Chapter 3

Connections

3.1 Frames

As we have already noted in chapter 1, the theory of curves in $\mathbf{R}^3$ can be elegantly formulated by introducing orthonormal triplets of vectors which we called Frenet frames. The Frenet vectors are adapted to the curves in such a manner that the rate of change of the frame gives information about the curvature of the curve. In this chapter we will study the properties of arbitrary frames and their corresponding rates of change in the direction of the various vectors in the frame. This concepts will then be applied later to special frames adapted to surfaces.

3.1 Definition A coordinate **frame** in $\mathbf{R}^n$ is an n-tuple of vector fields $\{e_1, \ldots, e_n\}$ which are linearly independent at each point $\mathbf{p}$ in the space.

In local coordinates $x^1, \ldots x^n$, we can always express the frame vectors as linear combinations of the standard basis vectors

$$e_i = \partial_j A^j{}_i, \tag{3.1}$$

where $\partial_j = \frac{\partial}{\partial x^1}$ We assume the matrix $A = (A^j{}_i)$ to be nonsingular at each point. In linear algebra, this concept is referred to as a change of basis, the difference being that in our case, the transformation matrix A depends on the position. A frame field is called **orthonormal** if at each point,

$$< e_i, e_j >= \delta_{ij}. \tag{3.2}$$

Throughout this chapter, we will assume that all frame fields are orthonormal. Whereas this restriction is not necessary, it is very convenient because it simplifies considerably the formulas to compute the components of an arbitrary vector in the frame.

3.2 Proposition If $\{e_1, \ldots, e_n\}$ is an orthonormal frame, then the transformation matrix is orthogonal (ie: $AA^T = I$)

Proof: The proof is by direct computation. Let $e_i = \partial_j A^j{}_i$. Then

$$\begin{aligned}
\delta_{ij} &= < e_i, e_j > \\
&= < \partial_k A^k{}_i, \partial_l A^l{}_j > \\
&= A^k{}_i A^l{}_j < \partial_k, \partial_l > \\
&= A^k{}_i A^l{}_j \delta_{kl}
\end{aligned}$$

$$
\begin{aligned}
&= A^k{}_i A_{kj} \\
&= A^k_i (A^T)_{jk}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(A^T)_{jk} A^k_i &= \delta_{ij} \\
(A^T)^j{}_k A^k_i &= \delta^j_i \\
A^T A &= I
\end{aligned}
$$

Given a frame vectors e_i, we can also introduce the corresponding dual coframe forms θ_i by requiring

$$
\theta^i(e_j) = \delta^i_j \tag{3.3}
$$

since the dual coframe is a set of 1-forms, they can also be expressed in of local coordinates as linear combinations

$$
\theta^i = B^i{}_k dx^k.
$$

It follows from equation(3.3), that

$$
\begin{aligned}
\theta^i(e_j) &= B^i_k dx^k(\partial_l A^l{}_j) \\
&= B^i_k A^l{}_j dx^k(\partial_l) \\
&= B^i_k A^l{}_j \delta^k{}_l \\
\delta^i{}_j &= B^i_k A^k{}_j.
\end{aligned}
$$

Therefore we conclude that $BA = I$, so $B = A^{-1} = A^T$. In other words, when the frames are orthonormal we have

$$
\begin{aligned}
e_i &= \partial_k A^k_i \\
\theta^i &= A^i{}_k dx^k.
\end{aligned} \tag{3.4}
$$

3.3 Example Consider the transformation from Cartesian to cylindrical coordinates

$$
\begin{aligned}
x &= r\cos\theta, \\
y &= r\sin\theta, \\
z &= z.
\end{aligned}
$$

Using the chain rule for partial derivatives, we have

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \\
\frac{\partial}{\partial \theta} &= -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial z}
\end{aligned}
$$

From these equations we easily verify that the quantities

$$
\begin{aligned}
e_1 &= \frac{\partial}{\partial r} \\
e_2 &= \frac{1}{r}\frac{\partial}{\partial \theta} \\
e_3 &= \frac{\partial}{\partial z},
\end{aligned}
$$

are a triplet of mutually orthogonal unit vectors and thus constitute an orthonormal frame.

