EXTREMAL UNIVALENT POLYNOMIALS SUBORDINATING THE KOEBE FUNCTION

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Abstract. We prove a surprising relation between univalent polynomials constructed by Suffridge in 1969 and positive trigonometric polynomials discovered by Fejér in 1915. This helps us to establish a kind of distortion result about univalent polynomials with real coefficients and to respond, at least partially, a question of Greiner and Ruscheweyh.

1. Introduction

Let $D = \{z \ : \ |z| < 1\}$ be the unit disc in the complex plane and $\mathcal{A}(D)$ be the set of analytic functions in D. A function $f \in \mathcal{A}(D)$ is univalent in D if $f(z_1) \neq f(z_2)$ whenever $z_1, z_2 \in D, \ z_1 \neq z_2$. Andriecskii and Ruscheweyh [2] proved the following result about polynomial approximation to conformal maps of D.

Theorem A. *There exists a constant $c > 0$ such that, for each $f(z)$ univalent in D, there exists a sequence of polynomials $p_n(z)$, all univalent in D with $p_n(0) = f(0)$, such that*

$$f(\rho_n D) \subset p_n(D) \subset f(D), \quad \rho_n = 1 - c/n, \tag{1.1}$$

for every $n > 2c$.

If $f, g \in \mathcal{A}(D)$, the function g is called subordinate to f when there exists $\varphi \in \mathcal{A}(D)$, such that $|\varphi(z)| \leq |z|$ for any $z \in D$ and $f = g \circ \varphi$. If g is subordinate to f we write $g \prec f$. Since, when f is univalent, the subordination $g \prec f$ is equivalent to the fact that $g(0) = f(0)$ and $g(D) \subset f(D)$ hold simultaneously, then (1.1) can be rewritten in the form $f_{\rho_n} \prec p_n \prec f$, with $f_\rho(z) := f(\rho z)$. It turns out than $1/n$ is the correct order of approximation by subordinate polynomials because Greiner [12] proved that $c \leq 73$ and Greiner and Ruscheweyh [13] provided an example which shows that $c \geq \pi$. In order to do this Greiner and Ruscheweyh approximated the Koebe function $k(z) = z/(1-z)^2$ by univalent polynomials which are slight modifications of polynomials constructed by Suffridge [17].

Suffridge studied the classes of univalent polynomials

$$s_n(j; z) = \sum_{k=1}^{n} \frac{n-k+1}{n} \frac{\sin kj\pi/(n+1)}{\sin j\pi/(n+1)} z^k, \quad j \in \mathbb{N}.$$

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He established various extremal properties which show that $s_n(z) := s_n(1;z)$ are "good" approximations of $k(z)$. For example, they obey an "asymptotic Koebe 1/4-theorem", namely,

$$\inf_{z\in\partial D} |s_n(z)| \longrightarrow 1/4 \quad \text{as} \quad n\to\infty,$$

and, for each $n\in\mathbb{N}$, the above infimum is attained for $z=-1$.

The convolution (or Hadamard product) of

$$f(z)=\sum_{k=0}^{\infty} a_k z^k, \quad \text{and} \quad g(z)=\sum_{k=0}^{\infty} b_k z^k,$$

is defined by

$$(f*g)(z)=\sum_{k=0}^{\infty} a_k b_k z^k.$$

In [13], the polynomials

$$\lambda_n(z)=1+\frac{\cot(\pi/(2n+2))}{2n+2}\sum_{k=1}^{n}\frac{n+1-k}{k}\sin\frac{k\pi}{n+1}z^k$$

were considered. It is easily seen that $(k*\lambda_n)(z)=z\lambda_n'(z)$ and this polynomial is a constant multiple of $s_n(z)$, where the constant of normalization is chosen in such a way that $(k*\lambda_n)(-1)=-1/4$. Then, it was pointed out in [13], that $k_{e_n}\prec k*\lambda_n\prec k$ for

$$e_n=\frac{1-\sin(\pi/(2n+2))}{1+\sin(\pi/(2n+2))}. \tag{1.2}$$

Since $k_\varepsilon(D)\not\subset(k*\lambda_n)(D)$ for every $\varepsilon>e_n$, then $c\geq\pi$ at least when one considers subordination of starlike functions. Recall that $f(z)$ is starlike (with respect to the origin) if f is univalent in D and, together with any of its points w, the image $f(D)$ contains the entire segment $\{tw\ :\ 0\leq t\leq 1\}$. This result motivated Greiner and Ruscheweyh to formulate the following

