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Monotonicity results for Dirichlet L-functions

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ABSTRACT

We present some monotonicity results for Dirichlet L-functions associated to real primitive characters. We show in particular that these Dirichlet L-functions are far from being logarithmically completely monotonic. Also, we show that, unlike in the case of the Riemann zeta function, the problem of comparing the signs of $\frac{d^k}{ds^k}\log L(s,\chi)$ at any two points $s_1, s_2 > 1$ is subtler.

1. Introduction

A function f is said to be completely monotonic on $[0,\infty)$ if $f \in C[0,\infty)$, $f \in C^\infty(0,\infty)$ and $(-1)^k f^{(k)}(t) \geqslant 0$ for $t > 0$ and $k = 0, 1, 2, \dots$, i.e., the successive derivatives alternate in sign. The following result due to S.N. Bernstein and D. Widder gives a complete characterization of completely monotonic functions [9, p. 95]:

A function $f : [0,\infty) \to [0,\infty)$ is completely monotonic if and only if there exists a non-decreasing bounded function γ such that $f(t) = \int_0^\infty e^{-st}\, d\gamma(s)$.

Lately, the variety of completely monotonic functions, which have found applications in different areas of mathematics, has been greatly expanded to include several special functions, for example, functions associated to gamma and psi functions by Chen [8], B.-N. Guo, S. Guo and F. Qi [15] and quotients of K-Bessel functions by Ismail [16]. A conjecture that certain quotients of Jacobi theta functions are completely monotonic was formulated by the first author and Solynin in [12], and slightly corrected later by the present authors in [13]. The problem on monotonicity of such quotients first appeared in the work of Solynin [23] on Gonchar's problem on the optimal distribution of n identical radial heating elements in a circular stove. Certain other classes of such functions were introduced by Alzer and Berg [1], Qi and Chen [20]. Completely monotonic functions have applications in diverse fields such as probability theory [17], physics [4], potential theory [6], combinatorics [3] and numerical and asymptotic analysis [14], to name a few.

A close companion to the class of completely monotonic functions is the class of logarithmically completely monotonic functions. This was first studied, although implicitly, by Alzer and Berg [2]. A function $f : (0,\infty) \to (0,\infty)$ is said to be logarithmically completely monotonic [5] if it is C^∞ and $(-1)^k[\log f(x)]^{(k)} \geqslant 0$, for $k = 0, 1, 2, 3, \dots$. Moreover, a function is said to be strictly logarithmically completely monotonic if $(-1)^k[\log f(x)]^{(k)} > 0$. The following is true:

Every logarithmic completely monotonic function is completely monotonic.

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The reader is referred to Alzer and Berg [2], Qi and Guo [21], and Qi, Guo and Chen [22] for proofs of this statement. See [19] for a survey article on completely monotonic functions.

One goal of this paper is to study the Dirichlet L-functions from the point of view of logarithmically complete monotonicity. For $\operatorname{Re} s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Consider $s > 1$. Since $\log \zeta(s) > 0$ and

$$(-1)^k \frac{d^k}{ds^k} \log \zeta(s) = (-1)^k \frac{d^{k-1}}{ds^{k-1}} \left(\frac{\zeta'(s)}{\zeta(s)} \right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)(\log n)^{k-1}}{n^s},$$

where $\Lambda(n) \geqslant 0$ is the von Mangoldt function, $(-1)^k \frac{d^k}{ds^k} \log \zeta(s) > 0$ for all $s > 1$. This implies that $\zeta(s)$ is a logarithmically completely monotonic function for $s > 1$ (in fact, strictly logarithmically completely monotonic). But this approach fails in the case of $L(s, \chi)$ with $s > 1$ and χ, a real primitive Dirichlet character modulo q, since

$$(-1)^k \frac{d^k}{ds^k} \log L(s, \chi) = (-1)^k \frac{d^{k-1}}{ds^{k-1}} \left(\frac{L'(s, \chi)}{L(s, \chi)} \right) = \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)(\log n)^{k-1}}{n^s}$$

may change sign for different values of s as $\chi(n)$ takes the values $-1, 0$ or 1. Hence, we need to consider a different approach for studying $L(s, \chi)$ in the context of logarithmically complete monotonicity. This naturally involves studying the zeros of derivatives of $\log L(s, \chi)$.

There have been several studies made on the number of zeros of $\zeta^{(k)}(s)$ and $L^{(k)}(s, \chi)$, one of which dates back to Spieser [24], who showed that the Riemann Hypothesis is equivalent to the fact that $\zeta'(s)$ has no zeros in $0 < \operatorname{Re} s < 1/2$. Spira [25,26] conjectured that

$$N(T) = N_k(T) + \left[\frac{T \log 2}{2\pi} \right] \pm 1,$$

where $N_k(T)$ denotes the number of zeros of $\zeta^{(k)}(s)$ with positive imaginary parts up to height T, and $N(T) = N_0(T)$. Berndt [7] showed that for any $k \geqslant 1$, as $T \to \infty$,

$$N_k(T) = \frac{T \log T}{2\pi} - \left(\frac{1 + \log 4\pi}{2\pi} \right) T + O(\log T).$$

Levinson and Montgomery [18] proved a quantitative result implying that most of the zeros of $\zeta^{(k)}(s)$ are clustered about the line $\operatorname{Re} s = 1/2$ and also showed that the Riemann Hypothesis implies that $\zeta^{(k)}(s)$ has at most finitely many non-real zeros in $\operatorname{Re} s < 1/2$. Their results were further improved by Conrey and Ghosh [10]. Analogues of several of the above-mentioned results for Dirichlet L-functions were given by Yildirim [28]. Our results in this paper are related to the zeros of $\log L(s, \chi)$ and its derivatives.

