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On q-analogues of double Euler sums

K. Dilcher [a], Kh. Hessami Pilehrood [a], T. Hessami Pilehrood [b,*]

[a] Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 4R2, Canada
[b] Department of Mathematics and Computing Science, Saint Mary's University, Halifax, Nova Scotia, B3H 3C3, Canada

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ABSTRACT

In this paper we study properties of a q-analogue of the function $\pi/\sin\pi z$ which is defined by means of Jackson's q-gamma function and a reflection formula for the gamma function. As application, we obtain evaluations for q-analogues of double Euler sums in terms of single q-zeta values. We also derive a q-analogue of a well-known formula for the multiple zeta-star value $\zeta^*(\{2\}^a)$.

1. Introduction

In recent years there has been significant interest in investigating multiple zeta values and their q-analogues. The subject of this paper is studying q-analogues of double Euler sums

$$S_{a,b}^{+-}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n(a)}{n^b},\qquad S_{a,b}^{-+}=\sum_{n=1}^{\infty}\frac{\overline{H}_n(a)}{n^b} \tag{1}$$

where

$$H_n(a)=\sum_{j=1}^{n}\frac{1}{j^a}\quad\text{and}\quad \overline{H}_n(a)=\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j^a}$$

are generalized harmonic numbers or partial sums of the series

$$\zeta(a)=\sum_{n=1}^{\infty}\frac{1}{j^a}\quad\text{and}\quad \bar{\zeta}(a)=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j^a}$$

defining the Riemann zeta function and the alternating zeta function, respectively. If a, b are positive integers and $a+b$ is odd, the sums (1) are expressible in terms of ζ and $\bar{\zeta}$ values as follows [3]:

* Corresponding author.
E-mail addresses: dilcher@mathstat.dal.ca (K. Dilcher), hessamik@gmail.com (Kh. Hessami Pilehrood), hessamit@gmail.com (T. Hessami Pilehrood).

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$$2S^{+-}_{a,b} = \big(1-(-1)^a\big)\zeta(a)\bar\zeta(b) + \bar\zeta(a+b) - 2\sum_{j+2k=a}\binom{b+j-1}{b-1}(-1)^j\bar\zeta(b+j)\bar\zeta(2k)$$
$$+ 2(-1)^a\sum_{j+2k=b}\binom{a+j-1}{a-1}\zeta(a+j)\bar\zeta(2k), \tag{2}$$
$$2S^{-+}_{a,b} = \big(1-(-1)^a\big)\bar\zeta(a)\zeta(b) + \bar\zeta(a+b) + 2(-1)^a\sum_{j+2k=a}\binom{b+j-1}{b-1}\zeta(b+j)\bar\zeta(2k)$$
$$+ 2\sum_{j+2k=b}\binom{a+j-1}{a-1}(-1)^j\bar\zeta(a+j)\bar\zeta(2k), \quad b>1. \tag{3}$$

In formulas (2), (3) it is understood that $\zeta(0) = -1/2$ in accordance with the analytic continuation of the Riemann zeta function, the value $\zeta(1)$ is interpreted as zero, $\bar\zeta(0) = 1/2$, and $\bar\zeta(1) = \log 2$.

In this paper we find a suitable q-analogue for alternating zeta values that allows us to obtain appropriate q-analogues of formulas (2) and (3). As an intermediate result, we derive a q-analogue of the well-known formula

$$\zeta^*\big(\{2\}^a\big) = 2\bar\zeta(2a) = 2\big(1-2^{1-2a}\big)\zeta(2a), \tag{4}$$

where

$$\zeta^*(s_1,\dots,s_a) = \sum_{1\leqslant k_1\leqslant\cdots\leqslant k_a}\frac{1}{k_1^{s_1}\cdots k_a^{s_a}}, \quad s_j\in\mathbb{N},\ s_a\geqslant 2,$$

is the multiple zeta-star value, $\zeta^*(\emptyset) = 1$, and $\{2\}^a$ denotes the a-tuple $(2,\dots,2)$.

Note that the proof of (4) is quite simple and is based on the representation of the function $\pi z/\sin\pi z$ as an infinite product, which gives a generating function for the zeta-star values $\zeta^*(\{2\}^a)$,

$$\frac{\pi z}{\sin\pi z} = \prod_{m=1}^{\infty}\frac{1}{1-z^2/m^2} = 1 + \sum_{a=1}^{\infty}\zeta^*\big(\{2\}^a\big)z^{2a} \tag{5}$$

on the one hand, and as an infinite partial fraction expansion, which leads to the Laurent series,

$$\frac{\pi z}{\sin\pi z} = 1 + 2z^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-z^2} = 1 + 2\sum_{a=1}^{\infty}\bar\zeta(2a)z^{2a} \tag{6}$$

on the other hand. Laurent series expansions of the function $\pi/\sin\pi z$ played an important role, as well, in the Flajolet and Salvy proof [3] of formulas (2) and (3) (other proofs of these formulas using an elementary partial fraction approach and Fourier expansions of Bernoulli polynomials can be found in [2,8]).

