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Asymptotics of zeros of polynomials arising from rational integrals ☆

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Abstract

We prove that the zeros of the polynomials $P_m(a)$ of degree m, defined by Boros and Moll via

$$P_m(a) = \frac{2^{m+3/2}}{\pi}(a+1)^{m+1/2}\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}},$$

approach the lemniscate $\{\zeta \in \mathbb{C}\colon |\zeta^2-1| = 1,\ \Re\zeta < 0\}$, as $m \to \infty$.

Keywords: Rational integral; Polynomial; Zeros; Asymptotic behavior; Lemniscate

1. Introduction

Recently Moll, Boros and their coauthors [1–3,6] investigated thoroughly the rational integral

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$$N(a,m)=\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{1}$$

its dependence both on the parameters m and a, and established various interesting properties of $N(a,m)$. We refer to a recent personal account of Moll [6] about the motivation for the study of the integrals (1) and about some unexpected interplays. Among the other facts, they proved that, for every fixed $m \in \mathbb{N}$,

$$P_m(a)=\frac{2^{m+3/2}}{\pi}(a+1)^{m+1/2}N(a,m)$$

is a polynomial in a of degree m. It was observed in [1] that the zeros of $P_m(a)$ possess a pretty regular asymptotic behavior and nice pictures in support of this observation were furnished in [1,6]. It was pointed out by Boros and Moll in [1], that their numerical experiments suggest that the zeros go to a lemniscate but they were not able to predict its equation. We prove that the limit curve for the zeros of $P_m(a)$ is the left half of the lemniscate of Bernoulli

$$L=\{\zeta \in \mathbb{C}\colon\ |\zeta^2-1|=1,\ \Re\zeta<0\}.$$

The polar equation of L is

$$\rho^2=2\cos 2\theta, \quad \theta \in (3\pi/4, 5\pi/4).$$

We also try to explain the lack of zeros around the part of L which is close to the origin, a fact which can be observed in the above mentioned pictures and in Fig. 1 which shows the zeros of $P_{70}(a)$.

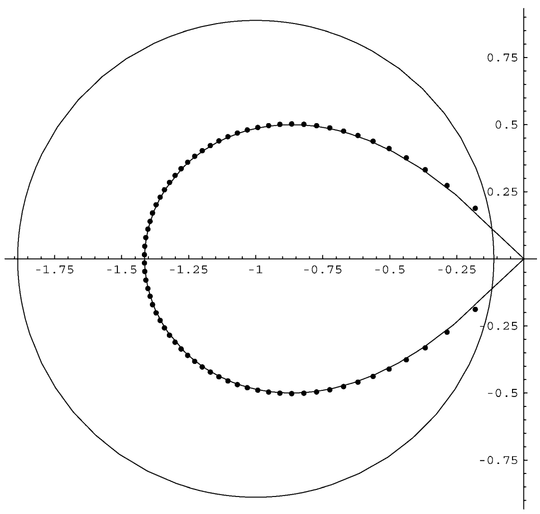

Fig. 1. The zeros of $P_{70}(a)$, the lemniscate L and the circumference C_{70}.

Theorem 1. *If Z_m is the set of the zeros of $P_m(a)$, then*

$$\max_{a\in Z_m} \min\{|a-\zeta|\colon \zeta\in\mathbb{C},\ |\zeta^2-1|=1,\ \Re\zeta<0\} \to 0 \quad \text{as } m\to\infty.$$

Moreover, if $m\in\mathbb{N}$ is fixed, then

$$|a+1| \leqslant 1-\frac{2(\sqrt{m(m+1)(4m+1)}-m)}{(2m+1)^2} \quad \textit{for every } a\in Z_m.$$

The zeros of $P_{70}(z)$, together with the lemniscate L and the circumference

$$C_m=\left\{\zeta\in\mathbb{C}\colon |\zeta+1|=1-\frac{2(\sqrt{m(m+1)(4m+1)}-m)}{(2m+1)^2}\right\},$$

for $m=70$, can be seen in Fig. 1.