3.4 Example For spherical coordinates(2.20)

$$\begin{aligned} x &= \rho\sin\theta\cos\phi \\ y &= \rho\sin\theta\sin\phi \\ z &= \rho\cos\theta, \end{aligned}$$

the chain rule leads to

$$\begin{aligned} \frac{\partial}{\partial\rho} &= \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z} \\ \frac{\partial}{\partial\theta} &= \rho\cos\theta\cos\phi\frac{\partial}{\partial x} + \rho\cos\theta\sin\phi\frac{\partial}{\partial y} + -\rho\sin\theta\frac{\partial}{\partial z} \\ \frac{\partial}{\partial\phi} &= -\rho\sin\theta\sin\phi\frac{\partial}{\partial x} + \rho\sin\theta\cos\phi\frac{\partial}{\partial y}. \end{aligned}$$

In this case, the vectors

$$\begin{aligned} e_1 &= \frac{\partial}{\partial\rho} \\ e_2 &= \frac{1}{\rho}\frac{\partial}{\partial\theta} \\ e_3 &= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi} \end{aligned} \tag{3.5}$$

also constitute an orthonormal frame.

The fact that the chain rule in the two situations above leads to orthonormal frames is not coincidental. The results are related to the orthogonality of the level surfaces $x^i = constant$. Since the level surfaces are orthogonal whenever they intersect, one expects the gradients of the surfaces to also be orthogonal. Transformations of this type are called triply orthogonal systems.

3.2 Curvilinear Coordinates

Orthogonal transformations such as Spherical and cylindrical coordinates appear ubiquitously in mathematical physics because the geometry of a large number of problems in this area exhibit symmetry with respect to an axis or to the origin. In such situations, transformation to the appropriate coordinate system often result in considerable simplification of the field equations involved in the problem. It has been shown that the Laplace operator which enters into all three of main classical fields, the potential, the heat and the wave equations, is separable in 12 coordinate systems. A simple and efficient method to calculate the Laplacian in orthogonal coordinates can be implemented by the use of differential forms.

3.5 Example In spherical coordinates the differential of arc length is given by (see equation 2.21)

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta d\phi^2.$$

Let

$$\begin{aligned} \theta^1 &= d\rho \\ \theta^2 &= \rho d\theta \\ \theta^3 &= \rho\sin\theta d\phi. \end{aligned} \tag{3.6}$$

Note that these three 1-forms constitute the dual coframe to the orthonormal frame which just derived in equation(3.5). Consider a scalar field $f = f(\rho, \theta, \phi)$. We now calculate the Laplacian of f in spherical coordinates using the methods of section 2.4. To do this, we first compute the differential df and express the result in terms of the coframe.

$$\begin{aligned} df &= \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi \\ &= \frac{\partial f}{\partial \rho} \theta^1 + \frac{1}{\rho}\frac{\partial f}{\partial \theta} \theta^2 + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi} \theta^3 \end{aligned}$$

The components df in the coframe represent the gradient in spherical coordinates. Continuing with the scheme of section 2.4, we first apply the Hodge-$*$ operator. Then we rewrite the resulting 2-form in terms of wedges of coordinate differentials so that we can apply the definition of the exterior derivative.

$$\begin{aligned} *df &= \frac{\partial f}{\partial \rho} \theta^2 \wedge \theta^3 - \frac{1}{\rho}\frac{\partial f}{\partial \theta} \theta^1 \wedge \theta^3 + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi} \theta^1 \wedge \theta^2 \\ &= \rho^2 \sin\theta \frac{\partial f}{\partial \rho} d\theta \wedge d\phi - \rho \sin\theta \frac{1}{\rho}\frac{\partial f}{\partial \theta} d\rho \wedge d\phi + \rho \sin\theta \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi} d\rho \wedge d\theta \\ &= \rho^2 \sin\theta \frac{\partial f}{\partial \rho} d\theta \wedge d\phi - \sin\theta \ \frac{\partial f}{\partial \theta} d\rho \wedge d\phi + \frac{\partial f}{\partial \phi} d\rho \wedge d\theta \\ d * df &= \frac{\partial}{\partial \rho}(\rho^2 \sin\theta \frac{\partial f}{\partial \rho}) d\rho \wedge d\theta \wedge d\phi - \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) d\theta \wedge d\rho \wedge d\phi + \frac{\partial}{\partial \phi}(\frac{\partial f}{\partial \phi}) d\phi \wedge d\rho \wedge d\theta \\ &= \left[\sin\theta \frac{\partial}{\partial \rho}(\rho^2 \frac{\partial f}{\partial \rho}) + \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial^2 f}{\partial \phi^2} \right] d\rho \wedge d\theta \wedge d\phi. \end{aligned}$$

