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Monodromy at Infinity and the Weights of Cohomology

ALEXANDRU DIMCA[1] and MORIHIKO SAITO[2]
[1]Laboratoire d'Analyse et Géométrie, CNRS, Université Bordeaux I, 33405 Talence Cedex, France. e-mail: dimca@math.u-bordeaux.fr
[2]RIMS Kyoto University, Kyoto 606-8502, Japan. e-mail: msaito@kurims.kyoto-u.ac.jp

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Abstract. We show that for a polynomial map, the size of the Jordan blocks for the eigenvalue 1 of the monodromy at infinity is bounded by the multiplicity of the reduced divisor at infinity of a good compactification of a general fiber. The existence of such Jordan blocks is related to global invariant cycles of the graded pieces of the weight filtration. These imply some applications to period integrals. We also show that such a Jordan block of size greater than 1 for the graded pieces of the weight filtration is the restriction of a strictly larger Jordan block for the total cohomology group. If there are no singularities at infinity, we have a more precise statement on the monodromy.

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Introduction

Let $X = \mathbb{C}^{n+1}$, $S = \mathbb{C}$, and $f\colon X \to S$ be a polynomial map. Set $X_s = f^{-1}(s)$ for $s \in S$. Then there is a Zariski-open subset U of S such that the $H^j(X_s, \mathbb{Q})$ for $s \in U$ form a local system on U. It is known that the behavior of the local monodromy at infinity of this local system is rather different from the local monodromy around the points in S, see [9, 16, 17], etc. Among others, it is often observed that the size of the Jordan blocks for the eigenvalue 1 is smaller than the size for the other eigenvalues. The latter is bounded by $j+1$ due to (a generalization of) the monodromy theorem, and this is optimal for the eigenvalues different from 1.

For a general $s \in U$, let $\bar{X}_s$ be a good compactification of X_s such that $\bar{X}_s$ is smooth and the (reduced) divisor at infinity $D_s := \bar{X}_s \setminus X_s$ is a divisor with normal crossings. Let m_s be the maximum of the multiplicity (i.e. the number of local irreducible components) of D_s. This is independent of the choice of a general $s \in U$. In this paper, we show

THEOREM 0.1. *The size of the Jordan blocks for the eigenvalue* 1 *of the local monodromy at infinity is bounded by* m_s *and also by* j *for* $j > 0$. *In particular, this size is* 1 *if* X_s *admits a smooth compactification with a smooth divisor at infinity* (*e.g. if the hypersurface in* $\mathbb{P}^n$ *defined by the highest degree part of* f *is reduced and smooth*).

More precisely, the size is bounded by the difference m'_s between the maximal weight of $H^j(X_s, \mathbb{Q})$ and j, see (0.4). Note that $m_s \geqslant m'_s$ in general by [7], and we have the strict inequality, for example if $n = 2$, $m_s = 2$ and the dual graph of D_s has no cycle.

Another interesting fact is that there is a certain condition on the relation between the Jordan blocks for the eigenvalue 1 and the weight filtration W of the natural mixed Hodge structure [7] on $H^j(X_s, \mathbb{Q})$, and such Jordan blocks are closely related to global invariant cycles of the graded pieces of the weight filtration. Let G be the monodromy group which is the image of the monodromy representation $\pi_1(U, s) \to \operatorname{Aut} H^j(X_s, \mathbb{Q})$. Note that W is stable by the action of G, because W gives the weight filtration on the local system $\{H^j(X_s, \mathbb{Q})\}_{s \in U}$ which underlies a variation of mixed Hodge structures. (This may be considered to be one of basic examples of geometric variations of mixed Hodge structures defined on a Zariski-open subset of $\mathbb{C}$.) Let T_∞ denote the monodromy at infinity. This is an element of G, and is defined by choosing a path between s and ∞ in U.

THEOREM 0.2. *For an integer i, assume the monodromy at infinity of* $\mathrm{Gr}_i^W H^j(X_s, \mathbb{Q})$ *has a Jordan block of size* $r\,(>0)$ *for the eigenvalue* 1. *Then* $\mathrm{Gr}_{i'}^W H^j(X_s, \mathbb{Q})$ *has a nonzero global invariant cycle* (*i.e.* $(\mathrm{Gr}_{i'}^W H^j(X_s, \mathbb{Q}))^G \neq 0$) *with* $i' = i + r + 1 \leqslant j + m'_s\,(\leqslant \min\{2j, j + m_s\})$, *or* $i' = i > j$ *and* $r = 1$. *In particular, we have natural isomorphisms*

$$(\mathrm{Gr}_i^W H^j(X_s, \mathbb{Q}))^G = (\mathrm{Gr}_i^W H^j(X_s, \mathbb{Q}))^{T_\infty} \quad \textit{for } i - j > m'_s - 2.$$

In the case $i' = i + r + 1$ (*e.g. if* $r > 1$), *the given Jordan block is the restriction of a strictly larger Jordan block for the monodromy of* $H^j(X_s, \mathbb{Q})$ *to the graded piece* Gr_i^W.

This is a special case of (2.4-5). If $n = 2$, it implies that the size of the Jordan blocks for the eigenvalue 1 of the monodromy at infinity on $\mathrm{Gr}_i^W H^j(X_s, \mathbb{Q})$ is at most 1 (compare to the example in [13] mentioned after (0.4) below). When $n = 1$, (0.2) follows from [9] (because $\mathrm{Gr}_1^W H^1(X_s, \mathbb{Q})$ coincides with the cohomology of a smooth compactification). The last assertion of (0.2) means that if the restriction of T_∞ to Gr_i^W has a Jordan block of size >1 for the eigenvalue 1, then there is a strictly larger Jordan block for T_∞ on $H^j(X_s, \mathbb{Q})$. In some special case, this was observed in [10]. Note that the relation between the Jordan blocks of the local monodromies and the weight filtration is rather complicated in general, and the above assertion does not follow from the conditions of admissible variation of mixed Hodge structure. See (2.9) for an application to period integrals.

If $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ does not have singularities at infinity (more precisely, if f is cohomologically tame [28]), then the situation becomes quite simple. We have $\tilde{H}^j(X_s, \mathbb{Q}) = 0$ for $s \in U$ and $j \neq n$. Let $m(s, \lambda, r)$ denote the number of Jordan blocks of the local monodromy of $\{H^n(X_s, \mathbb{Q})\}_{s \in U}$ at s with eigenvalue λ and size r, and similarly for $m'(s, \lambda, r)$ with $H^n(X_s, \mathbb{Q})$ replaced by $\mathrm{Gr}_n^W H^n(X_s, \mathbb{Q})$. Let $r_i = \dim \mathrm{Gr}_i^W H^n(X_s, \mathbb{Q})$ for $s \in U$. The following assertion (except for the one about the monodromy around $s \in S$) was obtained in [10], 4.3-5 under an additional mild assumption.

THEOREM 0.3. *Assume* $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ *is cohomologically tame* [28] *with* $n \geqslant 1$. *Then the local systems* $\{\mathrm{Gr}_i^W H^n(X_s, \mathbb{Q})\}_{s\in U}$ *are constant for* $i \neq n$ *and*

$$r_{n+1} = m(\infty, 1, 1) \geqslant \dim \mathrm{IH}^1(\mathbb{P}^1, R^n f_* \mathbb{Q}_X|_U),$$
$$r_{n+r+1} = m'(\infty, 1, r) = m(\infty, 1, r+1) \quad \text{for } r > 0,$$
$$m(s, \lambda, r) = m'(s, \lambda, r) + \delta_{\lambda,1}\delta_{r,1} \sum_{i\neq n} r_i \quad \text{for } s \in S\setminus U.$$

Furthermore, the difference between r_{n+1} *and* $\dim \mathrm{IH}^1(\mathbb{P}^1, R^n f_*\mathbb{Q}_X|_U)$ *is given by the length of the direct factor with discrete support of the perverse sheaf* Gr_{n+1}^W ${}^pR^{n+1}f_*\mathbb{Q}_X$ (*see also* [12], 0.8 *for the case* $n = 1$).