2. On a q-analogue of the function $\pi/\sin\pi z$ and its properties

Let q be a real number with $0<q<1$. For any complex number α, the basic number is defined as

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}.$$

For a positive integer n, put

$$(\alpha)_0 := (\alpha;q)_0 := 1, \qquad (\alpha)_n := (\alpha;q)_n := \prod_{k=0}^{n-1}\big(1-\alpha q^k\big), \qquad (\alpha)_\infty := (\alpha;q)_\infty := \prod_{k=0}^{\infty}\big(1-\alpha q^k\big).$$

For defining a q-analogue of $\pi/\sin\pi z$ it is convenient to use the reflection formula for the Euler gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}.$$

Let us consider the function

$$f(z) = -\frac{(q;q)_\infty^2}{(z;q)_\infty(qz^{-1};q)_\infty} = \frac{1}{z-1}\prod_{k=1}^{\infty}\frac{(1-q^k)^2}{(1-q^kz)(1-q^kz^{-1})}. \tag{7}$$

Since

$$\frac{(1-q^k)^2}{(1-q^kz)(1-q^kz^{-1})} = 1 + \frac{q^k(1-z)^2}{z(1+q^{2k}) - q^k(1+z^2)},$$

by the Weierstrass test we conclude that the infinite product (7) converges uniformly on any compact subset of $\mathbb{C}$ not containing 0 and q^n, $n \in \mathbb{Z}$, and defines a meromorphic function on $\mathbb{C} \setminus \{0\}$ with simple poles at q^n, $n \in \mathbb{Z}$, and residues

$$\operatorname*{res}_{z=q^n} f(z) = (-1)^n q^{n(n+1)/2}, \quad n \in \mathbb{Z}. \tag{8}$$

It is easily seen that

$$(q-1) f\left(q^x\right) = \Gamma_q(x)\Gamma_q(1-x),$$

where $\Gamma_q(x)$ is Jackson's q-gamma function,

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}.$$

From [6, Appendix B] it follows that

$$\lim_{q\to 1}(q-1) f\left(q^x\right) = \lim_{q\to 1}\Gamma_q(x)\Gamma_q(1-x) = \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Lemma 2.1. *For any integer n, the function $f(z)$ satisfies the functional equation*

$$f\left(zq^n\right) = (-1)^n z^n q^{n(n-1)/2} f(z). \tag{9}$$

Proof. The formula is trivial for $n = 0$. Suppose n is positive. Then

$$f\left(zq^n\right) = \frac{1}{zq^n - 1}\prod_{k=1}^{\infty}\frac{(1-q^k)^2}{(1-q^{k+n}z)(1-q^{k-n}z^{-1})} = -zf(z)\prod_{k=1}^{n-1}\frac{1-q^kz}{1-q^{k-n}z^{-1}}$$

$$= -zf(z)\prod_{k=1}^{n-1}(1-q^kz)\prod_{k=1}^{n-1}\frac{zq^{n-k}}{zq^{n-k}-1} = (-1)^n z^n f(z) q^{n(n-1)/2},$$

and Eq. (9) is true in this case. Since

$$f\left(z^{-1}\right) = -zf(z), \tag{10}$$

we have

$$f\left(zq^{-n}\right) = f\left(\left(q^n/z\right)^{-1}\right) = -\frac{q^n}{z} f\left(q^n z^{-1}\right).$$

Now applying Eq. (9) with z replaced by $1/z$ and $n \in \mathbb{N}$ we obtain

$$f\left(zq^{-n}\right) = -\frac{q^n}{z}(-1)^n z^{-n} q^{n(n-1)/2} f\left(z^{-1}\right) = (-1)^{n-1} z^{-n-1} q^{n(n+1)/2} f\left(z^{-1}\right)$$

which by (10) gives

$$f\left(zq^{-n}\right) = (-1)^{n-1} z^{-n-1} q^{n(n+1)/2}\left(-zf(z)\right) = (-1)^n z^{-n} q^{n(n+1)/2} f(z),$$

and therefore Eq. (9) is valid for all $n \in \mathbb{Z}$. $\square$

We consider the following q-analogues of zeta, alternating zeta, and multiple zeta-star values:

$$\mathfrak{z}_q[b;a] := \sum_{n=1}^{\infty}\frac{q^{n(n-1)/2+an}}{[n]_q^b}, \quad a,b \in \mathbb{Z},$$

$$\mathfrak{z}_q^{\wedge}[b;a] := \sum_{n=1}^{\infty}\frac{(1+q^n)q^{n(n-1)/2+an}}{[n]_q^b} = \mathfrak{z}_q[b;a] + \mathfrak{z}_q[b;a+1],$$

$$\bar{\mathfrak{z}}_q[b;a] := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n-1)/2+an}}{[n]_q^b},$$

$$\mathfrak{z}_q^{\wedge}[b;a] := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^n)q^{n(n-1)/2+an}}{[n]_q^b} = \mathfrak{z}_q[b;a] + \mathfrak{z}_q[b;a+1],$$

$$\zeta_q^*[b_1,\ldots,b_m] := \sum_{1\leqslant k_1\leqslant\cdots\leqslant k_m} \frac{q^{k_1}}{[k_1]_q^{b_1}}\cdots\frac{q^{k_m}}{[k_m]_q^{b_m}}, \quad b_1,\ldots,b_m\in\mathbb{Z},$$

with the convention $\zeta_q^*[\emptyset]=1$.