Throughout the paper, we assume that s is a real number and χ is a real primitive Dirichlet character modulo q. Let $F(s, \chi) := \log L(s, \chi)$, and for $s > 1$, define

$$A_{\chi,k} := \{ s\colon\ F^{(k)}(s, \chi) = 0 \}. \tag{1.1}$$

Then we obtain the following result:

Theorem 1.1. *Let χ be a real primitive character modulo q and $L(s, \chi) \neq 0$ for $0 < s < 1$. Then there exists a constant c_χ such that $[c_\chi, \infty) \cap (\bigcup_{k=1}^{\infty} A_{\chi,k})$ is dense in $[c_\chi, \infty)$.*

Let us note that Theorem 1.1 shows in particular that $L(s, \chi)$ is not logarithmically completely monotonic on any subinterval of $[c_\chi, \infty)$. A stronger assertion is as follows:

For any subinterval of $[c_\chi, \infty)$, however small it may be, infinitely many derivatives $F^{(k)}(s, \chi)$ change sign in this subinterval.

Now consider any two points s_1, s_2 with $1 < s_1 < s_2$. In the case of the Riemann zeta function, if we compare the signs of the values of $\frac{d^k}{ds^k} \log \zeta(s)$ at s_1 and s_2 for all values of k, we see that they are always the same. Then a natural question arises – what can we say if we make the same comparison in the case of a Dirichlet L-function? We will see below that the answer is completely different (actually it is as different as it could be). We first define a function ψ_χ for a real primitive Dirichlet character modulo q as follows:

Let $\mathcal{B} := \{g\colon \mathbb{N} \to \{-1, 0, 1\}\}$. Define an equivalence relation $\sim$ on $\mathcal{B}$ by $g \sim h$ if and only if $g(n) = h(n)$ for all n large enough. Let $\hat{\mathcal{B}} = \mathcal{B}/\sim$. By abuse of notation, we define $\psi_\chi : (1,\infty) \to \hat{\mathcal{B}}$ to be a function whose image is a sequence given by $\{\operatorname{sgn}(F^{(k)}(s,\chi))\}$, i.e.,

$$\psi_\chi(s)(k) := \operatorname{sgn}\bigl(F^{(k)}(s,\chi)\bigr). \tag{1.2}$$

With this definition, we answer the above question in the form of the following theorem.

Theorem 1.2. *Let χ be a real primitive character modulo q and let ψ_χ be defined as above. Then there exists a constant C_χ with the following property*:

(a) *The Riemann Hypothesis for $L(s,\chi)$ implies that ψ_χ is injective on $[C_\chi,\infty)$.*
(b) *Let ψ_χ be injective on $[C_\chi,\infty)$. Then there exists an effectively computable constant D_χ such that if all the non-trivial zeros ρ of $L(s,\chi)$ up to the height D_χ lie on the critical line* $\operatorname{Re} s = 1/2$*, then the Riemann Hypothesis for $L(s,\chi)$ is true.*

2. Proof of Theorem 1.1

First we will compute $F^{(k)}(s,\chi)$ in terms of the zeros of $L(s,\chi)$. The logarithmic derivative of $L(s,\chi)$ satisfies [11, p. 83]

$$F'(s,\chi) = \frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'(s/2+b/2)}{\Gamma(s/2+b/2)} + B(\chi) + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{2.1}$$

where $B(\chi)$ is a constant depending on χ,

$$b = \begin{cases} 1 & \text{if } \chi(-1) = -1, \\ 0 & \text{if } \chi(-1) = 1 \end{cases} \tag{2.2}$$

and $\rho = \beta + i\gamma$ are the non-trivial zeros of $L(s,\chi)$. Since $B(\overline{\chi}) = \overline{B(\chi)}$ and χ is real, $B(\chi)$ is given by

$$B(\chi) = -\sum_\rho \frac{1}{\rho} = -2\sum_{\gamma>0}\frac{\beta}{\beta^2+\gamma^2} < \infty,$$

see [11, p. 83]. Note that $B(\chi)$ is negative. The Weierstrass infinite product for $\Gamma(s)$ is [11, p. 73]

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^{\infty}(1+s/n)^{-1}e^{s/n}, \tag{2.3}$$

with $s = 0, -1, -2, \dots$ being its simple poles. The functional equation for $\Gamma(s)$ is

$$\Gamma(s+1) = s\Gamma(s) \tag{2.4}$$

where as the duplication formula for $\Gamma(s)$ is

$$\Gamma(s)\Gamma(s+1/2) = 2^{(1-2s)}\pi^{1/2}\Gamma(2s), \tag{2.5}$$

see [11, p. 73]. The following can be easily derived from (2.4), (2.5) and the logarithmic derivative of (2.3):

$$\frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} = -\frac{\gamma}{2} - \frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right), \tag{2.6}$$

$$\frac{1}{2}\frac{\Gamma'(s/2+1/2)}{\Gamma(s/2+1/2)} = -\log(2) - \frac{\gamma}{2} - \sum_{n=0}^{\infty}\left(\frac{1}{s+2n+1} - \frac{1}{2n+1}\right), \tag{2.7}$$

From (2.1), (2.2), (2.6) and (2.7), we have

$$F'(s,\chi) = -\frac{1}{2}\log\frac{q}{\pi} + b\log 2 + \frac{\gamma}{2} + B(\chi) + \frac{1-b}{s} + \sum_{\rho\neq 0}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{2.8}$$

where ρ runs through all the zeros of $L(s,\chi)$. The successive differentiation of (2.8) gives for $k \geqslant 2$,

$$\begin{aligned} F^{(k)}(s,\chi) &= (-1)^{k-1}(k-1)!\left(\frac{1-b}{s^k} + \sum_{\substack{\rho\neq 0\\ \Lambda(\rho,\chi)=0}}\frac{1}{(s-\rho)^k}\right) \\ &= (-1)^{k-1}(k-1)!\sum_{L(\rho,\chi)=0}\frac{1}{(s-\rho)^k}. \end{aligned} \tag{2.9}$$