Here $\mathrm{IH}^1(\mathbb{P}^1, R^n f_*\mathbb{Q}_X|_U)$ is the intersection cohomology with coefficients in the local system $\{H^n(X_s, \mathbb{Q})\}_{s\in U}$ on U, and ${}^pR^if_* = {}^pH^if_*$ with the notation of [2] (see also (1.1.1) below). Theorem (0.3) means that each Jordan block of size r for the eigenvalue 1 of the local monodromy of $\{\mathrm{Gr}_n^W H^n(X_s, \mathbb{Q})\}_{s\in U}$ at ∞ has a nontrivial extension with a global section of $\{\mathrm{Gr}_{n+r+1}^W H^n(X_s, \mathbb{Q})\}_{s\in U}$, and gives a Jordan block of size $r+1$ of the local monodromy of $\{H^n(X_s, \mathbb{Q})\}_{s\in U}$. Otherwise there are no nontrivial extensions between the Jordan blocks of the graded pieces of the weight filtration. See (2.6).

In this paper we show that Theorem (0.1) is a special case of the following assertion on the relation between the local monodromy and the weights of the cohomology [7]:

THEOREM 0.4. *Let* $f\colon X \to S$ *be a morphism of complex algebraic varieties such that* $\dim X = n+1$ *and* S *is a smooth curve. Let* $\bar{S}$ *be the smooth compactification of* S. *Let* j *be a positive integer, and* r, r' *be integers such that* $r' < r$. *Assume* $H^j(X_s, \mathbb{Q})$ *for a general* $s \in S$ *has weights in* $[j+r', j+r]$ (*i.e.* $\mathrm{Gr}_k^W H^j(X_s, \mathbb{Q}) = 0$ *for* $k \notin [j+r', j+r]$) *and* $H^{j+1}(X, \mathbb{Q})$ *has weights* $\leqslant j+r$. *Then the Jordan blocks of the monodromies of* $H^j(X_s, \mathbb{Q})$ *around* $s_\infty \in \bar{S} \setminus S$ *for the eigenvalue* 1 *have size* $\leqslant r - r'$.

This means that the restriction on the weights of the cohomology of the total space and a general fiber implies a certain restriction on the monodromy at infinity. The converse of Theorem (0.4) is true in a weak sense if f is proper and X is smooth so that $H^j(X_s, \mathbb{Q})$ is pure of weight j for a general s. See Remark (ii) after (2.3).

For the proof of Theorems, we use the fact that the action of the nilpotent part N of the monodromy on the nearby cycles at infinity (endowed with the limit mixed Hodge structure) is a morphism of mixed Hodge structures of type $(-1, -1)$ (see [7]) so that the assertion is reduced to the estimate of the weights of the nearby cycles at infinity. Then the point is that we can estimate the weights of the *limit* mixed

Hodge structure on the nearby cycles in terms of the weights of the *natural* mixed Hodge structure [7, III] on the cohomology of the general fiber and the total space. We first consider the spectral sequence (1.6.1) which relates the hypercohomology over S of the (perverse) direct image of the constant sheaf by f to that of the graded pure pieces of the direct image. For each graded piece, the weight filtration on its nearby cycles is given by the monodromy filtration up to a shift. The weights on the nearby cycles can be estimated by using its higher direct image by the inclusion to the compactification of S (1.4.4), and then using its hypercohomology over S (1.5.1). Thus we can deduce the assertions. However, we do not discuss the number of the local irreducible components of the deleted fiber which may be assumed to be a divisor with normal crossings if X is smooth. (The condition on $H^{j+1}(X, \mathbb{Q})$ can be replaced by that on $H^0(S, {}^pR^{j+1}f_*\mathbb{Q}_X)$, see (2.3).)

Theorem (0.1) was proved in the case $n = 1$ by [9], and the estimate by j follows from [28], Corollary 10, if f is cohomologically tame in the sense of loc. cit. See also [16, 17]. For $n = j = 2$, there is an example such that the size of a Jordan block is j (see [13], Example (5.3.2): $f(x, y, z) = x + y + z + x^2y^2z^2$). Note that Theorem (0.1) does not hold for the monodromy around a point of $S\setminus U$ (consider, e.g., $f(x, y) = y^2 + x^3 - 3x$), although the local analogue is true (0.5).

The corresponding local assertion is more or less well-known. Let f be a holomorphic function on a complex manifold X (or, more generally, on an analytic space X which is a rational homology manifold). Then we have $N^{n+1} = 0$ on the nearby cycle sheaf $\psi_f\mathbb{Q}_X[n]$ (by reducing to the normal crossing case). See, e.g., [26]. Consequently, Jordan blocks of the monodromy on the jth cohomology of the Milnor fiber at any point of X have size $\leqslant j + 1$ (by restricting to a generic hyperplane and using the vanishing of certain relative cohomology [25]). See also [15]. Restricting to the eigenvalue 1 (and to the reduced cohomology), it is known that the size is bounded by j due to J. Steenbrink [33] (in the isolated singularity case), D. Barlet [1] (for $j = n$) and V. Navarro Aznar [27] (in general).

Actually, we can get a slightly better estimate (which is similar to Theorem (0.1), but is much easier), when the singularity has certain *equisingularity*.

PROPOSITION 0.5. *Let* $\{S_\alpha\}$ *be a Whitney stratification of* X *satisfying Thom's* A_f*-condition* (*which exists at least locally by* [20]). *Let* $r = \max \operatorname{codim} S_\alpha$. *Then the support of the perverse sheaves* $N^j\psi_f\mathbb{Q}_X[n]$ *and* $N^{j-1}\varphi_{f,1}\mathbb{Q}_X[n]$ *have dimension* $\leqslant n - j$, *and for* $j \geqslant r$ *we have* $N^j\psi_f\mathbb{Q}_X[n] = N^{j-1}\varphi_{f,1}\mathbb{Q}_X[n] = 0$. *See* (2.8).

The first assertion concerning the nearby cycles is equivalent to the assertion that $\dim \operatorname{supp} \mathrm{Gr}^W_{n-j}\psi_f\mathbb{Q}[n] \leqslant n - j$ by (1.4.2), and the last assertion is equivalent to the vanishing of $\mathrm{Gr}^W_{n-j}\psi_f\mathbb{Q}[n]$ for $j \geqslant r$. They imply the assertions on the vanishing cycle sheaf with unipotent monodromy $\varphi_{f,1}\mathbb{Q}_X[n]$, because the latter is isomorphic to $N\psi_{f,1}\mathbb{Q}_X[n]$ by the sheaf version of the *local invariant cycle theorem* (1.4.5). This gives also $\dim \operatorname{supp} \mathrm{Gr}^W_{n-j}\mathbb{Q}_{X_0}[n] \leqslant n - j$, and $\mathrm{Gr}^W_{n-j}\mathbb{Q}_{X_0}[n] = 0$ for $j \geqslant r$, where W is the weight filtration of the mixed Hodge Module [30]. Note that we can replace

r by the maximal number of the local irreducible components of an embedded resolution of $f^{-1}(0)$.

