. Then, $z''(t_0) \le 0$, $z(t_0) > 0$, and

$$\begin{aligned}
0 \ge z''(t_0) &= x^{*''}(t_0) - \beta''(t_0) \\
&\ge [k^2 x^*(t_0) - \phi(t_0) f^*(t_0, x^*(t_0))] - [k^2 \beta(t_0) - \phi(t_0) f^*(t_0, \beta(t_0))] \\
&= k^2[x^*(t_0) - \beta(t_0)] + \phi(t_0)[f^*(t_0, \beta(t_0)) - f^*(t_0, x^*(t_0))] \\
&= k^2[x^*(t_0) - \beta(t_0)] + \phi(t_0)[f(t_0, \beta(t_0)) - f(t_0, \beta(t_0))] \\
&= k^2[x^*(t_0) - \beta(t_0)] > 0,
\end{aligned}$$

which is contradictory. Then, $x^*(t) \le \beta(t), \forall\, t \in \mathbb{R}^+$. In the same way, we prove $\alpha(t) \le x^*(t)$. Finally, (2.5) implies that $\phi f \in L^1(0, 1)$; Lemma 2.1 yields that $x^*(+\infty) = 0$.

2.3 Existence result

Let $F(t, z) = f(t, e^{\theta t} z)$ and assume

$(\mathcal{H}_1)$ There exist $m \in C(\mathbb{R}^+, I)$ and $p \in C(I, I)$, such that

$$F(t, z) \le m(t) p(z), \quad \forall\, t \in \mathbb{R}^+,\ \forall\, z \in I. \tag{2.7}$$

There exists a decreasing function $q \in C(I, I)$, such that $\frac{p(z)}{q(z)}$ is increasing and

$$\int_0^{+\infty} \phi(s) m(s) q(c\widetilde{\gamma}(s)) \mathrm{d}s < +\infty,\ \forall\, c > 0. \tag{2.8}$$

$(\mathcal{H}_2)$ For any $c > 0$, there exists $\psi_c \in C(\mathbb{R}^+, I)$, such that

$$F(t, z) \ge \psi_c(t), \quad \forall\, t \in \mathbb{R}^+,\ \forall z \in (0, c]$$

with

$$\int_0^{+\infty} \phi(s) \psi_c(s) \mathrm{d}s < +\infty. \tag{2.9}$$

Using Theorem 2.1, we obtain the following.

Theorem 2.2 Besides Assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$, assume that there exist $M > 0$ and $h \in C(\mathbb{R}^+, I)$ such that

$$f(t, x) \le h(t),\ \ \forall\, (t, x) \in \mathbb{R}^+ \times [M, +\infty) \tag{2.10}$$

with

$$\int_0^{+\infty} \phi(s) h(s) \mathrm{d}s < +\infty. \tag{2.11}$$

Then, problem (1.1) has at least one positive solution $x^* \in C^2(\mathbb{R}^+, \mathbb{R})$.

Proof **Step 1** Choose a decreasing sequence $\{\varepsilon_n\}_{n\geq 1}$ with $\lim\limits_{n\to+\infty}\varepsilon_n=0$ and $\varepsilon_1<M$. Then, consider the sequence of boundary value problems

$$\begin{cases} x''(t)-k^2x(t)+\phi(t)f(t,x(t))=0, \\ x(0)=\varepsilon_n, \qquad \lim\limits_{t\to+\infty}x(t)=0. \end{cases} \tag{2.12}$$

We prove that, for each $n\geq 1$, (2.12) has at least one solution x_n. Let β be a solution of the boundary value problem

$$\begin{cases} x''(t)+\phi(t)h(t)=0, \quad t>0 \\ x(0)=M, \quad \lim\limits_{t\to+\infty}x'(t)=0, \end{cases}$$

that is,

$$\beta(t)=M+\int_0^{+\infty}L(t,s)\phi(s)h(s)\mathrm{d}s,$$

with Green's function

$$L(t,s)=\begin{cases} s, \quad 0\leq s\leq t<+\infty, \\ t, \quad 0\leq t\leq s<+\infty, \end{cases}$$

and h is as defined in (2.10). By (2.11), β is well defined. Obviously, $\beta(t)\geq M, \forall\, t\in\mathbb{R}^+$ and by (2.10), we have, for any $t>0$, $f(t,\beta(t))\leq h(t)$. Hence,

$$\begin{aligned} \beta''(t)-k^2\beta(t)+\phi(t)f(t,\beta(t)) &= -\phi(t)h(t)-k^2\beta(t)+\phi(t)f(t,\beta(t)) \\ &= -k^2\beta(t)+\phi(t)[f(t,\beta(t))-h(t)] \\ &\leq 0 \end{aligned}$$

and $\beta(0)=M\geq\varepsilon_n$, $\beta(\infty)\geq 0$. Consequently, for any $n\geq 1$, β is an upper solution of (2.12). In addition, if $\alpha_n(t)=\varepsilon_n e^{-kt}$, $t>0$, then,

$$\alpha_n''(t)-k^2\alpha_n(t)+\phi(t)f(t,\alpha_n(t))=\phi(t)f(t,\alpha_n(t))\geq 0,$$

and $\alpha_n(0)=\varepsilon_n$, $\alpha_n(\infty)=0$. Hence, α_n is a lower solution of (2.12), $\forall\, n\geq 1$. Moreover, for all $t\in\mathbb{R}^+$ and $\alpha_n(t)\leq x\leq\beta(t)$, we have $\varepsilon_n\widetilde{\gamma}(t)\leq xe^{-\theta t}\leq\|\beta\|_\theta$. The monotonicity of q and $\frac{p}{q}$ implies that

$$\begin{aligned} \sup_{x\in D^\beta_{\alpha_n}(t)} e^{-\theta t}f(t,x) &= \sup_{\alpha_n\leq x\leq\beta} e^{-\theta t}F(t,xe^{-\theta t}) \\ &\leq \sup_{\alpha_n\leq x\leq\beta} e^{-\theta t}m(t)p(xe^{-\theta t}) \\ &\leq \sup_{\alpha_n\leq x\leq\beta} e^{-\theta t}m(t)q(xe^{-\theta t})\frac{p(xe^{-\theta t})}{q(xe^{-\theta t})} \\ &\leq e^{-\theta t}m(t)q(\varepsilon_n\widetilde{\gamma}(t))\frac{p(\|\beta\|)}{q(\|\beta\|)}:=\delta(t). \end{aligned}$$

Using (2.8), we have $\int_0^{+\infty}e^{\theta s}\phi(s)\delta(s)\mathrm{d}s<+\infty$. By Theorem 2.1, problem (2.12) has at least one positive solution $x_n\in E$ with $\alpha_n(t)\leq x_n(t)\leq\beta(t)$, $\forall\, t\in\mathbb{R}^+$.

Step 2 The sequence $\{x_n\}_{n\geq 1}$ is relatively compact.