2. Proof of the theorem

We begin with a simple technical lemma which shows that the inverse polynomial of a hypergeometric polynomial is also hypergeometric. Recall that the hypergeometric function is defined by

$$F(\alpha,\beta;\gamma;x) := {}_2F_1(\alpha,\beta;\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!},$$

where $(\alpha)_k$ denotes the Pochhammer symbol, $(\alpha)_k := \alpha(\alpha+1)\dots(\alpha+k-1)$ for $k\in\mathbb{N}$, and $(\alpha)_0 := 1$.

Lemma 1. *If $n\in\mathbb{N}$, $\beta,\gamma\in\mathbb{C}$, with $\beta,\gamma\neq 0,-1,-2,\dots,-n$, then*

$$F(-n,\beta;\gamma;z)=\frac{(\beta)_n}{(\gamma)_n}(-z)^n F(-n,1-\gamma-n;1-\beta-n;1/z). \tag{2}$$

Proof. Calculating the coefficient d_k of z^k on the right-hand side of (2), we obtain

$$\begin{aligned} d_k &= (-1)^n\frac{(\beta)_n}{(\gamma)_n}\frac{(-n)_{n-k}(1-\gamma-n)_{n-k}}{(1-\beta-n)_{n-k}}\frac{1}{(n-k)!} \\ &= \frac{(-1)^k}{k!}\frac{(\beta)_k}{(\gamma)_k}n(n-1)\dots(n-k+1)=\frac{(\beta)_k}{(\gamma)_k}(-n)_k\frac{1}{k!} \end{aligned}$$

and the latter coincides with the coefficient of z^k on the left-hand side of (2). □

We shall need the following version of the classical Eneström–Kakeya theorem (see Exercise 2 on p. 137 in [5]):

Theorem A. *All the zeros of the polynomial $f(z)=c_0+c_1z+\dots+c_nz^n$, having positive coefficients c_j, lie in the ring*

$$\min_{0\leqslant k\leqslant n-1}(c_k/c_{k+1}) \leqslant |z| \leqslant \max_{0\leqslant k\leqslant n-1}(c_k/c_{k+1}).$$

Proof of Theorem 1. It was proved in [2,3] that $P_m(a)$ is a hypergeometric polynomial,

$$P_m(a) = \binom{2m}{m} F\bigl(-m, m+1; 1/2-m; (a+1)/2\bigr).$$

On using this representation and applying identity (2) for $n = m$, $\beta = m+1$, $\gamma = 1/2 - m$ and $z = (a+1)/2$, we obtain

$$P_m(a) = (-1)^m \binom{2m}{m} \frac{(m+1)_m}{(1/2-m)_m} \left(\frac{a+1}{2}\right)^m F\left(-m, \frac{1}{2}; -2m; \frac{2}{a+1}\right). \tag{3}$$

On the other hand, Driver and Möller [4] proved that, for any real positive b, the zeros of the polynomials

$$w^n F(-n, b; -2n; 1/w) \tag{4}$$

approach the Cassini curve

$$\bigl|(2w-1)^2 - 1\bigr| = 1 \tag{5}$$

as n diverges. Their result and the representation (3) immediately imply that the zeros of $P_m(a)$ tend to the lemniscate $|\zeta^2 - 1| = 1$ as m goes to infinity. In order to prove that these zeros remain in the disc D_m surrounded by the circumference C_m, we apply Theorem A. The coefficients c_k in the expansion

$$F\bigl(-m, m+1; 1/2-m; (a+1)/2\bigr) = \sum_{k=0}^{m} c_k (a+1)^k$$

are positive. Moreover

$$\Delta(k) := \frac{c_k}{c_{k+1}} = \frac{(k+1)(2m-2k-1)}{(m-k)(m+k+1)}.$$

Straightforward calculations show that

$$\Delta'(\kappa) = \frac{(2m-1)\kappa^2 - 2(2m^2+1)\kappa + 2m^3 - m^2 - m - 1}{(m-\kappa)^2(m+\kappa+1)^2},$$