In Section 1 we review some basic facts from the theory of mixed Hodge Modules [29, 30], and prove (0.1-5) in Section 2.

In this paper, cohomology of a complex algebraic variety means that of the associated analytic space.

1. Preliminaries

1.1. MIXED HODGE MODULES

For a complex algebraic variety X, let $\mathrm{MHM}(X)$ denote the category of mixed Hodge Modules on X. See [30, 4.2]. If X is smooth, an object $\mathcal{M}$ of $\mathrm{MHM}(X)$ consists of $((M, F, W), (K, W), \alpha)$ where (M, F) is a filtered $\mathcal{D}_X$-Module with the filtration W, (K, W) is a filtered perverse sheaf with rational coefficients on X^{an}, and α is an isomorphism of perverse sheaves $\mathrm{DR}(M) \simeq K \otimes_{\mathbb{Q}} \mathbb{C}$ compatible with W. They satisfy several good conditions. The filtrations F and W are called respectively the Hodge and weight filtrations. The category $\mathrm{MHM}(X)$ is an abelian category, and every morphism is strictly compatible with the two filtrations (F, W) in the strong sense [29]. Furthermore, the weight filtration W gives a filtration of mixed Hodge Modules such that the functor $\mathcal{M} \to \mathrm{Gr}_i^W \mathcal{M}$ is exact.

In general, $\mathrm{MHM}(X, \mathbb{Q})$ is defined by using closed embeddings of open subvarieties of X into smooth varieties. See [30, 31]. Then the underlying perverse sheaf K of a mixed Hodge Module $\mathcal{M}$ is globally well-defined, and the forgetful functor assigning K to $\mathcal{M}$ is faithful and exact. Note that the category of perverse sheaves $\mathrm{Perv}(X, \mathbb{Q})$ is an abelian category, and is a full subcategory of the bounded derived category of sheaves of $\mathbb{Q}$-vector spaces with algebraically constructible cohomologies $D_c^b(X, \mathbb{Q})$, see [2].

For morphisms f of complex algebraic varieties, we can construct canonical functors $f_*, f_!, f^*, f^!$ between the bounded derived categories of mixed Hodge Modules $D^b\mathrm{MHM}(X)$ [30, 4.3-4]. We will denote by $H^i\colon D^b\mathrm{MHM}(X) \to \mathrm{MHM}(X)$ the natural cohomology functor. This corresponds to the perverse cohomology functor [2] ${}^pH^j\colon D_c^b(X, \mathbb{Q}) \to \mathrm{Perv}(X, \mathbb{Q})$ by the forgetful functor. We define

$$ {}^pR^i f_* K = {}^pH^i \mathbf{R} f_* K \quad \text{for } K \in D_c^b(X, \mathbb{Q}). \tag{1.1.1}$$

This corresponds to $H^i f_* \mathcal{M}$ for $\mathcal{M} \in D^b\mathrm{MHM}(X)$ by the forgetful functor. Let $a_X\colon X \to pt$ denote the structure morphism. For $\mathcal{M} \in D^b\mathrm{MHM}(X)$, we define

$$ H^i(X, \mathcal{M}) = H^i(a_X)_* \mathcal{M}, \quad H_c^i(X, \mathcal{M}) = H^i(a_X)_! \mathcal{M}. \tag{1.1.2}$$

Remark. Let $f\colon X \to Y$ be a morphism of complex algebraic varieties, and $\mathcal{M}$ a bounded complex of mixed Hodge Module on X with a finite decreasing filtration G.

Then we have a spectral sequence in the category of mixed Hodge Modules

$$E_1^{j,i-j} = H^i f_* \mathrm{Gr}_G^j \mathcal{M} \Rightarrow H^i f_* \mathcal{M}. \tag{1.1.3}$$

Indeed, the direct image $f_*\mathcal{M}$ is represented by a complex of mixed Hodge Modules endowed with a filtration induced by the filtration G on $\mathcal{M}$ by the definition of direct image [30, 4.3]. In particular, Gr_G^i commutes with the direct image, and the spectral sequence follows (see, e.g., [7, (1.3.1)]). (We have a similar assertion for the pull-back functor f^*.)

If f is proper, $\mathcal{M}$ is a mixed Hodge Module, and G is the weight filtration W of $\mathcal{M}$, then we get the weight spectral sequence [7]:

$$E_1^{-i,i+j} = H^j f_* \mathrm{Gr}_i^W \mathcal{M} \Rightarrow H^j f_* \mathcal{M}. \tag{1.1.4}$$

Since $H^j f_* \mathrm{Gr}_i^W \mathcal{M}$ is pure of weight $i+j$ (see Remark after (1.2)), it degenerates at E_2, and its abutting filtration is the weight filtration of $H^j f_* \mathcal{M}$ by the same argument as in [7], see [30], 2.15.

Applying (1.1.3) to the truncation τ on $f_*\mathcal{M}$, we get the Leray spectral sequence

$$E_2^{p,q} = H^p g_* H^q f_* \mathcal{M} \Rightarrow H^{p+q}(gf)_* \mathcal{M}, \tag{1.1.5}$$

as in [7, (1.4.8)] for morphisms $f \colon X \to Y$ and $g \colon Y \to Z$.

1.2. INTERSECTION COMPLEXES

We say that $\mathcal{M}$ is pure of weight r if $\mathrm{Gr}_k^W \mathcal{M} = 0$ for $k \neq r$. A pure Hodge Module is also called a polarizable Hodge Module. It admits a decomposition by strict support

$$\mathcal{M} = \oplus_Z \mathcal{M}_Z, \tag{1.2.1}$$

where the direct sum is taken over irreducible closed subvarieties Z of X, and $\mathcal{M}_Z$ has support Z or $\emptyset$, but has no nontrivial subobject or quotient object with strictly smaller support. The underlying perverse sheaf K_Z of $\mathcal{M}_Z$ is an intersection complex with local system coefficients, i.e. there exist a dense open smooth affine subvariety U of Z with the inclusion $j \colon U \to Z$, and a local system L_Z on U such that K_Z is the intermediate direct image

$$j_{!*}(L_Z[\dim Z]) := \mathrm{Im}(j_!(L_Z[\dim Z]) \to j_*(L_Z[\dim Z])) \tag{1.2.2}$$

in the sense of [2]. It is also called the intersection complex $\mathrm{IC}_Z L_Z$, and its cohomology is called the intersection cohomology $\mathrm{IH}^\bullet(Z, L_Z)$.

If Z is a curve, then $K_Z[-1]$ is a sheaf in the usual sense, and is isomorphic to $j_* L_Z$ where the direct image is in the usual sense. In particular, we have

$$H^{-1}(Z, K_Z) = H^0(U, L_Z). \tag{1.2.3}$$

Remark. Let f be a proper morphism of complex algebraic varieties, and $\mathcal{M}$ a pure Hodge Modules of weight n on X. Then the cohomological direct image $H^j f_* \mathcal{M}$

is pure of weight $n+j$. See [30, 4.5]. A pure Hodge Module is also stable by the intermediate direct image (1.2.2), see loc. cit.