Lemma 2.2. *Let $a\in\mathbb{R}$, $n\in\mathbb{Z}$. Then the Laurent series expansion of $z^a f(z)$ about q^n is*

$$z^a f(z) = \frac{(-1)^n q^{n(n+1)/2+an}}{z-q^n} \sum_{j=0}^{\infty} q^{-nj}\left(z-q^n\right)^j \sum_{l=0}^{[j/2]} \binom{a+n-l}{j-2l} \frac{\zeta_q^*[\{2\}^l]}{(1-q)^{2l}}. \tag{11}$$

Proof. We start with rewriting Eq. (7) as

$$f(z) = \frac{1}{z-1}\prod_{k=1}^{\infty}\left(1-\frac{q^k}{(1-q^k)^2}\frac{(z-1)^2}{z}\right)^{-1}$$

which implies the following expansion valid in a neighborhood of the point $z=1$:

$$f(z) = \frac{1}{z-1}\sum_{l=0}^{\infty}\frac{\zeta_q^*[\{2\}^l]}{(1-q)^{2l}}\frac{(z-1)^{2l}}{z^l}.$$

Expanding z^{a-l} in powers of $z-1$,

$$z^{a-l} = \left(1+(z-1)\right)^{a-l} = \sum_{k=0}^{\infty}\binom{a-l}{k}(z-1)^k$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!},$$

we obtain

$$\begin{aligned} z^a f(z) &= \frac{1}{z-1}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{a-l}{k}\frac{\zeta_q^*[\{2\}^l]}{(1-q)^{2l}}(z-1)^{2l+k} \\ &= \frac{1}{z-1}\sum_{j=0}^{\infty}(z-1)^j\sum_{l=0}^{[j/2]}\binom{a-l}{j-2l}\frac{\zeta_q^*[\{2\}^l]}{(1-q)^{2l}}, \end{aligned} \tag{12}$$

which is exactly (11) in the case $n=0$.

Now suppose that n is a nonzero integer. Replacing z by zq^n and a by $a-n$ in (12), we obtain the following expansion in powers of $(z-q^{-n})$:

$$z^{a-n} f\left(zq^n\right) = \frac{q^{n(n-1)-an}}{z-q^{-n}}\sum_{j=0}^{\infty}q^{nj}\left(z-q^{-n}\right)^j\sum_{l=0}^{[j/2]}\binom{a-n-l}{j-2l}\frac{\zeta_q^*[\{2\}^l]}{(1-q)^{2l}}. \tag{13}$$

By Lemma 2.1, we have

$$z^a f(z) = (-1)^n q^{-n(n-1)/2} z^{a-n} f\left(zq^n\right)$$

which, by (13), implies the required expansion (11) if we replace n by $-n$. □

Theorem 2.1. *Let $a\in\mathbb{R}$ and b be a nonnegative integer. Then*

$$\mathfrak{z}_q[b;b-a] + (-1)^b\cdot\mathfrak{z}_q[b;a+1] = \sum_{l=0}^{[b/2]}\binom{a-l}{b-2l}(1-q)^{b-2l}\zeta_q^*\left[\{2\}^l\right]. \tag{14}$$

In particular, for any nonnegative integer a,

$$\zeta_q^*\left[\{2\}^a\right]=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(1+q^n)q^{n(n-1)/2+an}}{[n]_q^{2a}}=\bar{\mathfrak{z}}_q^{\wedge}[2a;a]. \tag{15}$$

Proof. Let $R>0$ be a sufficiently large real number distinct from q^k, $k\in\mathbb{Z}$, and N be the largest positive integer such that $q^N>1/R$. Then by the residue theorem, we have

$$\int_{|z|=R}\frac{z^a f(z)}{(z-1)^b}\,dz-\int_{|z|=1/R}\frac{z^a f(z)}{(z-1)^b}\,dz=\sum_{k=1}^{N}\operatorname*{res}_{z=q^k}\left(\frac{z^a f(z)}{(z-1)^b}\right)+\sum_{k=1}^{N}\operatorname*{res}_{z=q^{-k}}\left(\frac{z^a f(z)}{(z-1)^b}\right)+\operatorname*{res}_{z=1}\left(\frac{z^a f(z)}{(z-1)^b}\right) \tag{16}$$

where contours of integration on the left-hand side are taken in the counterclockwise direction. From an asymptotic formula for $(qz;q)_\infty$ with $|z|=R\to\infty$, due to Littlewood [7, §12], it follows that

$$\left|(qz;q)_\infty\right|\asymp e^{-\frac{1}{2}\frac{\ln^2 R}{\ln q}}R^{-1/2}.$$