and $\Delta'(\kappa) = 0$ only for

$$\kappa_{1,2} = \frac{2m^2 + 1 \pm \sqrt{m(m+1)(4m+1)}}{2m-1},$$

where $0 < \kappa_1 < m-1 < \kappa_2$. It is easy to see that κ_1 is a point of local and also of global maximum of $\Delta(\kappa)$ when κ varies in $[0, m-1]$. Then, by Theorem A we conclude that the zeros of $P_m(a)$ lie in the disc $|a+1| \leqslant \Delta(\kappa_1)$, where

$$\Delta(\kappa_1) = 1 - \frac{2(\sqrt{m(m+1)(4m+1)} - m)}{(2m+1)^2}.$$

This completes the proof of the theorem. □

3. Remarks and open questions

While the proof of the convergence of the zeros of $P_m(a)$ to L is a matter of simple transformation of Driver and Möller's result, the lack of zeros close to the origin seems to be an interesting phenomenon. Except for the fact that Z_m is a subset of D_m, one may obtain other regions which do not contain zeros of P_m.

An application of a generalization of Descartes' rule of signs, due to Obrechkoff [7] (see Theorem 41.3 in [5]), implies

$$\left|\arg(a+1)\right| \geqslant \pi/m \quad \text{for every } a \in Z_m.$$

In other words, the zeros of $P_m(a)$ are outside the infinite sector with vertex -1 which contains the ray $(-1, \infty)$, with angle $2\pi/m$.

Another method which is applicable to our case is the so-called Parabola theorem of Saff and Varga [8]. It guarantees that the zeros of certain polynomials that satisfy a three-term recurrence relation belong to a parabola region. It can be shown that the polynomials

$$q_k(z) = \frac{(1/2-m)_k}{(-1/2-2m)_k} F(-k, m+1, 1/2-m, 1+z)$$

satisfy the recurrence relation

$$q_k(z) = \left(\frac{z}{b_k}+1\right) q_{k-1}(z) - \frac{z}{c_k} q_{k-2}(z), \quad k = 1, \ldots, m,$$

where

$$b_k = \frac{2m-k+3/2}{m+k} \quad \text{and} \quad c_k = \frac{(2m-k+3/2)(2m-k+5/2)}{(k-1)(m-k+3/2)}.$$

Then if $a \in Z_m$, with $a = \xi + i\eta$, the parabola theorem yields

$$\eta^2 > \frac{2(4m+1)}{2m-1}\left(\xi + \frac{3}{4m-2}\right).$$

Let us apply the left-hand side estimate in Theorem A to the Maclaurin expansion

$$P_m(a) = \sum_{k=0}^{m} d_k(m) a^k.$$

In [1] Boros and Moll obtained the representation

$$d_k(m) = 2^{-2m} \sum_{j=k}^{m} 2^j \binom{2m-2j}{m-j}\binom{m+j}{m}\binom{j}{k}$$

for the coefficients of this expansion and proved that the first half of the sequence $\{d_k(m)\}_{k=0}^m$ is increasing and the second one is decreasing. Numerical experiments show that, for every fixed m, the minimum of $d_k(m)/d_{k+1}(m)$ is attained for $k=0$ and the values of $d_0(m)/d_1(m)$ behave like $1/m$ as m diverges.

Summarizing, these application of the results of Obrechkoff, of Saff and Varga and the Eneström–Kakeya theorem guarantee the lack of zeros in very "tiny" regions around the origin. More precisely, the circumference C_m cuts a larger portion of the lemniscate L

around the origin in comparison with the cuts with the above mentioned sector, parabola and circumference of radius $1/m$ around the origin. The author finds that a promising way of determining the precise behavior of the closest to the origin zero of $P_m(a)$ is the relation between Z_m and the zeros of the polynomials $H_n(b,u)$, defined by (4.4) in [4]. There, on p. 86, Driver and Möller formulated a challenging conjecture about the location of the zeros of $H_n(b,u)$. If true, and once it is established, the statement of that conjecture will provide precise information about the location of the zeros of $P_m(a)$.

We finish with an observation motivated by another result of Driver and Möller. Proposition 5.2 in [4] states that the zeros of the polynomials (4) lie outside the Cassini curve (5) for all $n \in \mathbb{N}$, provided $b \geqslant 1$. In our case $b = 1/2$, so we cannot conclude that the zeros of $P_m(a)$ lie outside L though numerical experiments show that they do.

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