1.3. VARIATION OF MIXED HODGE STRUCTURE

For $X = pt$, we have naturally an equivalence of categories

$$\mathrm{MHM}(pt) = \mathrm{MHS}^p, \tag{1.3.1}$$

where the right-hand side denotes the category of graded-polarizable mixed Hodge structures with rational coefficients [7] (and $F^p = F_{-p}$). See [30, (4.2.12)]. So mixed Hodge Modules on pt will be identified with graded-polarizable mixed Hodge structures.

More generally, a mixed Hodge Module $((M, F, W), (K, W); \alpha)$ on a smooth variety X such that $K[-d]$ is a local system can be naturally identified with an admissible variation of mixed Hodge structure [22, 34] by replacing K with $K[-d]$, and W with $W[-d]$, where $d = \dim X$ and $(W[-d])_k = W_{k+d}$. See [30, 3.27]. In particular, a polarizable Hodge Module of weight w such that the underlying perverse sheaf is a local system (up to a shift) can be identified with a polarizable variation of Hodge structure of weight $w-d$.

If X is smooth, we will denote by $\mathbb{Q}_X^H[d] \in \mathrm{MHM}(X)$ the pure Hodge Module of weight d corresponding to the constant variation of Hodge structure of type $(0,0)$. In general, $\mathbb{Q}_X^H$ is defined in the derived category of bounded complexes of mixed Hodge Modules $D^b\mathrm{MHM}(X)$. See [30]. By the direct image under the structure morphism $X \to pt$, we get a mixed Hodge structure on the cohomology of X. This coincides with Deligne's mixed Hodge structure [7]. See [31].

1.4. VANISHING CYCLES

Let g be a nonconstant function on X. Put $Y = g^{-1}(0)$. Then we have the nearby and vanishing cycle functors ψ_g and φ_g. They are exact functors from $\mathrm{MHM}(X, \mathbb{Q})$ to $\mathrm{MHM}(Y, \mathbb{Q})$, and correspond to the exact functors $\psi_g[-1]$, $\varphi_g[-1]$ on the underlying perverse sheaves [6].

The semisimple part T_s of the monodromy acts naturally on $\psi_g\mathcal{M}$, $\varphi_g\mathcal{M}$, and the submodules defined by $\mathrm{Ker}(T_s - 1)$ are denoted by $\psi_{g,1}\mathcal{M}$, $\varphi_{g,1}\mathcal{M}$. Let $N = (2\pi i)^{-1} \log T_u$, where T_u is the unipotent part of the monodromy. Then N gives morphisms of mixed Hodge Modules

$$N\colon \psi_g\mathcal{M} \to \psi_g\mathcal{M}(-1), \qquad N\colon \varphi_g\mathcal{M} \to \varphi_g\mathcal{M}(-1), \tag{1.4.1}$$

where (-1) is the Tate twist as in [7] (i.e. the Hodge filtration is shifted by -1 and the weight filtration by 2).

The weight filtration is given by the relative monodromy filtration in the sense of [8, (1.6.13)] and [34]. In particular, if $\mathcal{M}$ is pure of weight k, then we have isomorphisms

$$\begin{aligned} N^i &: \mathrm{Gr}^W_{k-1+i}\psi_g\mathcal{M} \xrightarrow{\sim} \mathrm{Gr}^W_{k-1-i}\psi_g\mathcal{M}(-i), \\ N^i &: \mathrm{Gr}^W_{k+i}\varphi_{g,1}\mathcal{M} \xrightarrow{\sim} \mathrm{Gr}^W_{k-i}\varphi_{g,1}\mathcal{M}(-i). \end{aligned} \tag{1.4.2}$$

Let $U = X\backslash Y$ with the inclusion morphisms $i\colon Y \to X$, $j\colon U \to X$. Since $\psi_g\mathcal{M}$ depends only on $\mathcal{M}|_U$, $\psi_g j_*\mathcal{M}$ for $\mathcal{M} \in \mathrm{MHM}(U)$ will be denoted by $\psi_g\mathcal{M}$. By [30, 2.24] we have a canonical isomorphism

$$i^*j_*\mathcal{M} = \mathrm{Cone}(N\colon \psi_{g,1}\mathcal{M} \to \psi_{g,1}\mathcal{M}(-1)). \tag{1.4.3}$$

Indeed, $\mathrm{Var}\colon \varphi_{g,1}j_*\mathcal{M} \to \psi_{g,1}\mathcal{M}(1)$ is an isomorphism (because Var corresponds to the action of t on the underlying $\mathcal{D}$-Module, see e.g. [29, 3.4.12]) and $\mathrm{can}\colon \psi_{g,1}\mathcal{M} \to \varphi_{g,1}j_*\mathcal{M}$ is identified with N. (This is related to [34] when X is a smooth curve and the local monodromies are unipotent.)

In particular, we have

$$\mathrm{Gr}^W_{k+1+i}H^0 i^*j_*\mathcal{M} = (P_N\mathrm{Gr}^W_{k-1+i}\psi_{g,1}\mathcal{M})(-1), \tag{1.4.4}$$

and $H^0 i^*j_*\mathcal{M}$ has weights $\geqslant k+1$. Here the right-hand side of (1.4.4) denotes the primitive part by the action of N. So the dimension of $H^0 i^*j_*\mathcal{M}$ coincides with the number of Jordan blocks for the eigenvalue 1 in the case $\dim X = 1$ and g is a local coordinate.

Remarks. (i) With the above notation, assume X smooth, or more generally, X is a rational homology manifold so that $\mathbb{Q}_X[\dim X]$ is the intersection complex. Since Y is a locally principal divisor, $\mathbb{Q}_Y[n]$ is a perverse sheaf, where $n = \dim Y$. So $\mathbb{Q}^H_Y[n]$ is a mixed Hodge Module on Y, and we have a short exact sequence of mixed Hodge Modules

$$0 \to \mathbb{Q}^H_Y[n] \to \psi_{g,1}\mathbb{Q}^H_X[n+1] \xrightarrow{\mathrm{can}} \varphi_{g,1}\mathbb{Q}^H_X[n+1] \to 0,$$

because the cokernel of can corresponds to the maximal quotient object supported on $g^{-1}(0)$, and it vanishes in this case. See [30, (2.4.3)]. Combined with [29, 5.1.7], this implies

$$\begin{aligned} &\mathbb{Q}^H_Y[n] = \mathrm{Ker}(N\colon \psi_{g,1}\mathbb{Q}^H_X[n+1] \to \psi_{g,1}\mathbb{Q}^H_X[n+1](-1)), \\ &\varphi_{g,1}\mathbb{Q}^H_X[n+1] = \mathrm{Coim}(N\colon \psi_{g,1}\mathbb{Q}^H_X[n+1] \to \psi_{g,1}\mathbb{Q}^H_X[n+1](-1)). \end{aligned} \tag{1.4.5}$$

The first isomorphism may be viewed as the sheaf version of the *local invariant cycle theorem*, and implies the second. These hold also in the case X and g are analytic by loc. cit.

(ii) Assume X is a smooth curve, and let $0 \in X$ with a local coordinate t such that $\{0\} = t^{-1}(0)$. Let $\mathcal{M}$ be an admissible variation of mixed Hodge structure on U, which is identified with a mixed Hodge Module on U. Assume the monodromy around 0 is unipotent. Then the nearby cycles $\psi_t\mathcal{M}$ can be defined as in [32, 34] by extending the Hodge bundles to Deligne's canonical extension [5] as subbundles, and then restricting them to the fiber at 0. (In general, we have to use the filtration V ([21, 26]) indexed by rational numbers as in [29].)