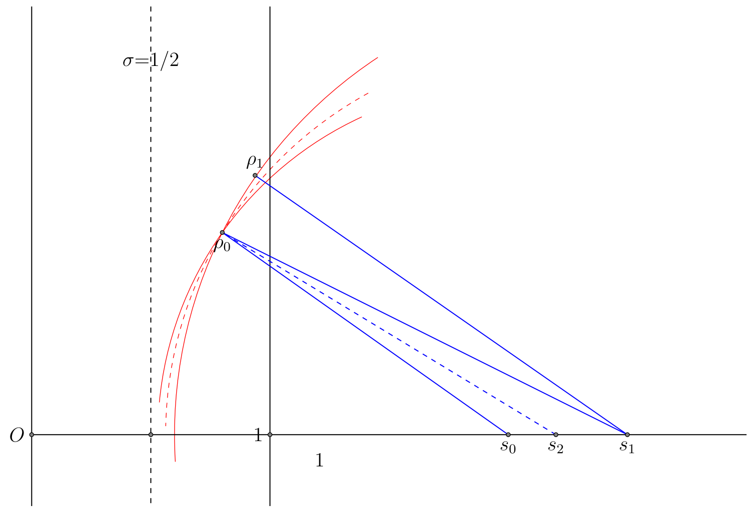

Fig. 1. Construction for identifying the unique ρ_0 at which $l(s)$ is attained for $s \in (s_0-\epsilon, s_0+\epsilon)$.

Let $s > 1/2$ and define

$$l(s) := \min\{|s-\rho|\colon\ L(\rho,\chi)=0\}. \tag{2.10}$$

If the minimum $l(s)$ is attained for a non-trivial zero ρ of $L(s,\chi)$, then since the non-trivial zeros are symmetric with respect to the line $\sigma = 1/2$, we have $\operatorname{Re}\rho \geqslant 1/2$. Let $\tilde{\rho}_0$ be the non-trivial zero of $L(s,\chi)$ with minimum but positive imaginary part, i.e., $\operatorname{Im}\tilde{\rho}_0 = \min\{\operatorname{Im}\rho > 0\colon\ L(\rho,\chi)=0,\ \operatorname{Re}\rho \geqslant 1/2\}$. Write $\tilde{\rho}_0 = \tilde{\beta}_0 + i\tilde{\gamma}_0$. Then for all $s > \tilde{\gamma}_0^2 + 1/4$, we have $s^2 > (s-1/2)^2 + \tilde{\gamma}_0^2 \geqslant |s-\tilde{\rho}_0|^2 \geqslant (l(s))^2$. Define

$$c_\chi := \operatorname{Inf}\{c > 1\colon\ s > c \Rightarrow |s| > l(s)\}. \tag{2.11}$$

The constant c_χ is defined in this way since we want $l(s)$ to be attained at a non-trivial zero of $L(s,\chi)$, as this will allow us to separate the two terms of the series in (2.9) corresponding to this zero and its conjugate, which together will give a dominating term essential in the proof. Note that if $\tilde{\gamma}_0 \leqslant \sqrt{3}/2$, $c_\chi = 1$, otherwise $1 \leqslant c_\chi \leqslant \tilde{\gamma}_0^2 + 1/4$.

Next we show that for any $s \geqslant c_\chi$, there is an $s' \in (s-\epsilon, s+\epsilon)$, $\epsilon > 0$, so that $l(s')$ is attained at a unique non-trivial zero ρ' of $L(s,\chi)$ with $\operatorname{Im}\rho' > 0$.

For any real number $s_0 > c_\chi$, consider the interval $(s_0-\epsilon, s_0+\epsilon) \subset [c_\chi, \infty)$ for some $\epsilon > 0$. Let

$$A := \{\rho'\colon\ \operatorname{Im}\rho' \geqslant 0 \text{ and } |s_0-\rho'| = l(s_0),\ L(\rho',\chi)=0\}, \tag{2.12}$$

that is, A is comprised of all non-trivial zeros on the circle with center s_0 and radius $l(s_0)$. Clearly A is a finite set since $|A| \leqslant N(l(s_0),\chi)$, where $N(T,\chi)$ denotes the number of zeros of $L(s,\chi)$ up to height T. As shown in Fig. 1, let $\rho_0 \in A$ with $\operatorname{Re}\rho_0 = \max\{\operatorname{Re}\rho\colon\ \rho \in A\}$. Then for any $s \in (s_0, s_0+\epsilon)$, $|s-\rho_0| < |s-\rho|$, for all $\rho \in A$, $\rho \neq \rho_0$. Fix one such s, say s_1, so that $s_0 < s_1 < s_0+\epsilon$. Now more than one zeros may lie on the circle with center s_1 and radius $|s_1-\rho_0|$. If there aren't any (apart from ρ_0), then we have constructed s' $(=s_1)$ that we sought. If there are more than one, we select the one among them, say ρ_1, which has the minimum real part, i.e., $\operatorname{Re}\rho_1 = \min\{\operatorname{Re}\rho\colon\ |s_1-\rho_0| = |s_1-\rho|,\ \rho \neq \rho_0,\ L(\rho,\chi)=0\}$. Note that $\operatorname{Im}\rho_1 > \operatorname{Im}\rho_0$, otherwise it will contradict the fact that the minimum $l(s_0)$ is attained at ρ_0.

