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On a class of starlike functions defined in a halfplane

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Abstract. Let $D = \{z : \operatorname{Re} z > 0\}$ and let $S^*(D)$ be the class of univalent functions normalized by the conditions $\lim_{D \ni z \to \infty}(f(z) - z) = a$, a a finite complex number, $0 \notin f(D)$, and mapping D onto a domain $f(D)$ starlike with respect to the exterior point $w = 0$. Some estimates for $|f(z)|$ in the class $S^*(D)$ are derived. An integral formula for f is also given.

Introduction. The theory of functions regular and univalent in the unit disc is well developed, and various results concerning different subclasses of univalent functions have been obtained. A natural problem is to obtain specific properties for univalent functions defined in other domains, for example in a halfplane. The present paper is dedicated to such type of problems.

We start with the definition of some specific classes of functions which play the role of the well known classes of functions defined in the unit disc.

Let $\widetilde{H} = \widetilde{H}(D)$ be the class of all functions that are regular in the right halfplane $D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Since the point at infinity lies on the boundary of D we can distinguish different subclasses of $\widetilde{H}$ using the asymptotic behavior of the functions.

By $H = H(D)$ we shall denote the subclass consisting of all functions $f(z)$ in $\widetilde{H}$ with the normalization

$$\lim_{D \ni z \to \infty}(f(z) - z) = a,$$

where a is an arbitrary finite complex number.

By $S^* = S^*(D)$ we shall denote the subclass of those nowhere vanishing functions $f(z)$ in H for which

$$\operatorname{Re} \frac{f'(z)}{f(z)} > 0, \qquad z \in D.$$

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We also need the following two classes of functions with positive real part in D:

$$\mathcal{P} = \{p \in H : \operatorname{Re} p(z) > 0,\ z \in D\},$$

$$\mathcal{Q} = \{q \in \widetilde{H} : \operatorname{Re} q(z) > 0,\ z \in D,\ \lim_{D \ni z \to \infty} z(q(z) - 1/z) = 0\}.$$

Preliminary results

Lemma 1 [1]. *If $p \in \mathcal{P}$ then*

$$\operatorname{Re} p(z) \ge \operatorname{Re} z, \qquad z \in D,$$

where equality is attained only for the functions

$$p(z) = z + it, \qquad t \in \mathbb{R}.$$

It was proved [1] that functions in S^* map the halfplane D onto a domain which is starlike with respect to the exterior point $w = 0$, i.e. if $w \in f(D)$, then $tw \in f(D)$ for every $t \ge 1$.

Further, we have [2]:
— if $p \in \mathcal{P}$ then $q(z) = 1/p(z)$ belongs to $\mathcal{Q}$;
— if $f \in S^*$ then $q(z) = f'(z)/f(z)$ belongs to $\mathcal{Q}$.

Lemma 2 [2]. *If $q \in \mathcal{Q}$ then*

$$\left| q(z) - \frac{1}{2\operatorname{Re} z} \right| \le \frac{1}{2\operatorname{Re} z}, \qquad z \in D,$$

and in particular

$$0 \le \operatorname{Re} q(z) \le |q(z)| \le \frac{1}{\operatorname{Re} z}, \qquad z \in D,$$

$$|\operatorname{Im} q(z)| \le \frac{1}{2\operatorname{Re} z}, \qquad z \in D.$$

The results are sharp for the functions

$$q_t(z) = \frac{1}{z + it}, \qquad t \text{ real}.$$

Lemma 3 [2]. *A function $f \in H$, with $f(z) \ne 0$ for $z \in D$, belongs to S^* if and only if it may be represented in the form*

$$f(z) = z \exp \int_{\infty}^{z} (q(\zeta) - 1/\zeta)\, d\zeta \tag{1}$$

for some $q \in \mathcal{Q}$.

If we fix $z_0 \in D$ we obtain another representation of functions in S^*.