Finally, rewriting the differentials back in terms of the the coframe, we get

$$d * df = \frac{1}{\rho^2 \sin\theta} \left[\sin\theta \frac{\partial}{\partial \rho}(\rho^2 \frac{\partial f}{\partial \rho}) + \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial^2 f}{\partial \phi^2} \right] \theta^1 \wedge \theta^2 \wedge \theta^3.$$

So, the Laplacian of f is given by

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial f}{\partial \rho} \right] + \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial^2 f}{\partial \phi^2} \right] \tag{3.7}$$

The derivation of the expression for the spherical Laplacian through the use of differential forms is elegant and leads naturally to the operator in Sturm Liouville form.

The process above can also be carried out for general orthogonal transformations. A change of coordinates $x^i = x^i(u^k)$ leads to an orthogonal transformation if in the new coordinate system u^k, the line metric

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2 \tag{3.8}$$

only has diagonal entries. In this case, we choose the coframe

$$\begin{aligned} \theta^1 &= \sqrt{g_{11}} du^1 = h_1 du^1 \\ \theta^2 &= \sqrt{g_{22}} du^2 = h_2 du^2 \\ \theta^3 &= \sqrt{g_{33}} du^3 = h_3 du^3 \end{aligned}$$

The quantities $\{h_1, h_2, h_3\}$ are classically called the weights. Please note that in the interest of connecting to classical terminology we have exchanged two indices for one and this will cause small discrepancies with the index summation convention. We will revert to using a summation symbol when these discrepancies occur. To satisfy the duality condition $\theta^i(e_j) = \delta^i_j$, we must choose the corresponding frame vectors e_i according to

$$
\begin{aligned}
e_1 &= \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial u^1} = \frac{1}{h_1}\frac{\partial}{\partial u^1} \\
e_2 &= \frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial u^2} = \frac{1}{h_2}\frac{\partial}{\partial u^2} \\
e_3 &= \frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial u^3} = \frac{1}{h_3}\frac{\partial}{\partial u^3}
\end{aligned}
$$

Gradient. Let $f = f(x^i)$ and $x^i = x^i(u^k)$. Then

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x^k}dx^k \\
&= \frac{\partial f}{\partial u^i}\frac{\partial u^i}{\partial x^k}dx^k \\
&= \frac{\partial f}{\partial u^i}du^i \\
&= \sum_i \frac{1}{h^i}\frac{\partial f}{\partial u^i}\theta^i \\
&= e_i(f)\theta^i.
\end{aligned}
$$

As expected, the components of the gradient in the coframe θ^i are the just the frame vectors

$$
\nabla = \left(\frac{1}{h_1}\frac{\partial}{\partial u^1}, \frac{1}{h_2}\frac{\partial}{\partial u^2}, \frac{1}{h_3}\frac{\partial}{\partial u^3}\right) \tag{3.9}
$$

Curl. Let $F = (F_1, F_2, F_3)$ be a classical vector field. Construct the corresponding one form $F = F_i\theta^i$ in the coframe. We calculate the curl using the dual of the exterior derivative.

$$
\begin{aligned}
F &= F_1\theta^1 + F_2\theta^2 + F_3\theta^3 \\
&= (h_1F_1)du^1 + (h_2F_2)du^2 + (h_3F_3)du^3 \\
&= (hF)_i du^i, \quad \text{where } (hF)_i = h_iF_i \\
dF &= \frac{1}{2}\left[\frac{\partial(hF)_i}{\partial u^j} - \frac{\partial(hF)_j}{\partial u^i}\right] du^i \wedge du^j \\
&= \frac{1}{h_ih_j}\left[\frac{\partial(hF)_i}{\partial u^j} - \frac{\partial(hF)_j}{\partial u^i}\right] d\theta^i \wedge d\theta^j \\
*dF &= \epsilon^{ij}{}_k\left[\frac{1}{h_ih_j}[\frac{\partial(hF)_i}{\partial u^j} - \frac{\partial(hF)_j}{\partial u^i}]\right]\theta^k = (\nabla \times F)_k\theta^k.
\end{aligned}
$$