For any $s \in (s_0, s_1)$, $|s-\rho_0| < |s-\rho_1|$. Now fix one such s, say $s_2 \in (s_0, s_1)$, and find a ρ_2 so that $\operatorname{Re}\rho_2 = \min\{\operatorname{Re}\rho\colon\ |s_2-\rho_0| = |s_2-\rho|,\ \rho \neq \rho_0,\ L(\rho,\chi)=0\}$. Since there are only finitely many zeros in the rectangle $[0,1] \times [\operatorname{Im}\rho_0, \operatorname{Im}\rho_1]$, repeating the argument allows us to find an $s' \in \mathbb{R}$ and $s_0 < s' < s_1 < s_0+\epsilon$, so that ρ_0 is the only non-trivial zero of $L(s,\chi)$ with $\operatorname{Im}\rho_0 \geqslant 0$ and $|s'-\rho_0| = \min\{|s'-\rho|, \operatorname{Im}\rho \geqslant 0 \text{ and } L(\rho,\chi)=0\}$, i.e., the circle with center s' and radius $|s'-\rho_0|$ does not contain any zero other than ρ_0 itself. Note that for any $s \in (s_0, s')$, ρ_0 is the only zero at which $l(s)$ is attained.

Next, let $B = \{\rho'\colon\ \rho' \neq \rho_0,\ |s_0-\rho'| < |s_0-\rho|\}$, where ρ, ρ' are zeros of $L(s,\chi)$. Note that B is also a finite set. Arguing in a similar way as above, we can find a $\tilde{\rho} \in B$ and $s'' \in (s_0, s_0+\epsilon)$ so that for all $s \in (s_0, s'')$, $|s-\tilde{\rho}| \leqslant |s-\rho|$ for $\rho \neq \rho_0$.

Therefore we can find a closed interval $[c,d] \subset (s_0-\epsilon, s_0+\epsilon)$ so that for all $s \in [c,d]$, we have

$$l(s) = |s-\rho_0| = |s-\bar{\rho}_0| < |s-\rho|, \quad \rho \neq \rho_0, \overline{\rho_0}, \tag{2.13}$$

$$|s-\tilde{\rho}| = |s-\bar{\tilde{\rho}}| \leqslant |s-\rho|, \quad \rho \neq \rho_0, \overline{\rho_0}, \tilde{\rho}, \bar{\tilde{\rho}}. \tag{2.14}$$

Now let $s-\rho_0 = r_s e^{i\theta_s}$ for all $c \leqslant s \leqslant d$. Then from (2.9) and the fact that the zeros of $L(s,\chi)$ are symmetric with respect to the real axis, we have

$$\begin{aligned}
F^{(k)}(s,\chi) &= (-1)^{k-1}(k-1)!\left(\frac{1}{(s-\rho_0)^k}+\frac{1}{(s-\bar\rho_0)^k}+\sum_{\rho\neq\rho_0,\bar\rho_0}\frac{1}{(s-\rho)^k}\right)\\
&= (-1)^{k-1}(k-1)!\left(\frac{2}{r_s^k}\cos(k\theta_s)+\sum_{\rho\neq\rho_0,\bar\rho_0}\frac{1}{(s-\rho)^k}\right)\\
&= \frac{(-1)^{k-1}(k-1)!}{r_s^k}\big(2\cos(k\theta_s)+f(s)\big), \qquad (2.15)
\end{aligned}$$

where $f(s) := r_s^k\sum_{\rho\neq\rho_0,\bar\rho_0}\frac{1}{(s-\rho)^k}$ and $k \geqslant 2$. Since the series $\sum_{\rho\neq\rho_0,\bar\rho_0}\frac{1}{(s-\rho)^k}$ converges absolutely for $k \geqslant 1$, $f(s)$ is a differentiable function for $s > 1$. Now,

$$\begin{aligned}
\big|f(s)\big| &\leqslant 2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0\\ \operatorname{Im}\rho\geqslant 0}}\frac{r_s^k}{|s-\rho|^k} = 2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0,\\ \operatorname{Im}\rho\geqslant 0}}\frac{|s-\rho_0|^k}{|s-\rho|^k}\\
&= 2|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0,\\ \operatorname{Im}\rho\geqslant 0}}\frac{1}{|s-\rho|^2}\frac{|s-\rho_0|^{k-2}}{|s-\rho|^{k-2}}\\
&\leqslant 2|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0,\\ \operatorname{Im}\rho\geqslant 0}}\frac{1}{|s-\rho|^2}\frac{|s-\rho_0|^{k-2}}{|s-\tilde\rho|^{k-2}}\\
&\leqslant 2|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0,\\ \operatorname{Im}\rho\geqslant 0}}\frac{1}{|s-\rho|^2}\operatorname{Sup}_{c\leqslant s\leqslant d}\left\{\frac{|s-\rho_0|^{k-2}}{|s-\tilde\rho|^{k-2}}\right\}, \qquad (2.16)
\end{aligned}$$

where in the penultimate step we use (2.14). Let $h(s) := \frac{|s-\rho_0|}{|s-\tilde\rho|}$. Then $h(s)$ is a continuous function on $[c,d]$ and hence attains its supremum on $[c,d]$. Thus there exists an $x \in [c,d]$ such that

$$\eta := \operatorname{Sup}_{c\leqslant s\leqslant d}\left\{\frac{|s-\rho_0|}{|s-\tilde\rho|}\right\} = \frac{|x-\rho_0|}{|x-\tilde\rho|}. \qquad (2.17)$$

Therefore by (2.13), $\eta < 1$. Combining (2.16) and (2.17), we have

$$\big|f(s)\big| \leqslant 2\eta^{k-2}|s-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0\\ \operatorname{Im}\rho\geqslant 0}}\frac{1}{|s-\rho|^2} \leqslant 2\eta^{k-2}|d-\rho_0|^2\sum_{\substack{\rho\neq\rho_0,\bar\rho_0\\ \operatorname{Im}\rho\geqslant 0}}\frac{1}{|c-\rho|^2} \leqslant C_{c,d,\chi}\eta^{k-2}. \qquad (2.18)$$