THEOREM 1. *Let* z_0 *be fixed*, $\operatorname{Re} z_0 > 0$. *For every* $f \in S^*$ *there exists* $q \in \mathcal{Q}$ *such that*

$$f(z) = f(z_0) \exp \int_{z_0}^{z} q(\zeta)\, d\zeta. \tag{2}$$

Proof. From (1) we have

$$f(z_0) = z_0 \exp \int_{\infty}^{z_0} (q(\zeta) - 1/\zeta)\, d\zeta, \qquad q \in \mathcal{Q}. \tag{3}$$

Dividing (1) by (3) we obtain

$$\begin{aligned}\frac{f(z)}{f(z_0)} &= \frac{z}{z_0} \exp\Big(\int_{z_0}^{z} (q(\zeta) - 1/\zeta)\, d\zeta - \int_{\infty}^{z_0} (q(\zeta) - 1/\zeta)\, d\zeta\Big)\\ &= \frac{z}{z_0} \exp \int_{z_0}^{z} (q(\zeta) - 1/\zeta)\, d\zeta = \frac{z}{z_0} \exp\Big(\int_{z_0}^{z} q(\zeta)\, d(\zeta) - \log \frac{z}{z_0}\Big)\\ &= \exp \int_{z_0}^{z} q(\zeta)\, d\zeta.\end{aligned}$$

This proves the theorem.

Estimates in S^*. It is clear that the integrals in (1) and (2) do not depend on the curves joining ∞ to z and z_0 to z respectively. Unfortunately, the estimates we shall obtain depend essentially on the path of integration. It remains an open problem to choose the best one.

Fix $z_0 \in D$. We start with some estimates obtained by integration along the straight-line segment joining z_0 and a variable point z.

LEMMA 4. *Let* $q \in \mathcal{Q}$. *For* $z, z_0, \zeta \in D$

$$\frac{\operatorname{Re}(z - z_0) - |z - z_0|}{2 \operatorname{Re} \zeta} \le \operatorname{Re}\{(z - z_0) q(\zeta)\} \le \frac{\operatorname{Re}(z - z_0) + |z - z_0|}{2 \operatorname{Re} \zeta},$$

$$\frac{\operatorname{Im}(z - z_0) - |z - z_0|}{2 \operatorname{Re} \zeta} \le \operatorname{Im}\{(z - z_0) q(\zeta)\} \le \frac{\operatorname{Im}(z - z_0) + |z - z_0|}{2 \operatorname{Re} \zeta}.$$

Proof. The assertion is an immediate consequence of Lemma 2.

THEOREM 2. *Let* $z_0 \in D$ *be fixed and let* $f \in S^*(D)$. *Then*

$$\begin{aligned}\Big(\frac{\operatorname{Re} z}{\operatorname{Re} z_0}\Big)^{(1 - |z - z_0| / \operatorname{Re}(z - z_0))/2} &\le \Big|\frac{f(z)}{f(z_0)}\Big|\\ &\le \Big(\frac{\operatorname{Re} z}{\operatorname{Re} z_0}\Big)^{(1 + |z - z_0| / \operatorname{Re}(z - z_0))/2} \qquad \textit{if } \operatorname{Re}(z - z_0) \ne 0,\end{aligned} \tag{4}$$

$$(5)\quad \exp\frac{-|\operatorname{Im}(z-z_0)|}{2\operatorname{Re}z_0}\le\left|\frac{f(z)}{f(z_0)}\right|\le\exp\frac{|\operatorname{Im}(z-z_0)|}{2\operatorname{Re}z_0}\qquad \textit{if } \operatorname{Re}(z-z_0)=0.$$

P r o o f. We use (2) integrating along the straight-line segment $[z_0, z]$, i.e. $\zeta = z_0 + (z-z_0)t$, $0 \le t \le 1$. By Lemma 4 we obtain

$$\begin{aligned}\left|\frac{f(z)}{f(z_0)}\right| &= \exp\operatorname{Re}\int_{z_0}^{z} q(\zeta)\,d\zeta = \exp\operatorname{Re}\int_0^1 (z-z_0)q(\zeta)\,dt\\ &\le \exp\int_0^1 \frac{\operatorname{Re}(z-z_0)+|z-z_0|}{2(\operatorname{Re}z_0+t\operatorname{Re}(z-z_0))}\,dt\\ &= \begin{cases}\left(\dfrac{\operatorname{Re}z}{\operatorname{Re}z_0}\right)^{(1+|z-z_0|/\operatorname{Re}(z-z_0))/2} & \text{if}\quad \operatorname{Re}(z-z_0)\ne 0,\\ \exp\dfrac{|\operatorname{Im}(z-z_0)|}{2\operatorname{Re}z_0} & \text{if}\quad \operatorname{Re}(z-z_0)=0.\end{cases}\end{aligned}$$

The left-hand inequalities in (4) and (5) are analogously obtained.