In particular, the set of integers p such that $\mathrm{Gr}_F^p \neq 0$ does not change by passing to the limit Hodge filtration. This is used for an estimate of the size of Jordan blocks, see (2.7).

LEMMA 1.5. *Let S be a smooth affine curve, and $\bar{S}$ be the smooth compactification of S with the inclusions $i\colon \Sigma := \bar{S} \setminus S \to \bar{S}$, $j\colon S \to \bar{S}$. Let $\mathcal{M}$ be a pure Hodge Module of weight k on S, and let $j_{!*}\mathcal{M} = \mathrm{Im}(j_!\mathcal{M} \to j_*\mathcal{M})$, the intermediate direct image. Then we have an exact sequence of mixed Hodge structures*

$$\begin{aligned} 0 &\to H^0(\bar{S}, j_{!*}\mathcal{M}) \to H^0(S, \mathcal{M}) \to H^0(\Sigma, i^*j_*\mathcal{M}) \\ &\to H^1(\bar{S}, j_{!*}\mathcal{M}) \to 0, \end{aligned} \tag{1.5.1}$$

and an isomorphism

$$H^{-1}(\bar{S}, j_{!*}\mathcal{M}) = H^{-1}(S, \mathcal{M}). \tag{1.5.2}$$

In particular, $H^{-1}(S, \mathcal{M})$ is pure of weight $k-1$, and is isomorphic to the dual of $H^1(\bar{S}, j_{!*}\mathcal{M})$ up to a Tate twist.

Proof. Since S is affine, it is enough to show the short exact sequence of mixed Hodge Modules

$$0 \to j_{!*}\mathcal{M} \to j_*\mathcal{M} \to i_*H^0 i^* j_*\mathcal{M} \to 0.$$

Indeed, $H^i(\bar{S}, j_{!*}\mathcal{M})$ is pure of weight $k+i$ (see Remark after (1.2)), and $\mathcal{M}$ is self-dual up to a Tate twist due to the polarization.

Let K be the underlying perverse sheaf of $\mathcal{M}$. Replacing $\bar{S}$ with an open neighborhood of $\bar{S} \setminus S$, we may assume that $\mathcal{M}$ is a variation of mixed Hodge structure, i.e. $K[-1]$ is a local system. Then the underlying $\mathbb{Q}$-complex of $(j_{!*}\mathcal{M})[-1]$ is $j_*(K[-1])$ where j_* is the direct image in the usual sense. So the assertion follows from the distinguished triangle

$$\to j_*(K[-1]) \to \mathbf{R}j_*(K[-1]) \to R^1 j_*(K[-1]) \to,$$

which gives the exact sequence of the underlying perverse sheaves after shifting the complexes by 1. □

Remark. If every local monodromy of $K[-1]|_U$ is unipotent, we can prove the assertion by using [34].

LEMMA 1.6. *With the notation of* (1.5)*, we have spectral sequences of mixed Hodge structures*

$$E_1^{-k,k+m} = H^m(S, \mathrm{Gr}_k^W \mathcal{M}) \Rightarrow H^m(S, \mathcal{M}), \tag{1.6.1}$$

$$E_1^{-k,k+m} = H^m(\Sigma, i^* j_* \mathrm{Gr}_k^W \mathcal{M}) \Rightarrow H^m(\Sigma, i^* j_* \mathcal{M}), \tag{1.6.2}$$

together with a natural morphism of the spectral sequence (1.6.1) *to* (1.6.2).

Proof. The first spectral sequence is clear by (1.1.3), and the argument is similar for the second. The morphism of spectral sequences follows from the canonical morphism $j_*\mathcal{M} \to i_*i^*j_*\mathcal{M}$ which is compatible with the filtration induced by W on $\mathcal{M}$. □

Remark. The functor i^*j_* calculates the cohomology of the punctured neighborhood of the points at infinity of S, see (1.4.3). For a perverse sheaf K (whose restriction to U is a local system shifted by 1), $H^{-1}i^*j_*K$ and $H^0i^*j_*K$ give respectively the local invariant and coinvariant cycles (i.e. the kernel and cokernel of the variation $T_\infty - id$).

If the differential $d_r\colon E_r^{-k-r,k+r-1} \to E_r^{-k,k}$ of the spectral sequence (1.6.2) is nonzero, some element of Coker $(\mathrm{Gr}_k^W T_\infty - id)$ belongs to the image of $T_\infty - id$, and the corresponding Jordan block for $\mathrm{Gr}_k^W T_\infty$ is the restriction of a bigger Jordan block for T_∞.

2. Proof of Theorems

PROPOSITION 2.1. *Let S be a smooth affine curve, and $\mathcal{M}$ a mixed Hodge Module on S with the weight filtration W. Let U be a dense open subvariety of S on which $\mathcal{M}$ is a variation of mixed Hodge structure. If $H^0(S,\mathcal{M})$ has weights $\leqslant m$ and $\mathcal{M}|_U$ has weights $\leqslant m+1$, then $H^0(S,\mathrm{Gr}_k^W\mathcal{M})$ has weights $\leqslant m$. More precisely, if $H^0(S,\mathcal{M})$ has weights $\leqslant m$ and $\mathrm{Gr}_i^W H^0(S,\mathrm{Gr}_k^W\mathcal{M}) \neq 0$ for an integer $i > m$, then $H^{-1}(S,\mathrm{Gr}_{i+1}^W\mathcal{M}) \neq 0$, and Gr_i^W of the differential $d_{i+1-k}\colon E_{i+1-k}^{-i-1,i} \to E_{i+1-k}^{-k,k}$ of the spectral sequence* (1.6.1) *is surjective.*

Proof. Consider the spectral sequence (1.6.1). Since S is affine, we see that $E_1^{p,q} = 0$ if $p+q < -1$ or $p+q > 0$. By (1.5), $E_1^{-k,k-1}$ is pure of weight $k-1$, and $E_1^{-k,k-1} = 0$ for $k-1 > m$ using the decomposition (1.2.1) applied to $\mathrm{Gr}_k^W\mathcal{M}$, because the direct factor of $\mathrm{Gr}_k^W\mathcal{M}$ with discrete support does not contribute to $H^{-1}(S,\mathrm{Gr}_k^W\mathcal{M})$. So the assertion follows from the strict compatibility of the differential with the weight filtration, see [7]. □

PROPOSITION 2.2. *With the above notation, assume $\mathcal{M}$ is a pure Hodge Module of weight k on S. For $s \in \bar{S}\setminus S$, let t be a local coordinate at s. If $H^0(S,\mathcal{M})$ has weights $\leqslant m$, then $\psi_{t,1}\mathcal{M}$ has weights between* $\min\{k-1, 2k-m\}$ *and* $\max\{k-1, m-2\}$. *Conversely, if the monodromy of $\psi_{t,1}\mathcal{M}$ has a Jordan block of size r for the eigenvalue 1, then it gives a nonzero element of $\mathrm{Gr}_{k+r}^W H^0(\Sigma, i^*j_*\mathcal{M})$ which induces a nonzero element of $\mathrm{Gr}_{k+r}^W H^0(S,\mathcal{M})$ or $\mathrm{Gr}_{k+r}^W H^1(\bar{S}, j_{!*}\mathcal{M})$ by* (1.5.1). *Furthermore, we have $r = 1$ if the last group is nonzero.*

Proof. We have the symmetry of the weights of nearby cycles by (1.4.2). So it is enough to estimate the maximal weight for the first assertion, and it is verified by using the exact sequence (1.5.1) and taking H^0 of (1.4.3). For the last assertion, the existence of a Jordan block of size r corresponds to a nonzero element of the primitive part of $\mathrm{Gr}_{k+r}^W H^0(\Sigma, i^*j_*\mathcal{M})$ by (1.4.4). Then it corresponds by (1.5.1) to a nonzero element of $\mathrm{Gr}^W_{k+r}H^0(S,\mathcal{M})$ or $\mathrm{Gr}^W_{k+r}H^1(\bar{S},j_{!*}\mathcal{M})$. In the second case, we have $r=1$ because $H^1(\bar{S},j_{!*}\mathcal{M})$ is pure of weight $k+1$. So the assertion follows.