Note that the constant term depends only on c, d and χ. Hence for sufficiently large k, we have $|f(s)| < 1$. Let $c-\rho_0 = r_c e^{i\theta_c}$ and $d-\rho_0 = r_d e^{i\theta_d}$. Then $\theta_c > \theta_d$. For k large enough, we can write $2\pi < k(\theta_c-\theta_d)$. Since for $s \in [c,d]$, we have $\theta_d \leqslant \theta_s \leqslant \theta_c$, for a sufficiently large k, $\cos(k\theta_s)$ attends all the values in the interval $[-1,1]$. So from (2.16) and (2.18) we conclude that for each large enough k there will be an s in $[c,d] \subset (s_0-\epsilon, s_0+\epsilon)$ so that $F^{(k)}(s,\chi) = 0$. This shows that $\bigcup_{k=1}^{\infty} A_{\chi,k}$ has a non-empty intersection with $(s_0-\epsilon, s_0+\epsilon)$ for any $s_0 > c_\chi$. This completes the proof of the theorem.

Remark. Let χ be a real nonprincipal Dirichlet character. If $L(s,\chi)$ has a Siegel zero, call it β, and if every zero of $L(s,\chi)$ has real part $\leqslant \beta$, then for any $s > 1$, (2.9) implies

$$F^{(k)}(s,\chi) = \frac{(-1)^{k-1}(k-1)!}{(s-\beta)^k}\left(1+\sum_{\substack{\rho\neq\beta\\ L(\rho,\chi)=0}}\left(\frac{s-\beta}{s-\rho}\right)^k\right). \qquad (2.19)$$

Arguing as in the proof of Theorem 1.1, we see that there exists an integer M such that for all $k \geqslant M$, the series in (2.19) is less than 1. This means that for those k, $F^{(k)}(s,\chi)$ maintains the same sign for all $s > 1$. This is why we include the condition that $L(s,\chi) \neq 0$ for $0 < s < 1$ in the hypotheses of Theorem 1.1.

3. Proof of Theorem 1.2

Assume that the Riemann Hypothesis holds for $L(s,\chi)$. Let $\gamma_0 := \operatorname{Im}\rho_0 = \min\{\operatorname{Im}\rho \geqslant 0{:}\ L(\rho,\chi) = 0\}$, where ρ_0, ρ are non-trivial zeros of $L(s,\chi)$. Then $\rho_0 = 1/2 + i\gamma_0$. We show that the function ψ_χ is injective on $[C_\chi, \infty)$, where the constant C_χ will be determined later.

Let $s > c_\chi$, where c_χ is defined in (2.11). Then $l(s) < |s|$ and $l(s) = |s-\rho_0| < |s-\rho|$ for $\rho \neq \rho_0, \bar{\rho}_0$. Let $s-\rho_0 = r_s e^{i\theta_s}$. From (2.15), we have for $k \geqslant 2$,

$$\begin{aligned}
|f(s)| &\leqslant \sum_{\rho\neq\rho_0,\bar{\rho}_0} \frac{r_s^k}{|s-\rho|^k} = |s-\rho_0|^2 \sum_{\rho\neq\rho_0,\bar{\rho}_0} \frac{1}{|s-\rho|^2}\cdot\frac{|s-\rho_0|^{k-2}}{|s-\rho|^{k-2}} \\
&\leqslant |s-\rho_0|^2 \sum_{\rho\neq\rho_0,\bar{\rho}_0} \frac{1}{|s-\rho|^2}\cdot \operatorname{Sup}_\rho \left\{\frac{|s-\rho_0|^{k-2}}{|s-\rho|^{k-2}}\right\} \\
&= |s-\rho_0|^2 \eta_s^{k-2} \sum_{\rho\neq\rho_0,\bar{\rho}_0} \frac{1}{|s-\rho|^2} \\
&= O_{s,\chi}\big(\eta_s^{k-2}\big).
\end{aligned} \tag{3.1}$$

Here in the penultimate step,

$$\eta_s = \operatorname{Sup}_\rho \left\{\frac{|s-\rho_0|}{|s-\rho|}\right\} \leqslant \frac{|s-\rho_0|}{|s-\tilde{\rho}|} < 1,$$

and $\operatorname{Im}\rho_0 < \operatorname{Im}\tilde{\rho} \leqslant \operatorname{Im}\rho$, resulting from (2.13) and (2.14). Combining (2.15) and (3.1), we obtain

$$F^{(k)}(s,\chi) = \frac{2(-1)^{k-1}(k-1)!}{r_s^k}\big(\cos(k\theta_s) + f(s)\big), \tag{3.2}$$

where $f(s) = O_{s,\chi}(\eta_s^{k-2})$.

Next, we show that there are infinitely many k for which $\cos(k\theta_s)$, which we view as the main term, dominate the error term. Since $\eta_s < 1$, for a fixed $s > 1$, we can bound the error term in $(-\epsilon, \epsilon)$ for all sufficiently large k and for all $0 < \epsilon < 1$. Write $\cos(k\theta_s) = \cos(\pi \frac{k\theta_s}{\pi}) = \cos(2\pi \frac{k\theta_s}{2\pi})$ and consider the cases when $\frac{\theta_s}{\pi}$ is rational and $\frac{\theta_s}{2\pi}$ is irrational.

If $\frac{\theta_s}{\pi}$ is a rational number, there are infinitely many $k \in \mathbb{N}$ so that $\frac{k\theta_s}{2\pi}$ is an even integer and hence $\cos(k\theta_s) = 1$.