R e m a r k. If $\operatorname{Im} z = \operatorname{Im} z_0$ and $\operatorname{Re} z > \operatorname{Re} z_0$, then $|z-z_0| = \operatorname{Re}(z-z_0)$. Hence for $f \in S^*$ and for such z we have

$$1 \le \left|\frac{f(z)}{f(z_0)}\right| \le \frac{\operatorname{Re}z}{\operatorname{Re}z_0}.$$

This estimate is sharp for $f(z) \equiv z$ and $\operatorname{Im} z = \operatorname{Im} z_0 = 0$.

THEOREM 3. *Let $z_0 \in D$ be fixed and let $f \in S^*(D)$. Then for every $z \in D$ we have*

$$(6)\quad \arg\frac{f(z)}{f(z_0)} \ge \begin{cases}\dfrac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re}(z-z_0)}\log\dfrac{\operatorname{Re}z}{\operatorname{Re}z_0} & \textit{if } \operatorname{Re}(z-z_0)\ne 0,\\ \dfrac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re}z_0} & \textit{if } \operatorname{Re}(z-z_0)=0,\end{cases}$$

and

$$(7)\quad \arg\frac{f(z)}{f(z_0)} \le \begin{cases}\dfrac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re}(z-z_0)}\log\dfrac{\operatorname{Re}z}{\operatorname{Re}z_0} & \textit{if } \operatorname{Re}(z-z_0)\ne 0,\\ \dfrac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re}z_0} & \textit{if } \operatorname{Re}(z-z_0)=0,\end{cases}$$

where we choose the branch of $\log(f(z)/f(z_0))$ *which is zero for* $z = z_0$.

P r o o f. We use (2) integrating along the straight-line segment $[z_0, z]$.

By Lemma 4 we obtain

$$\arg \frac{f(z)}{f(z_0)} = \operatorname{Im} \int_{z_0}^{z} q(\zeta)\, d\zeta = \int_0^1 \operatorname{Im}\{(z-z_0)q(z_0+t(z-z_0))\}$$
$$\geq \int_0^1 \frac{\operatorname{Im}(z-z_0)-|z-z_0|}{2(\operatorname{Re} z_0 + t\operatorname{Re}(z-z_0))}\, dt.$$

Thus for $\operatorname{Re}(z-z_0) \neq 0$ we have

$$\arg \frac{f(z)}{f(z_0)} \geq \frac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re}(z-z_0)} \log \frac{\operatorname{Re} z}{\operatorname{Re} z_0},$$

and for $\operatorname{Re}(z-z_0)=0$ we have

$$\arg \frac{f(z)}{f(z_0)} \geq \frac{\operatorname{Im}(z-z_0)-|z-z_0|}{2\operatorname{Re} z_0} = \begin{cases} 0 & \text{if } \operatorname{Im}(z-z_0) \geq 0, \\ -\dfrac{\operatorname{Im}(z-z_0)}{\operatorname{Re} z_0} & \text{if } \operatorname{Im}(z-z_0) < 0. \end{cases}$$

This proves (6).

On the other hand,

$$\arg \frac{f(z)}{f(z_0)} = \int_0^1 \operatorname{Im}\{(z-z_0)q(z_0+t(z-z_0))\}\, dt$$
$$\leq \int_0^1 \frac{\operatorname{Im}(z-z_0)+|z-z_0|}{2(\operatorname{Re} z_0 + t\operatorname{Re}(z-z_0))}\, dt.$$

Thus for $\operatorname{Re}(z-z_0) \neq 0$ we have

$$\arg \frac{f(z)}{f(z_0)} \leq \frac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re}(z-z_0)} \log \frac{\operatorname{Re} z}{\operatorname{Re} z_0},$$

and for $\operatorname{Re}(z-z_0)=0$ we have

$$\arg \frac{f(z)}{f(z_0)} \geq \frac{\operatorname{Im}(z-z_0)+|z-z_0|}{2\operatorname{Re} z_0} = \begin{cases} -\dfrac{\operatorname{Im}(z-z_0)}{\operatorname{Re} z_0} & \text{if } \operatorname{Im}(z-z_0) > 0, \\ 0 & \text{if } \operatorname{Im}(z-z_0) \leq 0. \end{cases}$$

This proves (7). The proof of Theorem 3 is complete.

Remark. For $\operatorname{Re}(z-z_0)=0$, $\operatorname{Im} z > \operatorname{Im} z_0$ we have

$$\arg \frac{f(z)}{f(z_0)} = \arg f(z) - \arg f(z_0) > 0,$$

which means that $\arg f(z_0+it)$ is increasing with t. This agrees with the definition of a starlike function in the halfplane D.

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