Since the infinite product

$$\frac{1}{|(qz^{-1};q)_\infty|}=\prod_{k=1}^{\infty}\frac{1}{|1-q^kz^{-1}|}=\prod_{k=1}^{\infty}\left|1+\frac{q^k}{z-q^k}\right|\leqslant\prod_{k=1}^{\infty}\left(1+\frac{q^k}{R-q^k}\right)$$

is bounded on $|z|=R$, we obtain

$$\left|\int_{|z|=R}\frac{z^a f(z)}{(z-1)^b}\,dz\right|=O\left(R^{a-b+1/2}e^{\frac{1}{2}\frac{\ln^2 R}{\ln q}}\right), \tag{17}$$

and therefore the absolute value of the integral in (17) tends to zero as $R\to\infty$. Similarly, on the circle $|z|=1/R\to 0$ we have

$$\left|\left(qz^{-1};q\right)_\infty\right|\asymp e^{-\frac{1}{2}\frac{\ln^2 R}{\ln q}}R^{-\frac{1}{2}},$$

and the product

$$\frac{1}{|(qz;q)_\infty|}=\prod_{k=1}^{\infty}\frac{1}{|1-q^kz|}=\prod_{k=1}^{\infty}\left|1+\frac{q^k}{1/z-q^k}\right|\leqslant\prod_{k=1}^{\infty}\left(1+\frac{q^k}{R-q^k}\right)$$

is bounded. This implies that

$$\left|\int_{|z|=1/R}\frac{z^a f(z)}{(z-1)^b}\,dz\right|=O\left(R^{-a-1/2}e^{\frac{1}{2}\frac{\ln^2 R}{\ln q}}\right) \tag{18}$$

and the absolute value of the integral tends to zero as $R\to\infty$. Therefore letting R tend to infinity in (16), by (17) and (18), we obtain

$$\sum_{k=1}^{\infty}\operatorname*{res}_{z=q^k}\left(\frac{z^a f(z)}{(z-1)^b}\right)+\sum_{k=1}^{\infty}\operatorname*{res}_{z=q^{-k}}\left(\frac{z^a f(z)}{(z-1)^b}\right)=-\operatorname*{res}_{z=1}\left(\frac{z^a f(z)}{(z-1)^b}\right).$$

The function $z^a f(z)/(z-1)^b$ has simple poles at q^k, $k\in\mathbb{Z}\setminus\{0\}$, and by (8), we have

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}q^{k(k-1)/2+(b-a)k}}{[k]_q^b}+\frac{(-1)^{k-1+b}q^{k(k+1)/2+ak}}{[k]_q^b}\right)=(1-q)^b\cdot\operatorname*{res}_{z=1}\left(\frac{z^a f(z)}{(z-1)^b}\right).$$

By Lemma 2.2 with $n=0$, we easily find that

$$\operatorname*{res}_{z=1}\left(\frac{z^a f(z)}{(z-1)^b}\right)=\sum_{l=0}^{[b/2]}\binom{a-l}{b-2l}\frac{\zeta_q^*[\{2\}^l]}{(1-q)^{2l}},$$

which implies the desired identity (14). Putting $b=2a$ in (14) and noting that $\binom{a-l}{2a-2l}$ is distinct from zero if and only if $l=a$, we obtain (15). □

Notice that formula (15) was also proved in [4, Corollary 1.1] by another method. Using Theorem 2.1, we can rewrite the expansion coefficients in (11) in terms of the alternating q-zeta values $\bar{\mathfrak{z}}_q$.

Corollary 2.1. *Let $a \in \mathbb{R}$, $n \in \mathbb{Z}$. Then the Laurent series expansion of $z^a f(z)$ about q^n is given by*

$$z^a f(z) = \frac{(-1)^n q^{n(n+1)/2+an}}{z-q^n} \sum_{j=0}^{\infty} \frac{q^{-nj}}{(1-q)^j} (z-q^n)^j \big(\bar{\mathfrak{z}}_q[j;\, j-a-n] + (-1)^j \bar{\mathfrak{z}}_q[j;\, a+n+1]\big).$$

Formula (15) also allows us to obtain a q-analogue of the infinite partial fraction expansion (6). Indeed, for the product $\Gamma_q(1+x)\Gamma_q(1-x)$, we easily derive the expansion

$$\Gamma_q(1+x)\Gamma_q(1-x) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{n+x})(1-q^{n-x})} = \prod_{n=1}^{\infty} \left(1 + \frac{q^n}{[n]_q^2}[x]_q[-x]_q\right)^{-1}$$
$$= 1 + \sum_{l=1}^{\infty} (-1)^l \zeta_q^*\big[\{2\}^l\big]\big([x]_q[-x]_q\big)^l, \tag{19}$$

which is a q-analogue of the generating function (5). Substituting formula (15) into the right-hand side of (19), changing the order of summation and simplifying, we get a q-analogue of the infinite partial fraction expansion for the function $\pi x/\sin \pi x$.

Corollary 2.2. *Let x be a real number satisfying $|x| < 1$. Then*

$$\Gamma_q(1+x)\Gamma_q(1-x) = 1 - [x]_q[-x]_q \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^n)q^{\frac{n(n+1)}{2}}}{[n]_q^2 + q^n [x]_q[-x]_q}.$$

Note that the expansion (19) in powers of $[x]_q$ and $[-x]_q$ is not unique, since $[x]_q$ and $[-x]_q$ are connected by the relation $[x]_q + [-x]_q = (1-q)[x]_q[-x]_q$. Using Heine's q-analogue of the Gauss summation formula for the ordinary hypergeometric function, Bradley [1, Theorem 16] proved another expansion for $\Gamma_q(1+x)\Gamma_q(1-x)$ in terms of the q-zeta values

$$\tilde{\zeta}_q[s_1, \ldots, s_m] = \sum_{k_1 > \cdots k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}},$$

which has the form

$$\Gamma_q(1+x)\Gamma_q(1-x) = 1 - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{m+n} \tilde{\zeta}_q\big[m+2, \{1\}^n\big][x]_q^{m+1}[-x]_q^{n+1}.$$