If $\frac{\theta_s}{\pi}$ is a rational number with odd numerator, then there are infinitely many $k \in \mathbb{N}$, namely the odd multiples of the denominator, so that $\frac{k\theta_s}{2\pi}$ is an odd integer and hence $\cos(k\theta_s) = -1$.

Let $\frac{\theta_s}{\pi} = \frac{2m}{n}$ be a rational number with even numerator and odd denominator. Since $(2m, n) = 1$, there exists an integer $l \in [1, n]$ such that $2ml \equiv 1 \pmod n$. For all $k \equiv l \pmod n$, $2mk \equiv 1 \pmod n$. Therefore for all $k \equiv l \bmod n$, since $2mk$ is even, we have $2mk = (2p+1)n + 1$. Hence there are infinitely many integers k for which $\cos(k\theta_s) = \cos(\pi(2p+1+\frac{1}{n})) = -\cos(\frac{\pi}{n})$.

If $\frac{\theta_s}{2\pi}$ is irrational, then we know from [27] that the sequence $\{\{\frac{k\theta_s}{2\pi}\}\}$ is dense in $[0,1]$, where $\{x\}$ denotes the fractional part of x. (Actually, Kronecker's approximation theorem is sufficient to prove the denseness.) Hence there are infinitely many $k \in \mathbb{N}$ with $\{\frac{k\theta_s}{2\pi}\}$ close to 1 and hence $\cos(k\theta_s) > 1 - \epsilon$ for any given $\epsilon > 0$. Likewise, there are infinitely many $k \in \mathbb{N}$ with $\{\frac{k\theta_s}{2\pi}\}$ close to $\frac{1}{2}$ and hence $\cos(k\theta_s) < -1 + \epsilon$.

Fix s_1 and s_2 such that $c_\chi < s_1 < s_2$. Then $l(s_1) = |s_1 - \rho_0|$ and $l(s_2) = |s_2 - \rho_0|$. Let θ_1 and θ_2 be such that $s_1 - \rho_0 = r_1 e^{i\theta_1}$ and $s_2 - \rho_0 = r_2 e^{i\theta_2}$. Note that $0 < \theta_2 < \theta_1 < \pi/2$. From (3.2), we have

$$F^{(k)}(s_1,\chi) = \frac{2(-1)^{k-1}(k-1)!}{r_1^k}\big(\cos(k\theta_1) + f(s_1)\big), \tag{3.3}$$

$$F^{(k)}(s_2,\chi) = \frac{2(-1)^{k-1}(k-1)!}{r_2^k}\big(\cos(k\theta_2) + f(s_2)\big), \tag{3.4}$$

where $f(s_1) = O_{s_1,\chi}(\eta_{s_1}^{k-2})$ and $f(s_2) = O_{s_2,\chi}(\eta_{s_2}^{k-2})$. Write $\theta_1 = \theta_2 + (\theta_1 - \theta_2)$.

We show that there exist infinitely many integers k such that the terminal rays of $k\theta_1$ and $k\theta_2$ stay away from the y-axis, that $\operatorname{sgn}(\cos k\theta_1) = -\operatorname{sgn}(\cos k\theta_2) \neq 0$, and that $\cos(k\theta_1)$ and $\cos(k\theta_2)$ dominate $f(s_1)$ and $f(s_2)$ in (3.3) and (3.4) respectively. We first determine the signs.

Case 1: If $\frac{\theta_1-\theta_2}{\pi}$ is rational with odd numerator then as we saw before, there are infinitely many positive integers k so that $k\frac{(\theta_1-\theta_2)}{\pi}$ is an odd integer and hence for those $k \in \mathbb{N}$, $\cos(k\theta_1) = \cos(k\theta_2 + \pi) = -\cos(k\theta_2)$.

Case 2: If $\frac{\theta_1-\theta_2}{\pi}$ is rational with even numerator and odd denominator n, there are infinitely many positive integers k so that $k\frac{(\theta_1-\theta_2)}{\pi} = 2p + 1 + 1/n$ for some $p \in \mathbb{N}$ and so $\cos(k\theta_1) = \cos(k\theta_2 + \pi + \pi/n) = -\cos(k\theta_2 + \pi/n)$.

Case 3: If $\frac{(\theta_1-\theta_2)}{2\pi}$ is irrational, there are infinitely many positive integers k so that $\{k\frac{(\theta_1-\theta_2)}{2\pi}\} \in [1/2, 1/2 + \epsilon/2\pi)$, for any given $\epsilon > 0$. So for any δ such that $0 < \delta < \epsilon$, we have $\cos(k\theta_1) = \cos(k\theta_2 + \pi + \delta) = -\cos(k\theta_2 + \delta)$. We can choose ϵ as small as we want and hence $0 < \delta < \epsilon < \pi/n$.

We first show that in Case 2, we have the terminal rays of the angles sufficiently away from the y-axis, with $\cos k\theta_1$ and $\cos k\theta_2$ dominating their corresponding terms $f(s_1)$ and $f(s_2)$. To that end, choose a constant $b_\chi > 1/2$ such that

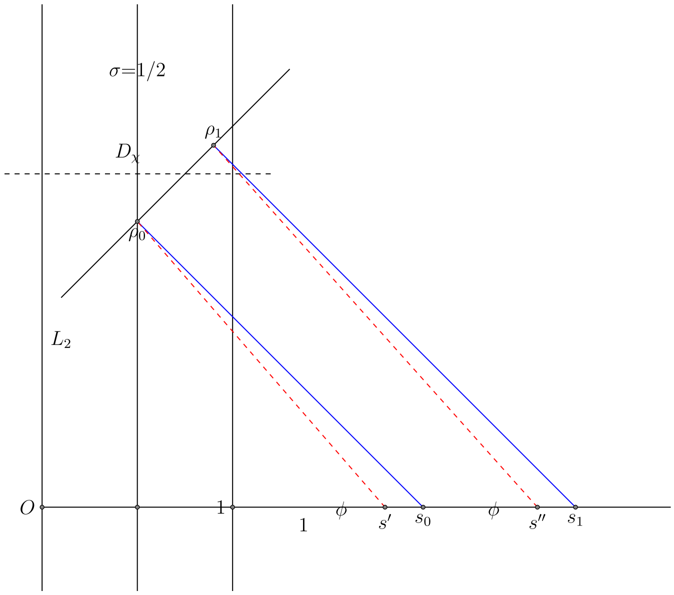

Fig. 2. Constructing the angle $\phi = 2\pi(a + b\sqrt{2})$.