(a) Given $\mu \in (k, \theta)$, we can prove that $\sup_{t\in\mathbb{R}^+} |\beta(t)|e^{-\mu t} < +\infty$. Moreover,

$$\|x_n\|_\mu = \sup_{t\in\mathbb{R}^+} |x_n(t)|e^{-\mu t} \leq \sup_{t\in\mathbb{R}^+} |\beta(t)|e^{-\mu t} = \|\beta\|_\mu.$$

Then, $\{x_n\}_n$ is uniformly bounded in sense of the norm $\|.\|_\mu$.

(b) From $(\mathcal{H}_2)$, there exists $\psi_{\|\beta\|_\mu} \in C(\mathbb{R}^+, I)$, such that

$$|F(t,z)| \geq \psi_{\|\beta\|_\mu}(t), \quad \text{for } t \in \mathbb{R}^+ \text{ and } z \in (0, \|\beta\|_\mu] \tag{2.13}$$

with $\int_0^{+\infty} \phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s < +\infty$. As

$$\int_0^{+\infty} e^{-\theta t} G(t,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s \leq \frac{1}{2k}\int_0^{+\infty} \phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s < +\infty,$$

then, the map $t \mapsto \int_0^{+\infty} e^{-\theta t}G(t,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s$ belongs to E. From (2.13) and Lemma 2.2, we find that, for all $\tau \geq 0$,

$$\begin{aligned}
x_n(t) &= \varepsilon_n e^{-kt} + \int_0^{+\infty} G(t,s)\phi(s)f(s, x_n(s))\mathrm{d}s \\
&= \varepsilon_n e^{-kt} + \int_0^{+\infty} G(t,s)\phi(s)F(s, x_n(s)e^{-\theta s})\mathrm{d}s \\
&\geq \varepsilon_n e^{-kt} + \int_0^{+\infty} G(t,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s \\
&\geq \int_0^{+\infty} G(t,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s \\
&\geq e^{\theta t}\gamma(t)\int_0^{+\infty} e^{-\theta\tau}G(\tau,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s \\
&\geq \gamma(t)\int_0^{+\infty} e^{-\theta\tau}G(\tau,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s.
\end{aligned}$$

Passing to the supremum over τ, we get the lower bound

$$x_n(t) \geq q^*\gamma(t),$$

where

$$q^* := \sup_{\tau\in\mathbb{R}^+} \int_0^{+\infty} e^{-\theta\tau}G(\tau,s)\phi(s)\psi_{\|\beta\|_\mu}(s)\mathrm{d}s.$$

Using $(\mathcal{H}_1)$ and the monotonicity of q and $\frac{p}{q}$, we obtain the upper bound

$$\begin{aligned}
G(t,s)e^{-\theta t}\phi(s)f(s,x_n(s)) &= G(t,s)e^{-\theta t}\phi(s)F(s, x_n(s)e^{-\theta s}) \\
&\leq G(t,s)e^{-\theta t}\phi(s)m(s)p(x_n(s)e^{-\theta s}) \\
&\leq \frac{1}{2k}\phi(s)m(s)q(x_n(s)e^{-\theta s})\frac{p(x_n(s)e^{-\theta s})}{q(x_n(s)e^{-\theta s})} \\
&\leq \frac{1}{2k}\phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(\|\beta\|_\mu)}{q(\|\beta\|_\mu)}.
\end{aligned} \tag{2.14}$$

Therefore, for any $T>0$ and $t,t'\in[0,T]$,

$$\begin{aligned}
&|x_n(t)-x_n(t')|\le \varepsilon_n|e^{-kt}-e^{-kt'}|\\
&+\int_0^{+\infty}|G(t,s)-G(t',s)|\phi(s)f(s,x_n(s))\mathrm{d}s\\
\le{}& \varepsilon_n|e^{-kt}-e^{-kt'}|+\int_0^{+\infty}|G(t,s)-G(t',s)|\phi(s)F(s,x_n(s)e^{-\theta s})\mathrm{d}s\\
\le{}& \varepsilon_n|e^{-kt}-e^{-kt'}|+\int_0^{T}|G(t,s)-G(t',s)|\phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(\|\beta\|_\mu)}{q(\|\beta\|_\mu)}\mathrm{d}s\\
&+\int_T^{+\infty}|G(t,s)-G(t',s)|\phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(\|\beta\|_\mu)}{q(\|\beta\|_\mu)}\mathrm{d}s\\
\le{}& \varepsilon_n|e^{-kt}-e^{-kt'}|+\int_0^{T}|G(t,s)-G(t',s)|m(s)\phi(s)q(\widetilde{\gamma}(s)q^*)\frac{p(\|\beta\|_\mu)}{q(\|\beta\|_\mu)}\mathrm{d}s\\
&+\frac{1}{2k}[(e^{kt}-e^{-kt})-(e^{kt'}-e^{-kt'})]\int_T^{+\infty}\phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(\|\beta\|_\mu)}{q(\|\beta\|_\mu)}\mathrm{d}s.
\end{aligned}$$

It follows that, for any $\varepsilon>0$ and $T>0$, there exists $\eta>0$ such that $|x_n(t)-x_n(t')|<\varepsilon$ for all t, $t'\in[0,T]$ with $|t-t'|<\eta$. Consequently, $\{x_n\}_{n\ge1}$ is equicontinuous, hence relatively compact by Lemma 1.1. Therefore, $\{x_n\}_{n\ge1}$ has a subsequence $\{x_{n_k}\}_{k\ge1}$ converging to some limit x^*, $k\to+\infty$. By continuity of f, we have, for each fixed positive s,

$$f(s,x_{n_k}(s))\longrightarrow f(s,x^*(s))\quad\text{as}\quad k\to+\infty.$$

Using (2.14) and the Lebesgue dominated convergence theorem, we finally deduce that, for any $t\in\mathbb{R}^+$,

$$\begin{aligned}
x^*(t)&=\lim_{k\to+\infty}x_{n_k}(t)\\
&=\lim_{k\to+\infty}[\varepsilon_{n_k}e^{-kt}+\int_0^{+\infty}G(t,s)\phi(s)f(s,x_{n_k}(s))\mathrm{d}s]\\
&=\int_0^{+\infty}G(t,s)\phi(s)f(s,x^*(s))\mathrm{d}s.
\end{aligned}$$

Then, x^* is a solution of problem (1.1) with $x^*\in E$, as claimed.

3 Two Positive Solutions for Problem (1.1)

Assume f is superlinear in x, that is,

$(\mathcal{H}_3)$ There exist $0<a<b<+\infty$, such that

$$\lim_{x\to+\infty}\frac{f(t,x)}{x}=+\infty\ \text{ uniformly in }\ t\in[a,b].$$

Given $f\in C(\mathbb{R}^+\times I,\mathbb{R}^+)$, define a sequence of functions $\{f_n\}_{n\ge1}$ by

$$f_n(t,x)=f(t,\max\{e^{\theta t}/n,x\}),\quad n\in\{1,2,\cdots\}.$$

Let $\mathcal{P}$ be the positive cone defined in E by

$$\mathcal{P}=\{x\in E:\ x(t)\ge0\ \text{ and }\ x(t)\ge\gamma(t)\|x\|_\theta,\ \forall\, t\ge0\}.$$

For $x \in \mathcal{P}$, define a sequence of operators by

$$A_n x(t) = \int_0^{+\infty} G(t,s)\phi(s) f_n(s, x(s))\mathrm{d}s, \quad n \in \{1, 2, \cdots\}.$$

We have

Lemma 3.1 Suppose $(\mathcal{H}_1)$ holds. Then, for each $n \geq 1$, the operator A_n sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.