3. Evaluation of q-analogues of double Euler sums

We start with the definition of q-analogues of the generalized harmonic numbers:

$$H_n[b;a] := \sum_{k=1}^{n} \frac{q^{ka}}{[k]_q^b}, \qquad \overline{H}_n[b;a] := \sum_{k=1}^{n} \frac{(-1)^{k-1}q^{ka}}{[k]_q^b},$$
$$H_n^{\wedge}[b;a] := \sum_{k=1}^{n} \frac{(1+q^k)q^{ka}}{[k]_q^b}, \qquad \overline{H}_n^{\wedge}[b;a] := \sum_{k=1}^{n} \frac{(-1)^{k-1}(1+q^k)q^{ka}}{[k]_q^b}$$

and single zeta values:

$$\zeta_q[b;a] := \sum_{k=1}^{\infty} \frac{q^{ka}}{[k]_q^b}, \qquad \bar{\zeta}_q[b;a] := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{ka}}{[k]_q^b}.$$

Define q-analogues of double Euler sums as

$$\mathfrak{z}_q^{+-}[b,d;a,c] := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n(n-1)/2+cn}}{[n]_q^d} H_n[b;a],$$
$$\mathfrak{z}_q^{-+}[b,d;a,c] := \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2+cn}}{[n]_q^d} \overline{H}_n[b;a],$$

$$\hat{\mathfrak{z}}_q^{+-}[b,d;a,c] := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^n)q^{n(n-1)/2+cn}}{[n]_q^d} H_n^{\wedge}[b;a],$$
$$\hat{\mathfrak{z}}_q^{-+}[b,d;a,c] := \sum_{n=1}^{\infty} \frac{(1+q^n)q^{n(n-1)/2+cn}}{[n]_q^d} \overline{H}_n^{\wedge}[b;a].$$

Consider the q-polygamma function

$$\psi_{a,b}(z) = \sum_{k=0}^{\infty} \frac{q^{ak}}{(1-zq^k)^b}, \quad a,b \in \mathbb{N},\ z \in \mathbb{C} \setminus \{1, q^{-1}, q^{-2}, \dots\},$$

and its alternating version

$$\overline{\psi}_{a,b}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{ak}}{(1-zq^k)^b}, \quad a,b \in \mathbb{N},\ z \in \mathbb{C} \setminus \{1, q^{-1}, q^{-2}, \dots\}.$$

The function $\psi_{a,b}$ is related to the Hurwitz zeta function by the limit relation

$$\lim_{q\to 1} (1-q)^b \psi_{a,b}(q^x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^b}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\dots\}.$$

Note that

$$q^a(1-q)^b \psi_{a,b}(q) = \zeta_q[b;a] \quad \text{and} \quad q^a(1-q)^b \overline{\psi}_{a,b}(q) = \bar{\zeta}_q[b;a].$$

Lemma 3.1. *Let a, b be positive integers. Then the Laurent series expansions of $\psi_{a,b}(z)$ and $\overline{\psi}_{a,b}(z)$ about q^n, $n \in \mathbb{N}$, are given by*

$$\psi_{a,b}(z) = \sum_{l=0}^{\infty} \binom{b+l-1}{l} \frac{q^{-n(a+l)}}{(1-q)^{b+l}} \big(\zeta_q[b+l;a+l] - H_{n-1}[b+l;a+l]\big)\big(z-q^n\big)^l,$$
$$\overline{\psi}_{a,b}(z) = (-1)^n \sum_{l=0}^{\infty} \binom{b+l-1}{l} \frac{q^{-n(a+l)}}{(1-q)^{b+l}} \big(\overline{H}_{n-1}[b+l;a+l] - \bar{\zeta}_q[b+l;a+l]\big)\big(z-q^n\big)^l,$$

and about q^{-n}, $n \in \mathbb{N} \cup \{0\}$, they are represented by

$$\psi_{a,b}(z) = \frac{(-1)^b q^{(a-b)n}}{(z-q^{-n})^b} + \sum_{l=0}^{\infty} \binom{b+l-1}{l} \frac{q^{n(a+l)}}{(1-q)^{b+l}} \big(\zeta_q[b+l;a+l] + (-1)^{b+l} H_n[b+l;b-a]\big)\big(z-q^{-n}\big)^l,$$
$$\overline{\psi}_{a,b}(z) = \frac{(-1)^{b+n} q^{(a-b)n}}{(z-q^{-n})^b} - (-1)^n \sum_{l=0}^{\infty} \binom{b+l-1}{l} \frac{q^{n(a+l)}}{(1-q)^{b+l}} \big(\bar{\zeta}_q[b+l;a+l] + (-1)^{b+l} \overline{H}_n[b+l;b-a]\big)\big(z-q^{-n}\big)^l.$$