Conjecture A. *Let f be a univalent mapping from D onto some domain $\Omega\subset\mathbb{C}$, starlike with respect to the origin. Then $f*\lambda_n$ is a univalent polynomial of degree n with*

$$f_{e_n}\prec f*\lambda_n\prec f, \tag{1.3}$$

where e_n is defined by (1.2)*. If f is a rotation of the Koebe function, then e_n cannot be replaced by any greater number.*

Being an extremal problem, with exact constants, for univalent functions, the entire statement of the conjecture seems rather hard to prove. The modest task of this paper is to establish another extremal property of $s_n(z)$. Our result may be interpreted as a partial affirmative answer to the last statement about the subordination of the Koebe function. Let $\mathcal{S}$ and $\mathcal{S}_n(\mathbb{R})$ be the classes of normalized univalent functions and polynomials with real coefficients of degree n,

$$\mathcal{S}=\{f(z)=z+\sum_{k=2}^{\infty}\alpha_k z^k\ :\ \alpha_k\in\mathbb{C},\ f \text{ is univalent in } D\},$$

$$\mathcal{S}_n(\mathbb{R})=\{p_n(z)=z+\sum_{k=2}^{n}\gamma_k z^k\ :\ p_n\in\mathcal{S},\ \gamma_k\in\mathbb{R}\}.$$

Theorem 1. *For every $p_n \in \mathcal{S}_n(\mathbb{R})$*

$$-\left(\cot\frac{\pi}{2n+2}\right)^2 p_n(1) \le p_n(-1) \le -\left(\tan\frac{\pi}{2n+2}\right)^2 p_n(1). \tag{1.4}$$

Moreover, equality in the right-hand side inequality (1.4) *is attained if and only if $p_n(z) = s_n(z)$ and the left-hand side inequality* (1.4) *reduces to equality if and only if $p_n(z) = -s_n(-z)$.*

It might be worth mentioning that $s_n(-z) = -s_n(n, z)$. Theorem 1 implies the following:

Corrollary 1. *For every polynomial of the form*

$$p_n(z) = \gamma_1 z + \gamma_2 z^2 + \cdots + \gamma_n z^n, \quad \text{with } \gamma_k \in \mathbb{R} \text{ for } k = 1, \dots, n, \text{ and } \gamma_1 > 0,$$

that is univalent in D and normalized by $p_n(-1) = -1/4$,

$$\left(\tan\frac{\pi}{2n+2}\right)^2 \le p_n(1) \le \left(\cot\frac{\pi}{2n+2}\right)^2. \tag{1.5}$$

Moreover, equality in the right-hand side inequality (1.5) *is attained if and only if $p_n(z) = -s_n(z)/(4s_n(-1))$ and the left-hand side inequality* (1.5) *reduces to equality if and only if $p_n(z) = -s_n(-z)/(4s_n(1))$.*

As another interesting consequence we conclude that the last statement of Conjecture A is true at least when one considers subordination of $k(z)$ by polynomials with real coefficients.

Corrollary 2. *If p_n is a polynomial with real coefficients that is univalent in D and satisfies $p_n(0) = 0$ and*

$$k_{\rho_n} \prec p_n \prec k,$$

then $\rho_n \le e_n$. Moreover, $\rho_n = e_n$ is attained if $p_n = k * \lambda_n$.