Proof The proof that A_n is continuous and completely continuous is similar to that of the operator A in Theorem 2.1; we omit it. Thus, we only show that $A_n\mathcal{P} \subseteq \mathcal{P}$. For $x \in \mathcal{P}$, we have $A_n x(t) \geq 0$, $\forall\, t \in \mathbb{R}^+$, and from condition $(\mathcal{H}_1)$, we have

$$\begin{aligned}
|A_n x(t)| e^{-\theta t} &= \int_0^{+\infty} G(t,s) e^{-\theta t}\phi(s) f_n(s, x(s))\mathrm{d}s \\
&= \int_0^{+\infty} G(t,s) e^{-\theta t}\phi(s) F(s, \max\{1/n, x(s)e^{-\theta s}\})\mathrm{d}s \\
&\leq \int_0^{+\infty} G(t,s) e^{-\theta t}\phi(s) m(s) p(\max\{1/n, x(s)e^{-\theta s}\})\mathrm{d}s \\
&\leq \frac{1}{2k}\int_0^{+\infty} \phi(s) m(s) q(\max\{1/n, x(s)e^{-\theta s}\}) \frac{p(\max\{1/n, x(s)e^{-\theta s}\})}{q(\max\{1/n, x(s)e^{-\theta s}\})}\mathrm{d}s \\
&\leq \frac{1}{2k}\int_0^{+\infty} \phi(s) m(s) q(\widetilde{\gamma}(s)\|x\|_\theta) \frac{p(\max\{1/n, \|x\|_\theta\})}{q(\max\{1/n, \|x\|_\theta\})}\mathrm{d}s \\
&= \frac{1}{2k}\frac{p(\max\{1/n, \|x\|_\theta\})}{q(\max\{1/n, \|x\|_\theta\})}\int_0^{+\infty} \phi(s) m(s) q(\widetilde{\gamma}(s)\|x\|_\theta)\mathrm{d}s < +\infty.
\end{aligned}$$

Then, $A_n x \in E$ and, from Lemma 2.2, it holds that

$$\begin{aligned}
A_n x(t) &= \int_0^{+\infty} G(t,s)\phi(s) f_n(s, x(s))\mathrm{d}s \\
&= e^{\theta t}\int_0^{+\infty} e^{-\theta t} G(t,s)\phi(s) f_n(s, x(s))\mathrm{d}s \\
&\geq e^{\theta t}\gamma(t)\int_0^{+\infty} e^{-\theta \tau} G(\tau,s)\phi(s) f_n(s, x(s))\mathrm{d}s \\
&\geq \gamma(t) e^{-\theta\tau} A_n x(\tau),\ \forall\, \tau \in \mathbb{R}^+ \\
&\geq \gamma(t)\|A_n x\|_\theta.
\end{aligned}$$

Therefore, $A_n\mathcal{P} \subseteq \mathcal{P}$.

Theorem 3.1 Assume that Assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$ hold. Then, there exists $k_0 > 0$ such that, for any $k > k_0$, problem (1.1) has at least two positive solutions.

Proof **Step 1** From Lemma 3.1, $A_n : P \longrightarrow P$ is completely continuous for each $n \geq 1$. Let $R > 0$ and

$$k_0 := \frac{\int_0^{+\infty} \phi(s) m(s) q(R\widetilde{\gamma}(s)) p(R)\mathrm{d}s}{2Rq(R)}.$$

Then, for any $k > k_0$, we have

$$2kRq(R) > \int_0^{+\infty} \phi(s) k(s) q(R\widetilde{\gamma}(s)) p(R)\mathrm{d}s. \tag{3.1}$$

Let $\Omega_1 = \{x \in E: \ \|x\|_\theta < R\}$. We claim that $x \neq \lambda A_n x$ for any $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0,1]$ and $n \geq n_0 > 1/R$. On the contrary, suppose that there exist $n \geq n_0$, $x_0 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$, such that $x_0 = \lambda_0 A_{n_0} x_0$. As $x_0 \in \partial\Omega_1 \cap \mathcal{P}$, one has $x_0(t) \geq \gamma(t)\|x_0\|_\theta = \gamma(t)R, \forall\, t \in \mathbb{R}^+$. Then, $x_0(t)e^{-\theta t} \geq \widetilde{\gamma}(t)\|x_0\|_\theta = \widetilde{\gamma}(t)R$. Therefore,

$$\begin{aligned}
R = \|x_0\|_\theta &= \|\lambda_0 A_{n_0} x_0\|_\theta \\
&\leq \sup_{t\geq 0} \int_0^{+\infty} e^{-\theta t} G(t,s)\phi(s) f_n(s, x_0(s))\mathrm{d}s \\
&\leq \sup_{t\geq 0} \int_0^{+\infty} e^{-\theta t} G(t,s)\phi(s) F(s, \max\{1/n, x_0(s)e^{-\theta s}\})\mathrm{d}s \\
&\leq \frac{1}{2k} \int_0^{+\infty} \phi(s)m(s)q(\max\{1/n, x_0(s)e^{-\theta s}\}) \frac{p(\max\{1/n, x_0(s)e^{-\theta s}\})}{q(\max\{1/n, x_0(s)e^{-\theta s}\})} \\
&\leq \frac{1}{2k} \int_0^{+\infty} \phi(s)m(s)q(\widetilde{\gamma}(s)R)\frac{p(R)}{q(R)} ds,
\end{aligned}$$

which implies

$$2kRq(R) \leq \int_0^{+\infty} \phi(s)m(s)q(R\widetilde{\gamma}(s))p(R)\mathrm{d}s,$$

contradicting (3.1). By Lemma 1.2, we infer that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \quad \text{for all} \ \ n \in \{n_0, n_0+1, \cdots\}. \tag{3.2}$$

Hence, there exists an $x_n \in \Omega_1 \cap \mathcal{P}$, such that $A_n x_n = x_n$, $\forall\, n \geq n_0$. As $\|x_n\|_\theta < R$, from $(\mathcal{H}_2)$, there exists $\psi_R \in C(\mathbb{R}^+, I)$ such that, for all $t \geq 0$,

$$f_n(t, x_n(t)) \geq \psi_R(t) \quad \text{ with } \quad \int_0^{+\infty} \phi(s)\psi_R(s)\mathrm{d}s < +\infty.$$

It is proved that the map $t \mapsto \int_0^{+\infty} G(t,s)\phi(s)\psi_R(s)\mathrm{d}s$ belongs to E. Then,

$$\begin{aligned}
x_n(t) &= A_n x_n(t) \\
&= \int_0^{+\infty} G(t,s)\phi(s) f_n(s, x_n(s))\mathrm{d}s \\
&\geq \int_0^{+\infty} G(t,s)\phi(s)\psi_R(s)\mathrm{d}s.
\end{aligned}$$