2.3. PROOF OF (0.4)

We may assume S connected and then affine (because otherwise $\bar{S}=S$). Let $\mathcal{M}=H^{j+1}f_*\mathbb{Q}_X^H$ so that ${}^pR^{j+1}f_*\mathbb{Q}_X$ is the underlying perverse sheaf of $\mathcal{M}$ (in particular, its restriction to U is $(R^jf_*\mathbb{Q}_X|_U)[1]$). Here ${}^pR^{j+1}f_*$ means ${}^pH^{j+1}\mathbf{R}f_*$, see (1.1.1).

By hypothesis $\mathcal{M}|_U$ has weights in $[j+r'+1, j+r+1]$ See (1.3) for the shift of weight. By the spectral sequence (1.1.5) (with $Z=pt$), we have an exact sequence of mixed Hodge structures

$$0\to H^0(S,H^{j+1}f_*\mathbb{Q}_X^H)\to H^{j+1}(X,\mathbb{Q})\to H^{-1}(S,H^{j+2}f_*\mathbb{Q}_X^H)\to 0, \qquad (2.3.1)$$

because S is an affine curve so that $E_2^{p,q}=0$ except for $p=-1,0$ and the spectral sequence degenerates at E_2. So $H^0(S,\mathcal{M})$ has weights $\leqslant j+r$, and hence $H^0(S,\mathrm{Gr}^W_k\mathcal{M})$ has weights $\leqslant j+r$ by (2.1).

This implies that the $\psi_{t,1}\mathrm{Gr}^W_k\mathcal{M}$ for $k\in[j+r'+1,j+r+1]$ (and hence $\psi_{t,1}\mathcal{M}$) have weights in $[j-r+2r'+1,j+r]$ by (2.2). This completes the proof of (0.4).

Remarks. (i) Theorem (0.1) follows from (0.4) by using (2.7) below.

(ii) Assume X is smooth and f is proper. Then $H^{j+1}(X,\mathbb{Q})$ has weights $\leqslant j+r+1$ if the Jordan blocks for the eigenvalue 1 of the local monodromies of $H^j(X_s,\mathbb{Q})$ at any points of $\bar{S}\setminus S$ have size $\leqslant r$. This follows by the same argument as above using (1.4.3), (1.5.1–2) and (2.3.1).

(iii) Assume f is a polynomial map. Let $\mathcal{M}=H^{j+1}f_*\mathbb{Q}_X^H$ for $j>0$. Then $H^i(S,\mathcal{M})=0$ for any i by (2.3.1). Let $\mathbb{D}\mathcal{M}$ be the dual of $\mathcal{M}$. Then $H^i_c(S,\mathbb{D}\mathcal{M})=0$ by the duality, and we get natural isomorphisms

$$H^i(S,\mathbb{D}\mathcal{M})=H^i(\Sigma,i^*j_*\mathbb{D}\mathcal{M}),$$

using the distinguished triangle $\to j_!\to j_*\to i_*i^*j_*\to$. For $i=-1$, this reproves a result of Dimca and Némethi [11]: $H_j(X_s,\mathbb{Q})^G=H_j(X_s,\mathbb{Q})^{T_\infty}$.

THEOREM 2.4. *Let $f\colon X\to S$ and $\bar{S}$ be as in* (0.4). *Assume the monodromy T_∞ of $\mathrm{Gr}^W_iH^j(X_s,\mathbb{Q})$ around $s_\infty\in\bar{S}\setminus S$ has a Jordan block of size r for the eigenvalue* 1, *and $H^{j+1}(X,\mathbb{Q})$ has weights $\leqslant i+r$. Let $j+m'_s$ be the maximal weight of $H^j(X_s,\mathbb{Q})$. Then $\mathrm{Gr}^W_{i'}H^j(X_s,\mathbb{Q})$ has a nonzero global invariant cycle (i.e.* $(\mathrm{Gr}^W_{i'}H^j(X_s,\mathbb{Q}))^G\neq 0$) *with $i'=i+r+1\leqslant j+m'_s\,(\leqslant\min\{2j,j+m_s\})$, or $i'=i$ and $r=1$. In the former case (e.g. if $r>1$), the given Jordan block is the restriction of a strictly larger Jordan block for the monodromy of $H^j(X_s,\mathbb{Q})$ to the graded piece* Gr^W_i. *In the latter case, letting $j+r'$ be the minimal weight of $H^j(X_s,\mathbb{Q})$, we have $i'=i>j+r'$ if $H^j(X,\mathbb{Q})$ has weights $>j+r'$.*

Proof. The given Jordan block of size r corresponds by (2.2) (with k replaced by

$i+1$) to a nonzero element u of

$$\mathrm{Gr}^W_{i+r+1} H^0(S, \mathrm{Gr}^W_{i+1}\mathcal{M}) \quad \text{or} \quad \mathrm{Gr}^W_{i+r+1} H^1(\bar{S}, j_{!*}\mathrm{Gr}^W_{i+1}\mathcal{M})$$

with the notation of (1.5) and (2.3) (in particular, $\mathcal{M} = H^{j+1} f_* \mathbb{Q}^H_X$). In the first case, we have $H^{-1}(S, \mathrm{Gr}^W_{i+r+2}\mathcal{M}) \neq 0$ by (2.1), and $i+r+1 \leqslant \min\{2j, j+m_s, 2n\}$ by [7, (8.2.4)] (see also (2.7) below). Furthermore, (2.1) says that u belongs to the image of Gr^W_{i+r+1} of the differential

$$d_{r+1} : E^{-i-r-2, i+r+1}_{r+1} \to E^{-i-1, i+1}_{r+1}$$

of the spectral sequence (1.6.1). The corresponding differential of (1.6.2) is also nonzero, because the image of u in the corresponding term of (1.6.2) does not vanish by the definition of u (i.e. it comes from the given Jordan block, see (2.2)). This nonvanishing implies the assertion on the extension of Jordan blocks, see Remark after (1.6). In the second case, we have $r = 1$ by (2.2), and $i > j + r'$ because the hypothesis on the weights of $H^j(X, \mathbb{Q})$ implies that $\{\mathrm{Gr}^W_{j+r'} H^j(X_s, \mathbb{Q})\}_{s\in U}$ has no nonzero global section (using (1.6.1)). So we get the assertion. □

COROLLARY 2.5. *Let $f: X \to S$ and T_∞ be as above. Assume $H^j(X_s, \mathbb{Q})$ and $H^{j+1}(X, \mathbb{Q})$ have weights $\leqslant m$. Then we have natural isomorphisms*

$$(\mathrm{Gr}^W_i H^j(X_s, \mathbb{Q}))^G = (\mathrm{Gr}^W_i H^j(X_s, \mathbb{Q}))^{T_\infty} \quad \textit{for } i > m-2 \tag{2.5.1}$$

and both are zero if $|\bar{S} \setminus S| > 1$.