2. Univalent polynomials and positive trigonometric sums

The main tool in the proof is a new relation between $\mathcal{S}_n(\mathbb{R})$ and the nonnegative trigonometric polynomials with real coefficients of certain order $n \in \mathbb{N}$,

$$\mathcal{T}_n^+ = \{\tau_n(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta) \ : \ a_k, b_k \in \mathbb{R},\ \tau_n(\theta) \ge 0,\ \theta \in \mathbb{R}\}.$$

Before we state it, we afford a short review about results of this nature. The interest in univalent functions and polynomials and in the positive trigonometric sums grew in the beginning of the twentieth century. It did not take a long time before the natural and deep interplay between them was discovered. The main motivation for the study of the positive trigonometric sums was Fejér's 1900 proof [7] of the uniform convergence of Cesaro means of the Fourier series and his interest in the Gibbs' phenomenon. Fejér himself observed that the sequence of convolutions $K_n * f(x) = (1/(2\pi)) \int_{-\pi}^{\pi} f(x-\theta) K_n(\theta) d\theta$ of a 2π-periodic continuous function f with nonnegative cosine polynomials $K_n(\theta)$ of order n converges uniformly to f provided the coefficient a_0 of $K_n(\theta)$ is equal to one and they converge uniformly to zero in the compact subsets of $[-\pi, \pi]$ that do not contain the origin. The sequences of cosine polynomials with these properties are called positive summability kernels. A vast number of such kernels were constructed already in the first three decades of twentieth century. To the best of our knowledge Alexander was the first to investigate subclasses of univalent functions studying their geometric properties. In 1915 he wrote the fundamental paper [1] illustrating his ideas by examples involving univalent polynomials. In the beginning of the thirties Dieudonné [5] provided a necessary and sufficient condition for an algebraic polynomial to be univalent. It appears as Theorem B below and, as is seen, already reveals the intimate connection of univalent polynomials with the trigonometric sums. Approximately at the same time Fejér [9, 10] initiated the study of the so-called "vertically convex" functions and his ideas took its final shape in a joint paper of Fejér and Szegő [11] in 1951. The main result in [11] is another interesting connection between nonnegative trigonometric polynomials and univalent algebraic polynomials. In 1958 Pólya and Schoenberg [16] studied the geometric properties of convolutions of univalent functions with the celebrated de la Vallée Poussin positive summability kernel. We stop with the review and refer to the papers [3, 6, 14, 15] for further information.

Our initial interest in the described interplay was to construct new summability kernels through certain extremal univalent polynomials. Such an attempt was made by Bertoni [4], where the Suffridge polynomials $s_n(z)$ were taken as a natural source. It was proved in [4] that

$$K_n(\theta) = \frac{2n\ (\sin \pi/(n+1))^2}{n+1} \frac{\Im[s_n(e^{i\theta})]}{\sin\theta}, \tag{2.1}$$

is a positive summability kernel. Moreover, it turned out that the order of approximation of the corresponding convolutions for any 2π-periodic function f is $\omega(f, 1/n)$, where $\omega(f,\delta)$ is the modulus of continuity of f. Thus another proof of the classical theorem of Jackson in Approximation Theory was furnished.

It is clear that, if a $p_n \in \mathcal{S}_n(\mathbb{R})$, then the trigonometric polynomial formed by the imaginary part of $p_n(\exp(i\theta))$ is nonnegative when $\theta \in [0,\pi]$. A formal proof may be given by the following characterization of the univalent polynomials due to Dieudonné [5].

Theorem B. *The polynomial* $\sum_{k=1}^{n} \gamma_k z^k$ *is univalent in* D *if and only if*

$$\sum_{k=1}^{n} \gamma_k z^{k-1} \frac{\sin k\theta}{\sin\theta} \neq 0 \text{ for every } |z|<1, \text{ and } 0 \le \theta \le \pi.$$

Lemma 1. *If*

$$p_n(z) = z + \sum_{k=2}^{n} \gamma_k z^k \ \in \mathcal{S}_n(\mathbb{R}), \tag{2.2}$$

then the cosine polynomial

$$c_{n-1}(\theta) = a_0 + 2\sum_{k=1}^{n-1} a_k \cos(k\theta),$$

where

$$\gamma_k = a_{k-1} - a_{k+1}, \quad k = 1, \dots, n, \tag{2.3}$$

with $\gamma_1 := 1$, $a_n := 0$ *and* $a_{n+1} := 0$, *is nonnegative for every* $\theta \in \mathbb{R}$.