Lemma 2.2 implies that

$$\begin{aligned}
x_n(t)e^{-\theta t} &\geq e^{-\theta t} \int_0^{+\infty} G(t,s)\phi(s)\psi_R(s)\mathrm{d}s \\
&\geq \gamma(t) \int_0^{+\infty} e^{-\theta\tau} G(\tau,s)\phi(s)\psi_R(s)\mathrm{d}s, \ \ \forall\, \tau \geq 0.
\end{aligned}$$

Passing to the supremum over τ, we get

$$x_n(t)e^{-\theta t} \geq \widetilde{\gamma}(t)q^*,$$

where

$$q^* := \sup_{\tau\in\mathbb{R}^+} \int_0^{+\infty} e^{-\theta\tau} G(\tau,s)\phi(s)\psi_R(s)\mathrm{d}s.$$

From $(\mathcal{H}_1)$, we have

$$\int_0^{+\infty} G(t,s)\phi(s)f_n(s,x_n(s))\mathrm{d}s \le \int_0^{+\infty} \phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(R)}{q(R)} < +\infty.$$

Step 2 The sequence $\{x_n\}_{n\ge n_0}$ is relatively compact.

(a) Given $\mu \in (k,\theta)$, from condition $(\mathcal{H}_1)$, we have

$$\begin{aligned}
\|x_n\|_\mu &= \sup_{\mathbb{R}^+} \int_0^{+\infty} G(t,s)e^{-\mu t}\phi(s)f_n(s,x_n(s))\mathrm{d}s \\
&\le \int_0^{+\infty} G(t,s)e^{-\mu t}\phi(s)F(s,\max\{1/n,x_n(s)e^{-\theta s}\})\mathrm{d}s \\
&\le \int_0^{+\infty} G(t,s)e^{-\mu t}\phi(s)m(s)p(\max\{1/n,x_n(s)e^{-\theta s}\})\mathrm{d}s \\
&\le \frac{1}{2k}\int_0^{+\infty} \phi(s)m(s)q(\max\{1/n,x_n(s)e^{-\theta s}\})\frac{p(\max\{1/n,x_n(s)e^{-\theta s}\})}{q(\max\{1/n,x_n(s)e^{-\theta s}\})}\mathrm{d}s \\
&\le \frac{1}{2k}\int_0^{+\infty} \phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(R)}{q(R)}\mathrm{d}s \\
&= \frac{1}{2k}\frac{p(R)}{q(R)}\int_0^{+\infty} \phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\mathrm{d}s < +\infty.
\end{aligned}$$

(b) For any $T>0$ and $t,t' \in [0,T]$, we have

$$\begin{aligned}
|x_n(t)-x_n(t')| &= \int_0^{+\infty} |G(t,s)-G(t',s)|\phi(s)f_n(s,x_n(s))\mathrm{d}s \\
&\le \int_0^{+\infty} |G(t,s)-G(t',s)|\phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(R)}{q(R)}\mathrm{d}s.
\end{aligned}$$

As in the proof of Theorem 2.2, we verify that, for any $\varepsilon>0$ and $T>0$, there exists $\eta>0$ such that $|x_n(t)-x_n(t')|<\varepsilon$ for all $t,t'\in[0,T]$ with $|t-t'|<\eta$. Then, $\{x_n\}_{n\ge n_0}$ is equicontinuous, hence relatively compact. Then, there exists a subsequence $\{x_{n_k}\}_{k\ge 1}$ with $\lim\limits_{k\to+\infty} x_{n_k} = x$. As $x_{n_k}(t) \ge \widetilde{\gamma}(s)q^*, \forall\, k\ge 1$, we have $x(t)\ge \widetilde{\gamma}(s)q^*, \forall\, t\in\mathbb{R}^+$. Consequently,

$$\int_0^{+\infty} G(t,s)\phi(s)f(s,x(s))\mathrm{d}s \le \int_0^{+\infty} \phi(s)m(s)q(\widetilde{\gamma}(s)q^*)\frac{p(R)}{q(R)} < +\infty.$$

The continuity of f implies that, for all $s\in\mathbb{R}^+$,

$$\begin{aligned}
\lim_{k\to+\infty} f_{n_k}(s,x_{n_k}(s)) &= \lim_{k\to+\infty} f(s,\max\{e^{\theta s}/n_k, x_{n_k}(s)\}) \\
&= f(s,\max\{0,x(s)\} = f(s,x(s)).
\end{aligned}$$

By Lebesgue dominated convergence theorem,

$$\begin{aligned}
x(t) = \lim_{k\to+\infty} x_{n_k}(t) &= \lim_{k\to+\infty}\int_0^{+\infty} G(t,s)\phi(s)f_{n_k}(s,x_{n_k}(s))\mathrm{d}s \\
&= \int_0^{+\infty} G(t,s)\phi(s)f(s,x(s))\mathrm{d}s.
\end{aligned}$$

It is clear that $\|x\|_\theta \le R$, and from (3.1), $\|x\|_\theta < R$. Then, x is a positive solution of (1.1).

Step 3 Let

$$N^*=1+\frac{1}{r\min\limits_{t\in[a,b]}\int_a^b G(t,s)e^{-\theta t}\phi(s)\mathrm{d}s},$$

where $r=\min\limits_{t\in[a,b]}\gamma(t)$. By $(\mathcal{H}_3)$, there exists an $R'>R$, such that

$$f(t,x)>N^*R',\quad \forall\, t\in[a,b],\ \forall\, x\geq R'.$$

Let

$$\Omega_2=\left\{x\in E:\ \|x\|_\theta<\frac{R'}{r}\right\}.$$

Without loss of generality, suppose that $R'>\max\{1,R\}$. We show that $A_nx\not\leq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$ and $n\in\{1,2,\cdots\}$. Suppose, on the contrary, that there exists an $n\in\{1,2,\cdots\}$ and $x_0\in\partial\Omega_2\cap\mathcal{P}$ with $A_nx_0\leq x_0$. As $x_0\in\mathcal{P}$, we have

$$x_0(t)\geq\gamma(t)\|x_0\|_\theta\geq\min_{s\in[a,b]}\gamma(s)\frac{R'}{r}\geq R',\ \forall\, t\in[a,b].$$

Then, for any $t\in[a,b]$, we have

$$\begin{aligned}
x_0(t)e^{-\theta t}&\geq A_nx_0(t)e^{-\theta t}\\
&=\int_0^{+\infty}G(t,s)e^{-\theta t}\phi(s)f_n(s,x_0(s))\mathrm{d}s\\
&\geq\int_a^b G(t,s)e^{-\theta t}\phi(s)f_n(s,x_0(s))\mathrm{d}s\\
&\geq\int_a^b G(t,s)e^{-\theta t}\phi(s)N^*\max\{\frac{e^{\theta s}}{n},x_0(s)\}\mathrm{d}s\\
&\geq\min_{t\in[a,b]}\int_a^b G(t,s)e^{-\theta t}\phi(s)N^*R'>\frac{R'}{r},
\end{aligned}$$

contradicting $\|x_0\|_\theta=\frac{R'}{r}$. Finally, Lemma 1.3 yields

$$\mathrm{i}(A_n,\Omega_2\cap\mathcal{P},\mathcal{P})=0,\quad\forall\, n\in\mathbb{N}^*,\tag{3.3}$$

while (3.2) and (3.3) imply

$$\mathrm{i}(A_n,(\Omega_2\setminus\overline{\Omega}_1)\cap\mathcal{P},\mathcal{P})=-1,\ \forall\, n\geq n_0.\tag{3.4}$$