Proof. For $i > m-2$, the restriction of T_∞ to the unipotent monodromy part (i.e. the generalized eigenspace for the eigenvalue 1) of $\mathrm{Gr}^W_i H^j(X_s, \mathbb{Q})$ is semisimple by (2.4). Hence $H^0 i^* j_* \mathrm{Gr}^W_{i+1}\mathcal{M}$ is pure of weight $i+2$, and

$$\begin{aligned}\bigoplus_{s_\infty \in \Sigma} \dim (\mathrm{Gr}^W_i H^j(X_s, \mathbb{Q}))^{T_\infty} &= \dim H^1(\bar{S}, j_{!*}\mathrm{Gr}^W_{i+1}\mathcal{M}) \\ &= \dim (\mathrm{Gr}^W_i H^j(X_s, \mathbb{Q}))^G\end{aligned}$$

by (1.5.1). In particular, $|\bar{S} \setminus S| = 1$ if both sides are nonzero. So the assertion follows. □

Remark. If f is a polynomial map and a general fiber X_s admits a smooth compactification such that the divisor at infinity is smooth, then $H^j(X_s, \mathbb{Q})$ has weights in $[j, j+1]$, and (2.4) implies that 1 is not an eigenvalue of the monodromy at infinity of $\mathrm{Gr}^W_j H^j(X_s, \mathbb{Q})$ and the size of the Jordan blocks for the eigenvalue 1 of the monodromy of $\mathrm{Gr}^W_{j+1} H^j(X_s, \mathbb{Q})$ is at most 1. In particular, the last assertion holds also for $H^j(X_s, \mathbb{Q})$. (Note that it follows also from Theorem (0.1).)

2.6. PROOF OF (0.3)

Recall that $f: X \to S$ is cohomologically tame [28] if there is an algebraic compactification $\bar{f}: \bar{X} \to S$ of f such that the support of the (shifted) perverse sheaf $\varphi_{\bar{f}-c}{}^p R^m \bar{j}_*(\mathbb{Q}_X[n+1])$ is contained in X for any $c \in \mathbb{C}$ and $m \in \mathbb{Z}$, where $\bar{j}: X \to \bar{X}$

denotes the inclusion and ${}^pR^m\bar{j}_*$ means ${}^pH^m\mathbf{R}\bar{j}_*$, see (1.1.1). Note that the condition implies that $\varphi_{\bar{f}-c}{}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1]) = 0$ for $m \neq 0$ (because ${}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1])$ is supported on $\bar{X} \setminus X$ for $m \neq 0$), and $\varphi_{\bar{f}-c}{}^pR^0\bar{j}_*(\mathbb{Q}_X[n+1])$ has discrete support. If furthermore $X = \mathbb{C}^{n+1}$ and $S = \mathbb{C}$, it is easy to show that ${}^pR^mf_*\mathbb{Q}_X = 0$ for $m \neq 1, n+1$.

Let W be the weight filtration on the perverse sheaf ${}^pR^{n+1}f_*\mathbb{Q}_X$ coming from the corresponding mixed Hodge Module. Note that there is a shift of index by 1 between this weight filtration and that on the cohomology $H^n(X_s, \mathbb{Q})$, see (1.3). We first show

$$\mathrm{Gr}_k^{W\,p}R^{n+1}f_*\mathbb{Q}_X \text{ is a constant sheaf if } k \neq n+1. \tag{2.6.1}$$

Consider the Leray spectral sequence

$$E_2^{i,m} = {}^pR^i\bar{f}_*{}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1]) \Rightarrow {}^pR^{i+m}f_*(\mathbb{Q}_X[n+1])$$

in the category of perverse sheaves on S. It underlies a spectral sequence of mixed Hodge Modules. Since the functor $\mathcal{M} \to \mathrm{Gr}_i^W\mathcal{M}$ is an exact functor of mixed Hodge Modules, we get a spectral sequence by applying this functor to the above spectral sequence. So it is enough to show that $\mathrm{Gr}_k^{W\,p}R^i\bar{f}_*{}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1])$ are locally constant sheaves for $k \neq n+1$, because S is simply connected. This is further reduced to the vanishing of the functor φ_{t-c} applied to these perverse sheaves on S for $c \in \mathbb{C}$. Here we may replace the perverse sheaves by ${}^pR^i\bar{f}_*\mathrm{Gr}_k^W({}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1]))$, using the weight spectral sequence (see (1.1.4))

$$\begin{aligned} E_1^{-k,k+i} &= {}^pR^i\bar{f}_*\mathrm{Gr}_k^W({}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1])) \\ &\Rightarrow {}^pR^i\bar{f}_*({}^pR^m\bar{j}_*(\mathbb{Q}_X[n+1])), \end{aligned}$$

together with the exactness of the vanishing cycle functor. Since this functor commutes also with ${}^pR^i\bar{f}_*$, the assertion (2.6.1) then follows from the hypothesis on the support of the vanishing cycle functor, because $\mathrm{Gr}_k^{W\,p}R^m\bar{j}_*(\mathbb{Q}_X[n+1])$ is supported on $\bar{X} \setminus X$ for $k \neq n+1$ or $m \neq 0$.

Now let $\mathcal{M} = H^0f_*(\mathbb{Q}_X^H[n+1])$, and consider the spectral sequence (1.6.1). By (2.6.1) we get

$$E_1^{-k,k+i} = 0 \quad \text{unless } i = -1, k > n+1 \text{ or } i = 0, k = n+1.$$

Furthermore $E_1^{-k,k-1}$ is pure of weight $k-1$, and $E_\infty^{-k,k+i} = 0$ for any i, k in (1.6.1) because $H^i(S, {}^pR^{n+1}f_*\mathbb{Q}_X) = 0$ for any i.

Let $\mathcal{M}' = \mathrm{Gr}_{n+1}^W\mathcal{M}$, and $\bar{S} = \mathbb{P}^1$ with the inclusion morphisms $i : \{\infty\} \to \bar{S}$, $j : S \to \bar{S}$. Then (1.6) gives commutative diagrams for $r \geqslant 1$:

$$\begin{array}{ccccc} & & H^{-1}(S, \mathrm{Gr}_{k+1}^W\mathcal{M}) & \xrightarrow{\sim} & H^{-1}i^*j_*\mathrm{Gr}_{k+1}^W\mathcal{M} \\ & & \downarrow{\scriptstyle d_r} & & \downarrow{\scriptstyle d_r} \\ \mathrm{Gr}_k^WH^0(\bar{S}, j_{!*}\mathcal{M}') & \longrightarrow & \mathrm{Gr}_k^WH^0(S, \mathcal{M}') & \longrightarrow & \mathrm{Gr}_k^WH^0i^*j_*\mathcal{M}' \end{array}$$

where $k = n + r$, and the bottom row is Gr_k^W of the exact sequence (1.5.1) with $H^1(\bar{S}, j_{!*}\mathcal{M}') = 0$. By the above argument the left vertical morphism d_r is an isomorphism for any $r \geqslant 1$. Since $H^0(\bar{S}, j_{!*}\mathcal{M}')$ is pure of weight $n+1$, and $H^0 i^* j_* \mathcal{M}'$ has weights $\geqslant n+2$ by (1.4), we see that the right vertical morphism, which is induced by d_r: $E_r^{-n-r-1,n+r} \to E_r^{-n-1,n+1}$ of (1.6.2), vanishes for $r = 1$, and is an isomorphism for $r > 1$. The first vanishing means the splitting of the extension between the Jordan blocks for the eigenvalue 1 of the local monodromy at infinity of $\mathrm{Gr}_n^W H^n(X_s, \mathbb{Q})$ and $\mathrm{Gr}_{n+1}^W H^n(X_s, \mathbb{Q})$. So the assertions on the local monodromy at infinity follows.