Proof. Suppose that $p_n \in \mathcal{S}_n(\mathbb{R})$ is of the form (2.2) and let $\theta \in (0,\pi)$ be fixed. Then, by Theorem B,

$$p(\theta,z) = 1 + \gamma_2 \frac{\sin(2\theta)}{\sin\theta} z + \cdots + \gamma_n \frac{\sin(n\theta)}{\sin\theta} z^{n-1}$$

does not vanish when $|z| < 1$, and in particular for real values of $z \in [0,1)$. Thus $p(\theta,z)$ does not change sign for $z \in [0,1)$. Since $p(\theta,0) = 1$, then $p(\theta,1) \geq 0$. On the other hand $p(\theta,1)$ is a cosine polynomial of order $n-1$,

$$c_{n-1}(\theta) = p(\theta,1) = a_0 + 2\sum_{k=1}^{n-1} a_k \cos(k\theta).$$

The relation between the coefficients γ_k and a_k in the two expansions of $p(\theta,1)$ is easily obtained. It is exactly as given in (2.3). □

3. Proof of the main results

In order to prove Theorem 1, we need some additional results. The first one is a technical Lemma which is due to Suffridge [17] who proved that the representation

$$s_n(j;z) = \frac{n+1}{2n(\cos\theta - \cos(j\pi/(n+1)))} + i\frac{\sin\theta(1-(-1)^j e^{i(n+1)\theta})}{2n(\cos\theta - \cos(j\pi/(n+1)))^2}$$

holds for every $n, j \in \mathbb{N}$, $1 \leq j \leq n$ and for each $\cos\theta \neq \cos(j\pi/(n+1))$. Then we obtain:

Lemma 2. *The representation*

$$\Im\left(s_n(e^{i\theta})\right) = \frac{\sin\theta(1+\cos((n+1)\theta)}{2n(\cos\theta - \cos\pi/(n+1))^2} \tag{3.1}$$

holds for every $n \in \mathbb{N}$.

The second fact we shall need is classical. After establishing the general representation of nonnegative trigonometric polynomials as squares of modulae of complex polynomials, with arguments varying on the unit circumference, jointly with Riesz, Fejér was interested in finding such trigonometric sum with certain extremal properties, especially those whose coefficients attain extremal values. One of the first problems of this nature, solved by Fejér is the one we need. He proved in [8] that the inequalities

$$-\cos\frac{\pi}{n+1} \leq \frac{a_1}{a_0} \leq \cos\frac{\pi}{n+1} \tag{3.2}$$

for the coefficients a_0 and a_1 of any nonnegative cosine polynomial of order $n-1$,

$$a_0 + 2\sum_{k=1}^{n-1} a_k \cos(k\theta),$$

holds and equality in the right-hand side inequality (3.2) is attained only for the positive constant multiples of the cosine polynomial

$$F^{+}_{n-1}(\theta) = 1 + \frac{2}{n+1}\sum_{k=1}^{n-1}\left((n-k)\cos\frac{k\pi}{n+1} + \frac{\sin((k+1)\pi/(n+1))}{\sin(\pi/(n+1))}\right)\cos k\theta. \tag{3.3}$$

Similarly, equality in the left-hand side inequality (3.2) is attained only for positive constant multiples of

$$F_{n-1}^{-}(\theta)=1+\frac{2}{n+1}\sum_{k=1}^{n-1}(-1)^k\left((n-k)\cos\frac{k\pi}{n+1}+\frac{\sin((k+1)\pi/(n+1))}{\sin(\pi/(n+1))}\right)\cos k\theta.$$

The following relation between the Suffridge univalent polynomials and the above Fejér's kernel is very much surprising and simultaneously it is clue observation which led us to the main result of this paper.