Therefore, A_n has another fixed point $y_n\in(\Omega_2-\overline{\Omega}_1)\cap\mathcal{P}$, $\forall\, n\geq n_0$. Consider the sequence $\{y_n\}_{n\geq n_0}$. Then, $y_n(t)\geq\gamma(t)R$, $\forall\, t\in\mathbb{R}^+$, and $\|y_n\|_\theta<\frac{R'}{r}$, $\forall\, n\geq n_0$. Arguing as above, we can show that $\{y_n\}_{n\geq n_0}$ has a convergent subsequence $\{y_{n_j}\}_{j\geq1}$ with $\lim\limits_{j\to+\infty}y_{n_j}=y_0$ to be a solution of (1.1). Moreover, $R<\|y_0\|_\theta<\frac{R'}{r}\cdot$ Hence, x_0 and y_0 are two distinct positive solutions to problem (1.1).

Remark 3.1 If the following condition holds

$(\mathcal{H}_4)$

$$\sup_{c>0}\frac{2kcq(c)}{p(c)\int_0^{+\infty}\phi(\tau)m(\tau)h(c\widetilde{\gamma}(\tau))d\tau}>1,$$

then, Theorem 3.1 holds for each $k > 0$.

Example 3.1 Consider the singular boundary value problem

$$\begin{cases} x''(t) - \dfrac{9}{4}x(t) + (e^{3t} - 1)e^{\frac{-17}{2}t}\dfrac{e^{-4t}x^2 + 1}{\sqrt{x}} = 0, \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} x(t) = 0. \end{cases} \tag{3.5}$$

If $\theta = 2$, then,

$$f(t, x) = e^{\frac{-17}{2}t}\frac{e^{-4t}x^2 + 1}{\sqrt{x}} \text{ and } F(t, z) = e^{\frac{-19}{2}t}\frac{z^2 + 1}{\sqrt{z}}\cdot$$

Let

$$m(t) = e^{\frac{-19}{2}t},\ p(z) = \frac{z^2 + 1}{\sqrt{z}},\ q(z) = \frac{1}{z}, \text{ and } \phi(t) = (e^{3t} - 1).$$

Then, Assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ are satisfied. Indeed

$(\mathcal{H}_1)$ The function $q \in C(\mathbb{R}_0^+, I)$ is decreasing and

$$F(t, z) \le m(t)p(z), \qquad \forall\, t \in \mathbb{R}^+,\ \forall\, z > 0.$$

The function $\frac{p(z)}{q(z)} = \sqrt{z}(z^2 + 1)$ is increasing, and for any $c > 0$,

$$\int_0^{+\infty} \phi(s)m(s)q(c\widetilde{\gamma}(s))\mathrm{d}s = \frac{1}{c} < +\infty.$$

$(\mathcal{H}_2)$ For any $c > 0$, there exists $\psi_c = \frac{1}{\sqrt{c}}e^{\frac{-19}{2}t} \in C(\mathbb{R}^+, I)$, such that

$$F(t, z) \ge \psi_c(t), \quad \forall\, t \in \mathbb{R}^+,\ \forall\, z \in (0, c] \text{ with } \int_0^{+\infty} \phi(s)\psi_c(s)\mathrm{d}s < +\infty.$$

$(\mathcal{H}_3)$ There exist $0 < a < b < +\infty$, such that

$$\lim_{x\to+\infty} \frac{f(t, x)}{x} = +\infty \text{ uniformly for } t \in [a, b].$$

Therefore, Theorem 3.1 implies that problem (3.5) has at least two positive solutions.

4 A Solution for Problem (1.2)

4.1 General setting

Notice that every solution of problem (1.2) is also a fixed point of the following integral operator:

$$By(t) = \int_0^{+\infty} K(t, s)\phi(s)g(s, y(s))\mathrm{d}s,$$

where $K(t, s)$ is the Green's function to the corresponding homogenous Dirichlet boundary value problem (1.2). It is seen that K is given by

$$K(t, s) = \frac{1}{r_1 - r_2}\begin{cases} e^{r_2 t}(e^{-r_2 s} - e^{-r_1 s}), & 0 \le s \le t < +\infty, \\ e^{-r_1 s}(e^{r_1 t} - e^{r_2 t}), & 0 \le t \le s < +\infty, \end{cases} \tag{4.1}$$

where

$$r_2=\frac{c-\sqrt{c^2+4\lambda}}{2}<0<r_1=\frac{c+\sqrt{c^2+4\lambda}}{2}\cdot$$

For some $\theta>r_1$, consider the weighted Banach space

$$E'=\{y\in C(\mathbb{R}^+,\mathbb{R}):\ \sup_{t\in\mathbb{R}^+}|y(t)|e^{-\theta t}<\infty\}$$

endowed with the Bielecki's sup-norm

$$\|y\|_\theta=\sup_{t\in\mathbb{R}^+}\{|y(t)|e^{-\theta t}\}.$$

Let $\mathcal{P}$ be the positive cone defined in E' by

$$\mathcal{P}=\{y\in E':\ y(t)\geq 0 \text{ and } y(t)\geq\gamma_1(t)\|y\|_\theta,\ \forall\, t\in\mathbb{R}^+\},$$

where

$$\gamma_1(t)=e^{(r_1-r_2)t}-1)e^{-(\theta+r_1-2r_2)t},\qquad \forall\, t\in\mathbb{R}^+.$$

Let $z=ye^{-\theta t}$, $\widetilde{g}(t,z)=g(t,e^{\theta t}z)$, and $\widetilde{\gamma_1}(t)=\gamma_1(t)e^{-\theta t}$.

Lemma 4.1 The Green's function K satisfies the estimates:

(a) $K(t,s)\geq 0,\quad \forall\, t,s\in\mathbb{R}^+$,

(b) $K(s,s)\leq\frac{1}{r_1-r_2},\quad \forall\, t,s\in\mathbb{R}^+$,

(c) $K(t,s)e^{-\mu t}\leq K(s,s)e^{-r_1 s},\quad \forall\, t,s\in I,\ \forall\,\mu\geq r_1$,

(d) $K(t,s)e^{-\mu t}\geq\gamma_1(t)K(\tau,s)e^{-\mu\tau},\quad \forall\, t,s,\tau\in\mathbb{R}^+,\quad \forall\,\mu\geq r_1$.