The triviality of local extensions at $s \in S \setminus U$ follows from the local classification of perverse sheaves or regular holonomic $\mathcal{D}$-modules ([3, 4]) which implies that locally there are no nontrivial extensions between intersection complexes with unipotent local monodromies. This completes the proof of (0.3).

2.7. GENERALIZATION OF THE MONODROMY THEOREM

Let f: $X \to S$ be a morphism of complex algebraic varieties such that $\dim S = 1$. By Remark (ii) after (1.4), the size of the Jordan blocks of the local monodromies does not exceed the maximal length of successive numbers p such that $\mathrm{Gr}_F^p H^j(X_s, \mathbb{C}) \neq 0$, because the $H^j(X_s, \mathbb{Q})$ for $s \in U$ form an admissible variation of mixed Hodge structure on a Zariski-open subset U of S ([14, 19, 34], etc.) The assertion was first shown in [32] when the generic fiber is proper smooth (see also [23]). Combined with Remark below, this gives a generalization of the monodromy theorem (see [18, 24] in the case the generic fiber is proper smooth).

Remark. Let Y be a complex algebraic variety of dimension n. Let $h^{j,p,q}(Y) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^j(Y, \mathbb{C})$. Then by [7, (8.2.4)], $h^{j,p,q}(Y) = 0$ except when $(p, q) \in [0, j] \times [0, j]$ with $j \leqslant n$, or $(p, q) \in [j-n, n] \times [j-n, n]$ with $j \geqslant n$. If Y is smooth, we have furthermore $h^{j,p,q}(Y) = 0$ for $p + q < j$ by loc. cit. In particular, $H^j(Y, \mathbb{Q})$ has weights in $[j, 2j]$ for $j \leqslant n$, and in $[j, 2n]$ otherwise.

2.8. PROOF OF (0.5)

It is well-known that $N^{n+1} = 0$ on the nearby cycle sheaf $\psi_f \mathbb{Q}_X[n]$. See e.g. [26]. This implies $N^n = 0$ on the vanishing cycles with unipotent monodromy $\varphi_{f,1}\mathbb{Q}_X[n]$ by (1.4.5). Now we take a Whitney stratification of X as in (0.5). (Here $f^{-1}(0)$ is assumed to be a union of strata.) For each stratum S_α in $f^{-1}(0)$, let X_α be a transversal space which is a locally closed complex submanifold of X. Applying the above argument to the restriction of f to X_α, we get the assertion on the dimension of the support of $\mathrm{Im}\, N^j$.

Remarks. (i) We can replace r in (0.5) by the maximal number of the local irreducible components of an embedded resolution of $f^{-1}(0)$. In this case, we can prove

(0.5), or rather the equivalent assertion after (0.5), by reducing to the normal crossing case and then using the calculation of nearby cycle sheaf as in [33] or [29, (3.6.10)]. (See also [15, 24].)

(ii) Proposition (0.5) implies the assertion that the size of the Jordan blocks for the eigenvalue 1 of the local monodromy on the jth reduced cohomology of the Milnor fiber is bounded by j (see [1, 27, 33]), because a perverse sheaf K on an analytic space Y satisfies $\mathcal{H}^i K = 0$ for $i < -\dim Y$. This can be verified by induction on $\dim Y$ using the transversal space to each stratum with positive dimension of a Whitney stratification of Y and also the long exact sequence associated to local cohomology.

As an application of Theorem (0.2), we have the following

PROPOSITION 2.9. *Let t be the coordinate of S. With the assumption of* (0.2)*, let i' be as there. Then there exists $\gamma \in H_j(X_s, \mathbb{Z})$ such that, for any algebraic differential j-form ω on X whose cohomology class in the de Rham cohomology of the generic fiber has weights $\leqslant i'$, the period integral $\int_{\gamma_t} \omega$ is a* (*univalent*) *rational function of t, where γ_t is a multivalued section of the local system consisting of the homology groups of general fibers, and is obtained by the parallel translation of γ using a local C^∞ trivialization of the restriction of f over U. This rational function is nonzero if ω is generic.*

Proof. Let W be the dual filtration on $H_j(X_s, \mathbb{Q}) = H^j(X_s, \mathbb{Q})^\vee$, i.e. $W_{-k}H_j(X_s, \mathbb{Q}) = (H^j(X_s, \mathbb{Q})/W_{k-1})^\vee$ for $k \in \mathbb{Z}$. The assumption and (0.2) imply that $(\mathrm{Gr}^W_{-i'} H_j(X_s, \mathbb{Q}))^G \neq 0$, because the local system $\{\mathrm{Gr}^W_{i'} H^j(X_s, \mathbb{Q})\}$ is selfdual by the polarization, and is identified with $\{\mathrm{Gr}^W_{-i'} H_j(X_s, \mathbb{Q})\}$. Take a nonzero element in $(\mathrm{Gr}^W_{-i'} H_j(X_s, \mathbb{Q}))^G$, which is represented by $\gamma \in W_{-i'} H_j(X_s, \mathbb{Q})$. Then, for an algebraic differential j-form ω on X such that the de Rham cohomology class of its restriction to the generic fiber of f is contained in $W_{i'}$, the period integral $\int_{\gamma_t} \omega$ is univalent, because the pairing factors through the pairing between $\mathrm{Gr}^W_{-i'} H_j(X_s, \mathbb{Q})$ and $\mathrm{Gr}^W_{i'} H^j(X_s, \mathbb{Q})$. It is a rational function by regularity, and is nonzero if ω is generic. So the assertion follows. □

Remarks. (i) Proposition (2.9) does not necessarily imply that γ is extended to a univalent section of the local system, because only $\mathrm{Gr}^W_{-i'}\gamma$ is extended in such a way. Note that the G-invariant cycles coincide with the T_∞-invariant cycles for homology (i.e. for the dual representation) by Dimca and Némethi [11]. However, every invariant cycle of $\mathrm{Gr}^W H_j(X_s, \mathbb{Q})$ does not necessarily come from an invariant cycle of $H_j(X_s, \mathbb{Q})$ in general (e.g. $f = x^2y^2z^2 - x^2y^2 + x^2 + y^2 + w^2$).

(ii) As another application, we have the the following consequence to the behavior of the period integral at infinity in general. For an algebraic differential form ω and $\gamma \in H_j(X_s, \mathbb{Z})$, consider the asymptotic expansion at infinity

$$\int_{\gamma_t} \omega \sim \sum_{\alpha \leqslant \alpha_0} \sum_{r=0}^{r(\alpha)} C(\alpha, r) t^\alpha (\log t)^r,$$

where $\alpha_0 \in \mathbb{Q}$, $r(\alpha) \in \mathbb{N}$ and $C(\alpha, r) \in \mathbb{C}$. Then, by the theory of Nilson class functions in [5], Theorem (0.1) implies

$$r(\alpha) \leqslant m'_s - 1 \,(\leqslant \min\{m_s - 1, j - 1\}) \quad \text{if } \alpha \in \mathbb{Z}. \tag{2.9.1}$$

Note that we have only $r(\alpha) \leqslant j$ for a general α by the monodromy theorem.

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