$\tan(\frac{\pi}{100}) = \frac{\gamma_0}{b_\chi - \frac{1}{2}}$, say. If $s - \rho_0 = r_s e^{i\theta_s}$ and $s > b_\chi$, then $0 < \theta_s < \pi/100$. So if we take $b_\chi < s_1 < s_2$, then $0 < \theta_2 < \theta_1 < \pi/100$. Since $\eta_{s_1}, \eta_{s_2} < 1$ there exists an integer K such that $|f(s_1)|, |f(s_2)| < \theta_2/4$ for all $k > K$. As we saw before, for infinitely many integers $k > K + 2$, we have $k\theta_1 = k\theta_2 + \pi + \pi/n$, where n depends on θ_1 and θ_2. We first note that all angles below are considered mod 2π. If $k\theta_2 \in (\pi/2 + \theta_2, \pi)$ then $k\theta_1 \in (-\pi/2 + \theta_2, \pi/2 - \theta_2)$. Thus $\cos(k\theta_1)\cos(k\theta_2) < 0$. Also $|\cos(k\theta_2)| > |\sin(\theta_2)| \geqslant \theta_2/2 > |f(s_2)|$ and $|\cos(k\theta_1)| = |\cos(k\theta_2 + \pi/n)| > |\sin(\theta_2)| \geqslant \theta_2/2 > |f(s_1)|$.

Similarly we see that $|\cos(k\theta_1)| > |f(s_1)|$ and $|\cos(k\theta_2)| > |f(s_2)|$ when $k\theta_2 \in (-\pi/2 + \theta_2, 0)$. If $k\theta_2 \in (0, \pi/2 - \theta_2)$ and $k\theta_1 \in (-\pi, -\pi/2 - \theta_2)$ in this case also $|\cos(k\theta_2)| > |\sin(\theta_2)| \geqslant \theta_2/2 > |f(s_2)|$ and $|\cos(k\theta_1)| = |\cos(k\theta_2 + \pi/n)| > |\sin(\theta_2)| \geqslant \theta_2/2 > |f(s_1)|$. Now let $k\theta_2 \in (0, \pi/2 + \theta_2)$ and $k\theta_1 \in (-\pi/2 - \theta_2, 0)$. Then since $\pi/n < \theta_1 < \pi/100$, it is easy to check that $(k-2)\theta_2 \in (0, \pi/2 - \theta_2)$ and $(k-2)\theta_1 = k\theta_2 + \pi + \pi/n - 2\theta_1 \in (-\pi, -\pi/2 - \theta_2)$. Hence $|\cos(k\theta_1)| > |f(s_1)|$ and $|\cos(k\theta_2)| > |f(s_2)|$. Similarly we have the same conclusion if $k\theta_2 \in (-\pi, -\pi/2 + \theta_2)$ and $k\theta_1 \in (\pi/2 - \theta_2, \pi)$.

Note that since $k\theta_2 + \pi + \pi/n > k\theta_2 + \pi + \delta$, for the values of θ_1 and θ_2 in Case 3 as well, one can similarly prove that $|\cos(k\theta_2)| > |f(s_2)|$ and $|\cos(k\theta_1)| > |f(s_1)|$. So is the case with the values of θ_1 and θ_2 in Case 1.

Let

$$C_\chi = \max\{c_\chi, b_\chi\}. \tag{3.5}$$

Then for any given real numbers s_1 and s_2 such that $C_\chi < s_1 < s_2$, we have shown that there exist infinitely many integers k such that $\cos(k\theta_1)$ and $\cos(k\theta_2)$ have opposite signs and $|\cos(k\theta_1)| > |f(s_1)|$ and $\cos(k\theta_2) > f(s_2)$. This implies that $F^{(k)}(s_1, \chi)$ and $F^{(k)}(s_2, \chi)$ have opposite signs and that in turn proves that the function ψ_χ is injective in $[C_\chi, \infty)$.

We now prove part (b) of Theorem 1.2. Let ρ_0 be the lowest zero of $L(s, \chi)$ above the real axis (so ρ_0 is not a real number). Let L_1 be the line passing through ρ_0 and perpendicular to the line which passes through ρ_0 and C_χ, where C_χ is defined in (3.5). Let $(1, D_\chi)$ be the point of intersection of the lines $\sigma = 1$ and L_1. We first show that if there is only one zero ρ_1 with $\operatorname{Im}\rho_1 \geqslant D_\chi$ off the critical line $\sigma = 1/2$, then this contradicts the injectivity of ψ_χ on $[C_\chi, \infty)$.