Proof As Properties (a)-(c) are easy to prove, we only check (d):

$$\frac{K(t,s)e^{-\mu t}}{K(\tau,s)e^{-\mu\tau}}=\begin{cases}\dfrac{e^{-r_1 s}(e^{r_1 t}-e^{r_2 t})}{e^{r_2\tau}(e^{-r_2 s}-e^{-r_1 s})}e^{\mu\tau}e^{-\mu t}, & t\leq s\leq\tau,\\ \dfrac{e^{r_2 t}(e^{-r_2 s}-e^{-r_1 s})}{e^{-r_1 s}(e^{r_1\tau}-e^{r_2\tau})}e^{\mu\tau}e^{-\mu t}, & \tau\leq s\leq t,\\ \dfrac{e^{r_2 t}}{e^{r_2\tau}}e^{\mu\tau}e^{-\mu t}, & s\leq t\leq\tau,\\ \dfrac{e^{r_1 t}-e^{r_2 t}}{e^{r_1\tau}-e^{r_2\tau}}e^{\mu\tau}e^{-\mu t}, & \tau\leq t\leq s,\\ \dfrac{e^{r_2 t}}{e^{r_2\tau}}e^{\mu\tau}e^{-\mu t}, & s\leq\tau\leq t,\\ \dfrac{e^{r_1 t}-e^{r_2 t}}{e^{r_1\tau}-e^{r_2\tau}}e^{\mu\tau}e^{-\mu t}, & t\leq\tau\leq s,\end{cases}$$

which implies

$$\frac{K(t,s)e^{-\mu t}}{K(\tau,s)e^{-\mu\tau}}=\begin{cases}\dfrac{1}{(e^{(r_1-r_2)s}-1)}e^{(\mu-r_2)\tau}(e^{(r_1-r_2)t}-1)e^{-(\mu-r_2)t}, & t\leq s\leq\tau,\\ \dfrac{(e^{(r_1-r_2)s}-1)}{(e^{(r_1-r_2)\tau}-1)}e^{(\mu-r_2)\tau}e^{-(\mu-r_2)t}, & \tau\leq s\leq t,\\ e^{(\mu-r_2)\tau}e^{-(\mu-r_2)t}, & s\leq t\leq\tau,\\ \dfrac{e^{(\mu-r_2)\tau}}{(e^{(r_1-r_2)\tau}-1)}(e^{(r_1-r_2)t}-1)e^{-(\mu-r_2)t}, & \tau\leq t\leq s,\\ e^{(\mu-r_2)\tau}e^{-(\mu-r_2)t}, & s\leq\tau\leq t,\\ \dfrac{e^{(\mu-r_2)\tau}}{(e^{(r_1-r_2)\tau}-1)}(e^{(r_1-r_2)t}-1)e^{-(\mu-r_2)t}, & t\leq\tau\leq s.\end{cases}$$

As $\mu \geq r_1$, we derive the bounds

$$\frac{K(t,s)e^{-\mu t}}{K(\tau,s)e^{-\mu\tau}} \geq \begin{cases} (e^{(r_1-r_2)t}-1)e^{-(\mu-r_2)t}, & t \leq s \leq \tau, \\ e^{-(\mu-r_2)t}, & \tau \leq s \leq t, \\ e^{-(\mu-r_2)t}, & s \leq t \leq \tau, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu-r_2)t}, & \tau \leq t \leq s, \\ e^{-(\mu-r_2)t}, & s \leq \tau \leq t, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu-r_2)t}, & t \leq \tau \leq s. \end{cases}$$

Consequently,

$$\frac{K(t,s)e^{-\mu t}}{K(\tau,s)e^{-\mu\tau}} \geq \begin{cases} (e^{(r_1-r_2)t}-1)e^{-(\mu+r_1-2r_2)t} = \gamma_1(t), & t \leq s \leq \tau, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu+r_1-2r_2)t} = \gamma_1(t), & \tau \leq s \leq t, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu+r_1-2r_2)t} = \gamma_1(t), & s \leq t \leq \tau, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu+r_1-2r_2)t} = \gamma_1(t), & \tau \leq t \leq s, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu+r_1-2r_2)t} = \gamma_1(t), & s \leq \tau \leq t, \\ (e^{(r_1-r_2)t}-1)e^{-(\mu+r_1-2r_2)t} = \gamma_1(t), & t \leq \tau \leq s. \end{cases}$$

4.2 Assumptions

In this section, consider the hypotheses:

$(\mathcal{A}_1)$ There exist some $k \in C(\mathbb{R}^+, I)$, $p \in C(I,I)$, and a decreasing function $q \in C(I,I)$, such that

$$\widetilde{g}(t,z) \leq k(t)p(z), \quad \forall\, t \in \mathbb{R}^+,\ \forall\, z \in I,$$

where $\frac{p(z)}{q(z)}$ is an increasing function and $\int_0^{+\infty} \phi(s)k(s)q(c\widetilde{\gamma}_1(s))\mathrm{d}s < +\infty$ for each $c > 0$.

$(\mathcal{A}_2)$ For any $c > 0$, there exists $\psi_c \in C(\mathbb{R}^+, I)$, such that

$$\widetilde{g}(t,z) \geq \psi_c(t), \quad \forall\, t \in \mathbb{R}^+,\ \forall\, z \in (0,c] \ \text{ with } \int_0^{+\infty} \phi(s)\psi_c(s)\mathrm{d}s < +\infty.$$

Consider the auxiliary boundary value problem

$$\begin{cases} -y''(t) + cy'(t) + \lambda y(t) = \phi(t)g(t,y(t)), \\ y(0) = h > 0, \quad \lim\limits_{t\to+\infty} y(t) = 0, \end{cases} \tag{4.2}$$

where $g \in C(\mathbb{R}^+ \times I, \mathbb{R})$ and $\phi \in C(I,I)$.

Definition 4.1 A function $\alpha \in C(\mathbb{R}^+, I) \cap C^2(I,\mathbb{R})$ is called a lower solution of (4.2) if it satisfies

$$\begin{cases} -\alpha''(t) + c\alpha'(t) + \lambda\alpha(t) \leq \phi(t)g(t,\alpha(t)), \\ \alpha(0) \leq h, \quad \lim\limits_{t\to+\infty} \alpha(t) \leq 0. \end{cases}$$

A function $\beta \in C(\mathbb{R}^+, I) \cap C^2(I,\mathbb{R})$ is called an upper solution of (4.2) if it satisfies

$$\begin{cases} -\beta''(t) + c\beta'(t) + \lambda\beta(t) \geq \phi(t)g(t,\beta(t)), \\ \beta(0) \geq h, \quad \lim\limits_{t\to+\infty} \beta(t) \geq 0. \end{cases}$$

If there exists an upper solution β and a lower solution α of (4.2) with $\alpha(t) \le \beta(t)$ for all $t \in \mathbb{R}^+$, then, we can define

$$D_\alpha^\beta(t) = \{y \in \mathbb{R} : \ \alpha(t) \le y \le \beta(t)\}, \quad t \in \mathbb{R}^+.$$

Theorem 4.1 Assume that α, β are respectively lower and upper solutions of problem (4.2) with $\alpha(t) \le \beta(t)$ for all $t \in \mathbb{R}^+$. Moreover, suppose that there exists $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\sup_{y \in D_\alpha^\beta(t)} e^{-r_1 t}|g(t,y)| \le \delta(t), \quad \forall\, t \in \mathbb{R}^+,$$

and $\int_0^{+\infty} e^{r_1 s}\phi(s)\delta(s)\mathrm{d}s < +\infty$. Then, problem (4.2) has at least one solution $y^* \in E'$ with $\alpha(t) \le y^*(t) \le \beta(t), \ \ t \in \mathbb{R}^+$.