Proof. Let n be a positive integer and $\sigma = \pm 1$. Consider the series

$$\psi_{a,b}(z,\sigma) := \sum_{k=0}^{\infty} \frac{\sigma^k q^{ak}}{(1-zq^k)^b} = \sum_{k=0}^{\infty} \frac{\sigma^k q^{ak}}{(1-q^{k+n})^b (1 - \frac{q^k}{1-q^{k+n}}(z-q^n))^b}$$
$$= \sum_{k=n}^{\infty} \frac{\sigma^{k-n} q^{a(k-n)}}{(1-q^k)^b (1 - \frac{q^{k-n}}{1-q^k}(z-q^n))^b}.$$

Expanding the last sum in powers of $(z-q^n)$, we obtain

$$\psi_{a,b}(z,\sigma) = \sum_{k=n}^{\infty} \frac{\sigma^{k-n} q^{a(k-n)}}{(1-q^k)^b} \sum_{l=0}^{\infty} \binom{b+l-1}{l} \frac{q^{(k-n)l}}{(1-q^k)^l} \big(z-q^n\big)^l$$
$$= \sum_{l=0}^{\infty} \binom{b+l-1}{l} \sigma^n q^{-n(a+l)} \big(z-q^n\big)^l \sum_{k=n}^{\infty} \frac{\sigma^k q^{k(a+l)}}{(1-q^k)^{b+l}},$$

which implies the first desired expansion if $\sigma = 1$, and the second one if $\sigma = -1$.

Now let n be a nonnegative integer. To expand $\psi_{a,b}(z,\sigma)$ in powers of $(z-q^{-n})$, we rewrite it as

$$\psi_{a,b}(z,\sigma)=\frac{\sigma^n q^{an}}{(1-zq^n)^b}+\sum_{k=0}^{n-1}\frac{\sigma^k q^{ak}}{(1-zq^k)^b}+\sum_{k=n+1}^{\infty}\frac{\sigma^k q^{ak}}{(1-zq^k)^b}.$$

Now changing the order of summation in the second and third sums, we obtain

$$\begin{aligned}\psi_{a,b}(z,\sigma)&=\frac{(-1)^b\sigma^n q^{(a-b)n}}{(z-q^{-n})^b}+\sum_{k=1}^{n}\frac{\sigma^{n-k}q^{a(n-k)}}{(1-zq^{n-k})^b}+\sum_{k=1}^{\infty}\frac{\sigma^{k+n}q^{a(k+n)}}{(1-zq^{k+n})^b}\\&=\frac{(-1)^b\sigma^n q^{(a-b)n}}{(z-q^{-n})^b}+\sum_{k=1}^{n}\frac{(-1)^b\sigma^{n-k}q^{an+k(b-a)}}{(1-q^k)^b(1+\frac{q^n}{1-q^k}(z-q^{-n}))^b}+\sum_{k=1}^{\infty}\frac{\sigma^{k+n}q^{a(k+n)}}{(1-q^k)^b(1-\frac{q^{k+n}}{1-q^k}(z-q^{-n}))^b}.\end{aligned}$$

Expanding the above series in powers of $(z-q^{-n})$, we find

$$\begin{aligned}\psi_{a,b}(z,\sigma)=&\frac{(-1)^b\sigma^n q^{(a-b)n}}{(z-q^{-n})^b}+\sum_{l=0}^{\infty}(-1)^{b+l}\binom{b+l-1}{l}\frac{\sigma^n q^{n(a+l)}(z-q^{-n})^l}{(1-q)^{b+l}}\sum_{k=1}^{n}\frac{\sigma^k q^{k(b-a)}}{[k]_q^{b+l}}\\&+\sum_{l=0}^{\infty}\binom{b+l-1}{l}\frac{\sigma^n q^{n(a+l)}}{(1-q)^{b+l}}(z-q^{-n})^l\sum_{k=1}^{\infty}\frac{\sigma^k q^{k(a+l)}}{[k]_q^{b+l}},\end{aligned}$$

which implies the third desired formula if we take $\sigma=1$, and the fourth one if $\sigma=-1$. □

Theorem 3.1. *Suppose $a,b,c,d\in\mathbb{Z}$, $a\geqslant 1$, $b\geqslant 1$, $d\geqslant 0$, and $b+d$ is odd. Then*

$$\mathfrak{z}_q^{+-}[b,d;a,c-a+1]+\mathfrak{z}_q^{+-}[b,d;b-a,a+d-c]$$

is given by

$$\begin{aligned}&\bar{\mathfrak{z}}_q[b+d;b+d-c]+\zeta_q[b;a]\big(\bar{\mathfrak{z}}_q[d;c-a+1]-(-1)^b\bar{\mathfrak{z}}_q[d;a+d-c]\big)\\&-\sum_{j=0}^{d}\binom{-b}{d-j}\zeta_q[b+d-j;a+d-j]\big(\bar{\mathfrak{z}}_q[j;c+1]+(-1)^j\bar{\mathfrak{z}}_q[j;j-c]\big)\\&+\sum_{j=0}^{b}\binom{-d}{b-j}\sum_{k=1}^{\infty}\frac{(-1)^k q^{k(k-1)/2+k(a+d-c)}}{[k]_q^{b+d-j}}\big(\bar{\mathfrak{z}}_q[j;j+k-c]+(-1)^j\bar{\mathfrak{z}}_q[j;c-k+1]\big).\end{aligned}$$