Lemma 3. *For every* $n\in\mathbb{N}$

$$\frac{\Im s_n(e^{i\theta})}{\sin\theta}=\frac{n+1}{2n(\sin(\pi/(n+1)))^2}\;F_{n-1}^{+}(\theta)$$

and

$$\frac{\Im s_n(n;e^{i\theta})}{\sin\theta}=\frac{n+1}{2n(\sin(\pi/(n+1)))^2}\;F_{n-1}^{-}(\theta).$$

The proof follows either by Lemma 2 and the well-known representation

$$F_{n-1}^{+}(\theta)=\frac{(\sin(\pi/(n+1)))^2}{n+1}\,\frac{(1+\cos((n+1)\theta)}{(\cos\theta-\cos\pi/(n+1))^2}$$

of the Fejér kernel or by checking the relations (2.3) for the coefficients of $s_n(z)$ and F_{n-1}^{+} and for those of $s_n(n;z)$ and F_{n-1}^{-}. We omit these technical details because the calculations are straightforward.

Proof of Theorem 1. Let $p_n\in\mathcal{S}_n(\mathbb{R})$. On using Lemma 1 we express the values of p_n at ±1 in terms of the coefficients a_k of the corresponding nonnegative cosine polynomial $c_{n-1}(\theta)$:

$$\begin{aligned}p_n(1)&=1+\gamma_2+\gamma_3+\cdots+\gamma_n=a_0+a_2,\\ p_n(-1)&=-1+\gamma_2-\gamma_3+\cdots+(-1)^n\gamma_n=a_2-a_0.\end{aligned}$$

Then

$$\frac{p_n(-1)}{p_n(1)}=\frac{a_2-a_0}{a_2+a_0}=\frac{a_2/a_0-1}{a_2/a_0+1}.$$

On the other hand, Fejér's inequalities (3.2) yield

$$-\left(\cot\frac{\pi}{2n+2}\right)^2=\frac{-1-\cos\frac{\pi}{n+1}}{1-\cos\frac{\pi}{n+1}}\le\frac{a_2/a_0-1}{a_2/a_0+1}\le\frac{\cos\frac{\pi}{n+1}-1}{1+\cos\frac{\pi}{n+1}}=-\left(\tan\frac{\pi}{2n+2}\right)^2$$

and equalities in the left and right-hand inequalities are attained only for F_{n-1}^{-} and F_{n-1}^{+}, respectively. Then Lemma 3 shows that the largest value $-(\tan(\pi/(2n+2)))^2$ of $p_n(-1)/p_n(1)$ is attained only for $p_n(z)=s_n(z)$ and the smallest value $-(\cot(\pi/(2n+2)))^2$ only for $p_n(z)=s_n(n;z)$. This completes the proof of the theorem.

It is worth mentioning that the converse statement of Lemma 1 is not true. Despite that (2.3) is a one-to-one relation between the coefficients $\{\gamma_k\}_1^n$ and $\{a_k\}_0^{n-1}$, there are nonnegative cosine polynomials $c_{n-1}(\theta)$ for which the corresponding polynomials $p_n(z)$ are not univalent in D. That is why the one-to-one relations between the Suffridge polynomials and the Fejér kernel, given in Lemma 3, is fundamental.

The statement of Corollary 1 is obtained by straightforward renormalization.

Proof of Corollary 2. Necessary conditions that an univalent polynomial $p_n(z)$ with real coefficients, normalized only by $p_n(0) = 0$, satisfies

$$k_{\rho_n} \prec p_n \prec k \quad \text{with } 0 < \rho_n < 1,$$

are

$$p_n(-1) \geq -1/4 \quad \text{and} \quad p_n(1) \geq k(\rho_n) = \rho_n/(1-\rho_n)^2.$$

These inequalities, together with the the right-hand side inequality (1.4), imply

$$\rho_n^2 - 2(1 + 2(\tan(\pi/(2n+2)))^2)\rho_n + 1 \geq 0.$$

Since the roots of this binomial are $\rho_n = e_n$ and $\rho_n = 1/e_n$, and $\rho_n < 1$, then we must have $\rho_n \leq e_n$. Moreover, by Corollary 1, the equality $\rho_n = e_n$ is possible only when $p_n(z) = -s_n(z)/(4s_n(-1)) = (k * \lambda_n)(z)$. The fact that $k_{e_n} \prec k * \lambda_n$ was established in [13].

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