Proof Let

$$g^*(t,y) = \begin{cases} g(t,\alpha(t)), & y < \alpha(t), \\ g(t,y), & \alpha(t) \le y \le \beta(t), \\ g(t,\beta(t)), & y > \beta(t). \end{cases}$$

Note that g^* is continuous which allows us to define the regular problem

$$\begin{cases} -y''(t) + cy'(t) + \lambda y(t) = \phi(t) g^*(t, y(t)), \\ y(0) = h > 0, \quad \lim\limits_{t \to +\infty} y(t) = 0. \end{cases} \tag{4.3}$$

To show (4.3) has at least one solution y^*, consider for $y \in E'$ the operator

$$By(t) = he^{r_2 t} + \int_0^{+\infty} K(t,s)\phi(s) g^*(s, y(s))\mathrm{d}s.$$

Step 1 For $y \in E'$ and $t \in \mathbb{R}^+$, we have

$$\begin{aligned} |By(t)|e^{-\theta t} &= he^{-(-r_2+\theta)t} + \int_0^{+\infty} e^{-\theta t} K(t,s)\phi(s)|g^*(s,y(s))|\mathrm{d}s \\ &\le h + \int_0^{+\infty} e^{-r_1 s} K(s,s)\phi(s)|g^*(s,y(s))|\mathrm{d}s \\ &\le h + \frac{1}{r_1 - r_2}\int_0^{+\infty} e^{-r_1 s}\phi(s)|g^*(s,y(s))|\mathrm{d}s \\ &\le h + \frac{1}{r_1 - r_2}\int_0^{+\infty} \phi(s) e^{-r_1 s}|g^*(s,y(s))|\mathrm{d}s \\ &\le h + \frac{1}{r_1 - r_2}\int_0^{+\infty} \phi(s)\delta(s)\mathrm{d}s < +\infty, \end{aligned}$$

proving that $By \in E'$ for all $y \in E'$.

Step 2 Assume that $\{y_n\}_{n \ge 1} \subseteq E'$ and $y_0 \in E'$ with $\lim\limits_{n \to +\infty} y_n = y_0$. Then, $y_n(t) \longrightarrow y_0(t)$ as $n \to +\infty$, $t \in \mathbb{R}^+$. Thus, the continuity of g^* implies that, for $t \ge 0$,

$$|g^*(s, y_n(s)) - g^*(s, y_0(s))| \longrightarrow 0, \ n \to +\infty.$$

Moreover, as $e^{-r_1 s}|g^*(s,y_n(s))-g^*(s,y_0(s))|\le 2\delta(s),\ \forall\, s\ge 0$, the Lebesgue dominated convergence theorem guarantees that

$$\begin{aligned}\|By_n-By_0\|_\theta &= \sup_{\mathbb{R}^+}|By_n(t)-By_0(t)|e^{-\theta t}\\ &\le \sup_{\mathbb{R}^+}\int_0^{+\infty} e^{-\theta t}K(t,s)\phi(s)|g^*(s,y_n(s))-g^*(s,y_0(s))|\mathrm{d}s\\ &\le \int_0^{+\infty} K(s,s)e^{-r_1 s}\phi(s)|g^*(s,y_n(s))-g^*(s,y_0(s))|\mathrm{d}s\\ &\le \frac{1}{r_1-r_2}\int_0^{+\infty}\phi(s)e^{-r_1 s}|g^*(s,y_n(s))-g^*(s,y_0(s))|\mathrm{d}s.\end{aligned}$$

As the right-hand term tends to 0 as $n\to\infty$, $B: E'\longrightarrow E'$ is continuous.

Step 3 The set $B(E')$ is relatively compact.

(a) Let $r_1<\mu<\theta$ and $q(t)=e^{-\mu t}$ in Lemma 1.1. Then, for $y\in E'$,

$$\begin{aligned}|By(t)|e^{-\mu t} &= he^{-(-r_2+\mu)t}+\int_0^{+\infty} e^{-\mu t}K(t,s)\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\le h+\int_0^{+\infty} e^{-r_1 s}K(s,s)\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\le h+\frac{1}{r_1-r_2}\int_0^{+\infty} e^{-r_1 s}\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\le h+\frac{1}{r_1-r_2}\int_0^{+\infty}\phi(s)\delta(s)\mathrm{d}s<+\infty.\end{aligned}$$

So, $B(E')$ is uniformly bounded in the sense of the norm $\|.\|_\mu$.

(b) For a given $T>0$, $y\in E'$, and $t,t'\in[0,T]$, we have

$$\begin{aligned}&|By(t)-By(t')|\le h|e^{r_2 t}-e^{r_2 t'}|\\ &\quad+\int_0^{+\infty}|K(t,s)-K(t',s)|\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\le h|e^{r_2 t}-e^{r_2 t'}|+\int_0^{T}|K(t,s)-K(t',s)|\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\quad+\int_T^{+\infty}|K(t,s)-K(t',s)|\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &= h|e^{r_2 t}-e^{r_2 t'}|+\int_0^{T}|K(t,s)-K(t',s)|\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\quad+\frac{1}{r_1-r_2}[(e^{r_1 t}-e^{r_2 t})-(e^{r_1 t'}-e^{r_2 t'})]\int_T^{+\infty} e^{-r_1 s}\phi(s)|g^*(s,y(s))|\mathrm{d}s\\ &\le h|e^{-kt}-e^{-kt'}|+\int_0^{T}|K(t,s)-K(t',s)|\phi(s)g^*(s,y(s))|\mathrm{d}s\\ &\quad+\frac{1}{r_1-r_2}[(e^{r_1 t}-e^{r_2 t})-(e^{r_1 t'}-e^{r_2 t'})]\int_T^{+\infty}\phi(s)\delta(s)\mathrm{d}s.\end{aligned}$$

Then, for any $\varepsilon>0$ and $T>0$, there exists $\eta>0$ such that $|By(t)-By(t')|<\varepsilon$ for all $t,\ t'\in[0,T]$ and $|t-t'|<\eta$. Hence, Lemma 1.1 implies that $B(E')$ is relatively compact. The

Schauder fixed point theorem guarantees that B has at least one fixed point $y^* \in E'$. Also, y^* is a solution of (4.3).