Proof. Let

$$g(z):=\frac{z^c f(z)\psi_{a,b}(z)}{(z-1)^d}.$$

Then by the residue theorem and a similar argument as in the proof of Theorem 2.1, we obtain

$$\sum_{k=1}^{\infty}\operatorname*{res}_{z=q^k}g(z)+\sum_{k=1}^{\infty}\operatorname*{res}_{z=q^{-k}}g(z)=-\operatorname*{res}_{z=1}g(z).\tag{20}$$

The function $g(z)$ has simple poles at $z=q^k$, $k\in\mathbb{N}$, poles of order $b+1$ at $z=q^{-k}$, $k\in\mathbb{N}$, and a pole of order $b+d+1$ at the point $z=1$. By Lemmas 2.2 and 3.1, we easily find that

$$\operatorname*{res}_{z=q^k}g(z)=\frac{(-1)^k q^{k(k+1)/2+k(c-a)}}{(1-q)^b(q^k-1)^d}\big(\zeta_q[b;a]-H_{k-1}[b;a]\big)$$

and therefore,

$$\sum_{k=1}^{\infty}\operatorname*{res}_{z=q^k}g(z)=\frac{(-1)^d}{(1-q)^{b+d}}\big(\mathfrak{z}_q^{+-}[b,d;a,c-a+1]-\zeta_q[b;a]\bar{\mathfrak{z}}_q[d;c-a+1]-\bar{\mathfrak{z}}_q[b+d;c+1]\big).\tag{21}$$

Taking into account that

$$(z-1)^{-d}=\big(1-q^{-k}\big)^{-d}\left(\frac{z-q^{-k}}{1-q^{-k}}-1\right)^{-d}=\frac{q^{kd}}{(1-q)^d[k]_q^d}\sum_{j=0}^{\infty}\binom{-d}{j}\frac{q^{kj}}{[k]_q^j}\frac{(z-q^{-k})^j}{(1-q)^j},$$

by Lemma 3.1 and Corollary 2.1, we obtain

$$\operatorname*{res}_{z=q^{-k}} g(z) = \frac{(-1)^k q^{k(k-1)/2+k(a+d-c)}}{(1-q)^{b+d}[k]_q^d}\big(\zeta_q[b;a] + (-1)^b H_k[b;b-a]\big) + \frac{(-1)^{b+k} q^{k(k-1)/2+k(a+d-c)}}{(1-q)^{b+d}[k]_q^{b+d}} \sum_{j=0}^{b} \binom{-d}{b-j} [k]_q^j \big(\mathfrak{z}_q[j;j+k-c] + (-1)^j \mathfrak{z}_q[j;c-k+1]\big)$$

and therefore,

$$\sum_{k=1}^{\infty} \operatorname*{res}_{z=q^{-k}} g(z) = \frac{(-1)^{b+1}}{(1-q)^{b+d}} \Bigg(\mathfrak{z}_q^{+-}[b,d;b-a,a+d-c] + (-1)^b \zeta_q[b;a]\mathfrak{z}_q[d;a+d-c] + \sum_{j=0}^{b} \binom{-d}{b-j} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{k(k-1)/2+k(a+d-c)}}{[k]_q^{b+d-j}} \big(\mathfrak{z}_q[j;j+k-c] + (-1)^j \mathfrak{z}_q[j;c-k+1]\big)\Bigg). \tag{22}$$

Finally, calculating the residue at $z=1$, we have

$$\operatorname*{res}_{z=1} g(z) = \frac{(-1)^b}{(1-q)^{b+d}} \Bigg(\mathfrak{z}_q[b+d;b+d-c] - \mathfrak{z}_q[b+d;c+1] - \sum_{j=0}^{d} \binom{-b}{d-j} \zeta_q[b+d-j;a+d-j]\big(\mathfrak{z}_q[j;c+1] + (-1)^j \mathfrak{z}_q[j;j-c]\big)\Bigg). \tag{23}$$

Now by (20)–(23), we readily obtain the theorem. □

Note that letting q tend to 1 in Theorem 3.1, we recover formula (2) of Flajolet and Salvy for $S_{b,d}^{+-}$ with $b+d$ odd. The next corollary gives two special cases of Theorem 3.1 when the parity of the parameters b and d is fixed.

Corollary 3.1. *Let $a, d \in \mathbb{Z}$, $a \geqslant 1$, $d \geqslant 0$. Then*

$$2\mathfrak{z}_q^{+-}[2a,2d+1;a,d+1] = \mathfrak{z}_q[2a+2d+1;a+d+1] - \sum_{j=0}^{2d+1} \binom{-2a}{2d+1-j} \zeta_q[2a+2d+1-j;2d+a+1-j] \times \big(\mathfrak{z}_q[j;a+d+1] + (-1)^j \mathfrak{z}_q[j;j-a-d]\big) + \sum_{j=0}^{2a} \binom{-2d-1}{2a-j} \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2+kd}}{[k]_q^{2a+2d+1-j}} \times \big(\mathfrak{z}_q[j;j+k-a-d] + (-1)^j \mathfrak{z}_q[j;a+d-k+1]\big), \tag{24}$$