Without loss of generality, let $\operatorname{Re}\rho_1 > 1/2$. As shown in Fig. 2, let L_2 be the line passing through ρ_0 and ρ_1. Let s_0 and s_1 be the points of intersection of the real axis with the lines perpendicular to L_2 and passing through ρ_0 and ρ_1 respectively. Clearly $s_1 > s_0 > C_\chi$. Note that by our construction, $l(s_0) = |s_0 - \rho_0|$ and $l(s_1) = |s_1 - \rho_1|$, where $l(s)$ is defined in (2.10), and there exists a θ such that $(s_0 - \rho_0) = r_{s_0} e^{i\theta}$ and $(s_1 - \rho_1) = r_{s_1} e^{i\theta}$. From the proof of Theorem 1.1, we know that there exists an $\epsilon > 0$ so that $l(s) = |s - \rho_0|$ for all $s \in (s_0 - \epsilon, s_0 + \epsilon)$ and $l(s) = |s - \rho_1|$ for all $s \in (s_1 - \epsilon, s_1 + \epsilon)$. Without loss of generality, we can assume that $s_0 + \epsilon < s_1 - \epsilon$. Therefore, there exists a $\delta > 0$ such that $\theta_s \in (\theta - \delta, \theta + \delta)$, where $s - \rho_0 = r_s e^{i\theta_s}$ and $l(s) = |s - \rho_0|$ for all $s \in (s_0 - \epsilon, s_0 + \epsilon)$, and such that $\theta_s \in (\theta - \delta, \theta + \delta)$, where $s - \rho_1 = r_s e^{i\theta_s}$ and $l(s) = |s - \rho_1|$ for all $s \in (s_1 - \epsilon, s_1 + \epsilon)$.

Since the sequence $\{\{n\sqrt{2}\}\}$ is dense in $[0, 1)$, and $\{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor$, there exist an integer a and an integer $b \neq 0$ such that $a + b\sqrt{2} \in (\frac{\theta - \delta}{2\pi}, \frac{\theta + \delta}{2\pi})$. Let $\phi = 2\pi(a + b\sqrt{2})$, $s' \in (s_0 - \epsilon, s_0 + \epsilon)$ and $s'' \in (s_1 - \epsilon, s_1 + \epsilon)$ be such that $s' - \rho_0 = r_{s'} e^{i\phi}$ and $s'' - \rho_1 = r_{s''} e^{i\phi}$. Therefore,

$$F^{(k)}(s', \chi) = \frac{2(-1)^{k-1}(k-1)!}{r_{s'}^k}\left(\cos(k\phi) + f(s')\right), \tag{3.6}$$

$$F^{(k)}\left(s'', \chi\right) = \frac{2(-1)^{k-1}(k-1)!}{r_{s''}^{k}}\left(\cos(k\phi) + f\left(s''\right)\right), \tag{3.7}$$

where $|f(s')| = O(\eta_{s'}^{k-2})$ and $|f(s'')| = O(\eta_{s''}^{k-2})$. Let $\eta = \min\{\eta_{s'}, \eta_{s''}\}$. Then $|f(s')|, |f(s'')| \leqslant C_{s',s''}\eta^{k-2}$ for some constant $C_{s',s''}$.

We next show that there exist positive constants $C_{a,b}$ and $K_{a,b}$ so that

$$\left|4k(a + b\sqrt{2}) + r\right| > \frac{C_{a,b}}{k}, \tag{3.8}$$

for any integers r and k, with $k > K_{a,b}$. Let $|4k(a + b\sqrt{2}) + r| \leqslant 1$. Then,

$$\left|4k(a - b\sqrt{2}) + r\right| \leqslant \left|4k(a + b\sqrt{2}) + r\right| + 8k|b|\sqrt{2} \leqslant 1 + 8k|b|\sqrt{2} < \frac{k}{C_{a,b}}. \tag{3.9}$$

Therefore for $k \geqslant 2$,

$$\left|4k(a + b\sqrt{2}) + r\right|\frac{k}{C_{a,b}} > \left|4k(a - b\sqrt{2}) + r\right|\left|4k(a + b\sqrt{2}) + r\right| = \left|(4ka + r)^2 - 2(4kb)^2\right| \geqslant 1, \tag{3.10}$$

since $b \neq 0$. If $|4k(a + b\sqrt{2}) + r| \geqslant 1$, then of course, there exists a $K_{a,b}$, such that for $k > K_{a,b}$, we have $|4k(a + b\sqrt{2}) + r| > \frac{C_{a,b}}{k}$. Hence in conclusion, for a large positive integer N and for all $k > N$, if we choose m so that $|4k(a + b\sqrt{2}) \pm 1 \pm 4m| < 1$, we have

$$\begin{aligned} |\cos k\phi| = \left|\sin\frac{\pi}{2}\left(4k(a + b\sqrt{2}) \pm 1 \pm 4m\right)\right| &\geqslant \sin\left(\frac{\pi C_{a,b}}{2k}\right) \\ &\geqslant \frac{\pi C_{a,b}}{4k} \\ &> C_{s',s''}\eta^{k-2}. \end{aligned} \tag{3.11}$$

Therefore for the above mentioned s' and s'' such that $s' \neq s''$, and for all $k > N$, $F^{(k)}(s', \chi)$ and $F^{(k)}(s'', \chi)$ have the same sign. This contradicts the injectivity of ψ_χ on $[C_\chi, \infty)$. Now if there is more than one zero ρ with $\operatorname{Im}\rho \geqslant D_\chi$ off the critical line, then we can choose the zero ρ_1 with the following properties:

i) The angle between the positive x-axis and the line L passing through the zeros ρ_0 and ρ_1 is smaller than the angle between the positive x-axis and the line passing through the zeros ρ_0 and $\rho \neq \rho_1$ and,

ii) $\operatorname{Im}\rho_1 = \min\{\operatorname{Im}\rho \geqslant D_\chi\colon\ \rho \text{ lies on the line } L\}$.

Then we can proceed similarly as above and again get a contradiction. Hence, all the zeros above the line $t = D_\chi$ lie on the critical line $\sigma = 1/2$. This completes the proof.

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