Step 4 Next, we show that y^* satisfies $\alpha(t) \le y^*(t) \le \beta(t), \forall\, t \in \mathbb{R}^+$, which implies that y^* is a solution of (4.2). Suppose there is some $t^* \in \mathbb{R}^+$ with $y^*(t^*) > \beta(t^*)$ and let $z(t) = y^*(t) - \beta(t)$. As $z(0) \le 0$, $z(\infty) \le 0$, and $z(t^*) > 0$, z must have positive maximum at some point $t_0 \in I$ and then $z''(t_0) \le 0$, $z'(t_0) = 0$, and $z(t_0) > 0$. Therefore,

$$\begin{aligned}
0 \ge z''(t_0) &= {y^*}''(t_0) - \beta''(t_0) \\
&\ge [c{y^*}'(t_0) + \lambda y^*(t_0) - \phi(t_0)g^*(t_0, y^*(t_0))] - [c\beta'(t_0) + \lambda\beta(t_0) - \phi(t_0)g^*(t_0, \beta(t_0))] \\
&= c[{y^*}'(t_0) - \beta'(t_0)] + \lambda[y^*(t_0) - \beta(t_0)] + \phi(t_0)[g^*(t_0, \beta(t_0)) - g^*(t_0, y^*(t_0))] \\
&= \lambda[y^*(t_0) - \beta(t_0)] + \phi(t_0)[g(t_0, \beta(t_0)) - g(t_0, \beta(t_0))] \\
&= \lambda[y^*(t_0) - \beta(t_0)] > 0,
\end{aligned}$$

which is a contradiction. Then, $y^*(t) \le \beta(t), \forall\, t \in \mathbb{R}^+$. Similarly, we can prove that $\alpha(t) \le y^*(t)$.

4.3 An existence result

Using Theorem 4.1, we obtain the following.

Theorem 4.2 Let $g \in C(\mathbb{R}^+ \times I, \mathbb{R}^+)$ and conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ hold. Also, assume that there exist $M > 0$ and $h \in C(\mathbb{R}^+, I)$, such that

$$g(t, y) \le h(t), \quad \forall\, (t, y) \in \mathbb{R}^+ \times [M, +\infty), \tag{4.4}$$

with

$$\int_0^{+\infty} \phi(s)h(s)\mathrm{d}s < +\infty. \tag{4.5}$$

Then, problem (1.2) has at least one positive solution $y^* \in C(\mathbb{R}^+, \mathbb{R}^+) \cap C^2(\mathbb{R}^+, \mathbb{R})$.

Proof Choose a decreasing sequence $\{\varepsilon_n\}_{n\ge1}$ with $\lim\limits_{n\to+\infty} \varepsilon_n = 0$ and $\varepsilon_1 < M$, and consider the sequence of boundary value problems

$$\begin{cases} -y''(t) + cy'(t) + \lambda y(t) = \phi(t)g(t, y(t)), \\ y(0) = \varepsilon_n, \quad \lim\limits_{t\to+\infty} y(t) = 0. \end{cases} \tag{4.6}$$

To show that (4.6) has at least one solution y_n, for $n \ge 1$, let β be a solution of the boundary value problem

$$\begin{cases} y''(t) + \phi(t)h(t) = 0, \quad t > 0 \\ y(0) = M, \quad \lim\limits_{t\to+\infty} y'(t) = 0. \end{cases}$$

From [8], we know that $\beta(t) \ge M$ and $\beta'(t) \ge 0$ for any $t \in \mathbb{R}^+$, which implies that β is an upper solution of (4.6), $\forall\, n \ge 1$. Let $\alpha_n(t) = \varepsilon_n e^{r_2 t}$, $t \ge 0$. Then, for all $n \ge 1$, α_n is a lower solution of (4.6). By Theorem 4.1, we infer that problem (4.6) has at least one positive solution $y_n \in E'$, $\forall\, n \ge 1$, with

$$\alpha_n(t) \le y_n(t) \le \beta(t), \quad \forall\, t \in \mathbb{R}^+.$$

Using similar arguments as in the proof of Theorem 2.2, we can prove that $\{y_n\}_{n\ge1}$ is relatively compact and then we can extract a convergent subsequence $\{y_{n_k}\}_{k\ge1}$ such that $y_{n_k} \to y^*$ as $k \to +\infty$ and y^* is a solution of problem (1.2).

5 Two Positive Solutions for Problem (1.2)

Theorem 5.1 Besides $(\mathcal{A}_1)-(\mathcal{A}_2)$, assume that

$(\mathcal{A}_3)$ there exist $0<a<b<+\infty$, such that

$$\lim_{y\to+\infty}\frac{g(t,y)}{y}=+\infty \ \text{ uniformly for } \ t\in[a,b].$$

Then, there exists $\xi_0>0$ such that, for any c and λ with $\sqrt{c^2+4\lambda}>\xi_0$, problem (1.2) has at least two positive solutions.

Proof The proof is similar to that of the proof of Theorem 3.1. We replace $2k$ by $\sqrt{c^2+4\lambda}=r_1-r_2$.

Remark 5.1 If the following condition holds:

$(\mathcal{A}_4)$

$$\sup_{c>0}\frac{(r_1-r_2)cq(c)}{p(c)\int_0^{+\infty}\phi(\tau)m(\tau)h(c\widetilde{\gamma}(\tau))d\tau}>1,$$

then, Theorem 5.1 holds for each $c,\lambda>0$.

Example 5.1 Consider the singular boundary value problem

$$\begin{cases}-y''(t)+y'(t)+2y(t)=(e^{3t}-1)e^{-\frac{17}{2}t}\frac{e^{-6t}y^2+2}{\sqrt{y}},\\ y(0)=0,\quad \lim\limits_{t\to+\infty}y(t)=0.\end{cases}\tag{5.1}$$

Here, $r_1=2$, $r_2=-1$, and $\theta=2$. Then,

$$\widetilde{g}(t,z)=e^{-10t}\frac{z^2+2}{\sqrt{z}},\ \phi(t)=(e^{3t}-1),\ \ \text{and}\ \ \gamma_1(t)=(e^{3t}-1)e^{-7t}.$$

Let

$$k(t)=e^{-10t},\ p(z)=\frac{z^2+1}{\sqrt{z}},\ \ \text{and}\ \ q(z)=\frac{1}{z}\cdot$$

Then, we check the validity of the assumptions:

$(\mathcal{A}_1)$ The function $q\in C(I,I)$ is decreasing and

$$\widetilde{g}(t,z)\le k(t)p(z),\quad \forall\, t\in\mathbb{R}^+,\ \forall\, z\in I,$$

where $\frac{p(z)}{q(z)}=\sqrt{z}(z^2+1)$ is an increasing function and for any $c>0$, it holds that

$$\int_0^{+\infty}\phi(s)k(s)q(c\widetilde{\gamma}_1(s))\mathrm{d}s=\frac{1}{c}<+\infty.$$

$(\mathcal{A}_2)$ There exists $\psi_c(t)=\frac{2}{\sqrt{c}}e^{-10t}\in C(\mathbb{R}^+,I)$, such that

$$\widetilde{g}(t,z)\ge\psi_c(t),\quad \forall\, t\in\mathbb{R}^+,\ \forall z\in(0,c]\ \text{ with }\ \int_0^{+\infty}\phi(s)\psi_c(s)\mathrm{d}s<+\infty.$$

$(\mathcal{A}_3)$ For any $0<a<b<+\infty$, we have

$$\lim_{x\to+\infty}\frac{g(t,x)}{x}=+\infty,\ \ \text{uniformly in}\ \ t\in[a,b].$$

Therefore, all conditions of Theorem 5.1 are fulfilled and so problem (5.1) has at least two positive solutions.

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