$$\hat{\mathfrak{z}}_q^{+-}[2a+1,2d;a,d] = 2\zeta_q[2a+1;a]\mathfrak{z}_q^{\wedge}[2d;d] + \mathfrak{z}_q^{\wedge}[2a+2d+1;a+d] + \sum_{j=0}^{2d} \binom{-2a-1}{2d-j} \zeta_q[2a+2d+1-j;a+2d-j]\big(\mathfrak{z}_q^{\wedge}[j;a+d] + (-1)^j \mathfrak{z}_q^{\wedge}[j;j-a-d]\big) + \sum_{j=0}^{2a+1} \binom{-2d}{2a+1-j} \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k-1)/2+kd}}{[k]_q^{2a+2d+1-j}} \big(\mathfrak{z}_q[j;j-a-d+k] + q^k \mathfrak{z}_q[j;j-a-d+k+1] + (-1)^j \big(\mathfrak{z}_q[j;a+d-k+1] + q^k \mathfrak{z}_q[j;a+d-k]\big)\big). \tag{25}$$

Proof. Replacing b by $2a$, d by $2d+1$, and c by $a+d$ in Theorem 3.1, we get identity (24). Considering the sum of two identities corresponding to the sets a,b,c,d and $a,b,c+1,d$ from Theorem 3.1, and then replacing b by $2a+1$, d by $2d$, and c by $a+d-1$, we get the desired identity (25). □

Similarly as in the proof of Theorem 3.1, evaluating residues of the function $\frac{z^c f(z)\overline{\psi}_{a,b}(z)}{(z-1)^d}$, we get a q-analogue of formula (3).

Theorem 3.2. *Suppose $a, b, c, d \in \mathbb{Z}$, $a \geqslant 1$, $b \geqslant 1$, $d \geqslant 0$, and $b+d$ is odd. Then*

$$\mathfrak{z}_q^{-+}[b, d; a, c-a+1] + \mathfrak{z}_q^{-+}[b, d; b-a, a+d-c]$$

is given by

$$\bar{\mathfrak{z}}_q[b+d; b+d-c] + \bar{\zeta}_q[b; a]\big(\mathfrak{z}_q[d; c-a+1] - (-1)^b \mathfrak{z}_q[d; a+d-c]\big)$$
$$+ \sum_{j=0}^{d} \binom{-b}{d-j} \bar{\zeta}_q[b+d-j; a+d-j]\big(\bar{\mathfrak{z}}_q[j; c+1] + (-1)^j \bar{\mathfrak{z}}_q[j; j-c]\big)$$
$$+ \sum_{j=0}^{b} \binom{-d}{b-j} \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2+k(a+d-c)}}{[k]_q^{b+d-j}} \big(\bar{\mathfrak{z}}_q[j; j+k-c] + (-1)^j \bar{\mathfrak{z}}_q[j; c-k+1]\big).$$

Fixing parities of the parameters b and d, we get the following corollary.

Corollary 3.2. *Let $a, d \in \mathbb{Z}$, $a \geqslant 1$, $d \geqslant 0$. Then*

$$2\mathfrak{z}_q^{-+}[2a, 2d+1; a, d+1] = \bar{\mathfrak{z}}_q[2a+2d+1; a+d+1] + \sum_{j=0}^{2d+1} \binom{-2a}{2d+1-j} \bar{\zeta}_q[2a+2d+1-j; 2d+a+1-j]$$
$$\times \big(\bar{\mathfrak{z}}_q[j; a+d+1] + (-1)^j \bar{\mathfrak{z}}_q[j; j-a-d]\big) + \sum_{j=0}^{2a} \binom{-2d-1}{2a-j} \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2+kd}}{[k]_q^{2a+2d+1-j}}$$
$$\times \big(\bar{\mathfrak{z}}_q[j; j+k-a-d] + (-1)^j \bar{\mathfrak{z}}_q[j; a+d-k+1]\big),$$

$$\hat{\mathfrak{z}}_q^{-+}[2a+1, 2d; a, d] = 2\bar{\zeta}_q[2a+1; a]\mathfrak{z}_q^{\wedge}[2d; d] + \bar{\mathfrak{z}}_q^{\wedge}[2a+2d+1; a+d]$$
$$+ \sum_{j=0}^{2d} \binom{-2a-1}{2d-j} \bar{\zeta}_q[2a+2d+1-j; a+2d-j]\big(\bar{\mathfrak{z}}_q^{\wedge}[j; a+d] + (-1)^j \bar{\mathfrak{z}}_q^{\wedge}[j; j-a-d]\big)$$
$$+ \sum_{j=0}^{2a+1} \binom{-2d}{2a+1-j} \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2+kd}}{[k]_q^{2a+2d+1-j}} \big(\bar{\mathfrak{z}}_q[j; j-a-d+k] + q^k \bar{\mathfrak{z}}_q[j; j-a-d+k+1]$$
$$+ (-1)^j \big(\bar{\mathfrak{z}}_q[j; a+d-k+1] + q^k \bar{\mathfrak{z}}_q[j; a+d-k]\big)\big).$$

For further possible applications of the formulas for double q-Euler sums proved in this section, we refer the reader to the paper [5].

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