Thus, the components of the curl are

$$
\left(\frac{1}{h_2h_3}[\frac{\partial(h_3F_3)}{\partial u^2} - \frac{\partial(h_2F_2)}{\partial u^3}], \frac{1}{h_1h_3}[\frac{\partial(h_3F_3)}{\partial u^1} - \frac{\partial(h_1F_1)}{\partial u^3}], \frac{1}{h_1h_2}[\frac{\partial(h_1F_1)}{\partial u^2} - \frac{\partial(h_2F_2)}{\partial u^1}].\right) \tag{3.10}
$$

Divergence. As before, let $F = F_i\theta^i$ and recall that $\nabla \cdot F = *d*F$. The computation yields

$$
\begin{aligned}
F &= F_1\theta^1 + F_2\theta^2 + F_3\theta^3 \\
*F &= F_1\theta^2 \wedge \theta^3 + F_2\theta^3 \wedge \theta^1 + F_3\theta^1 \wedge \theta^2 \\
&= (h_2h_3F_1)du^2 \wedge du^3 + (h_1h_3F_2)du^3 \wedge du^1 + (h_1h_2F_3)du^1 \wedge du^2 \\
d*dF &= \left[\frac{\partial(h_2h_3F_1)}{\partial u^1} + \frac{\partial(h_1h_3F_2)}{\partial u^2} + \frac{\partial(h_1h_2F_3)}{\partial u^3}\right] du^1 \wedge du^2 \wedge du^3.
\end{aligned}
$$

Therefore,

$$\nabla \cdot F = *d*F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 F_1)}{\partial u^1} + \frac{\partial(h_1 h_3 F_2)}{\partial u^2} + \frac{\partial(h_1 h_2 F_3)}{\partial u^3}\right]. \tag{3.11}$$

3.3 Covariant Derivative

In this section we introduce a generalization of directional derivatives. The directional derivative measures the rate of change of a function in the direction of a vector. What we want is a quantity which measures the rate of change of a vector field in the direction of another.

3.6 Definition Let X be an arbitrary vector field in $\mathbf{R}^n$. A map $\overline{\nabla}_X : T(\mathbf{R}^n) \longrightarrow T(\mathbf{R}^n)$ is called a **Koszul connection** if it satisfies the following properties.

1. $\overline{\nabla}_{fX}(Y) = f\overline{\nabla}_X Y$,
2. $\overline{\nabla}_{(X_1+X_2)}Y = \overline{\nabla}_{X_1}Y + \overline{\nabla}_{X_2}Y$,
3. $\overline{\nabla}_X(Y_1+Y_2) = \overline{\nabla}_X Y_1 + \overline{\nabla}_X Y_2$,
4. $\overline{\nabla}_X fY = X(f)Y + f\overline{\nabla}_X Y$,

for all vector fields $X, X_1, X_2, Y, Y_1, Y_2 \in T(\mathbf{R}^n)$ and all smooth functions f. The definition states that the map $\overline{\nabla}_X$ is linear on X but behaves like a linear derivation on Y. For this reason, the quantity $\overline{\nabla}_X Y$ is called the **covariant derivative** of Y in the direction of X.

3.7 Proposition Let $Y = f^i\frac{\partial}{\partial x^i}$ be a vector field in $\mathbf{R}^n$, and let X another C^∞ vector field. Then the operator given by

$$\overline{\nabla}_X Y = X(f^i)\frac{\partial}{\partial x^1}. \tag{3.12}$$

defines a Koszul connection. The proof just requires verification that the four properties above are satisfied, and it is left as an exercise. The operator defined in this proposition is called the **standard connection** compatible with the standard Euclidean metric. The action of this connection on a vector field Y yields a new vector field whose components are the directional derivatives of the components of Y.

3.8 Example Let

$$X = x\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y},\ Y = x^2\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y},$$

Then,

$$\begin{aligned}
\overline{\nabla}_X Y &= X(x^2)\frac{\partial}{\partial x} + X(xy^2)\frac{\partial}{\partial y} \\
&= [x\frac{\partial}{\partial x}(x^2) + xz\frac{\partial}{\partial y}(x^2)]\frac{\partial}{\partial x} + [x\frac{\partial}{\partial x}(xy^2) + xz\frac{\partial}{\partial y}(xy^2)]\frac{\partial}{\partial y} \\
&= 2x^2\frac{\partial}{\partial x} + (xy^2 + 2x^2yz)\frac{\partial}{\partial y}
\end{aligned}$$

3.9 Definition A Koszul connection $\overline{\nabla}_X$ is compatible with the metric $g(Y,Z) = <Y,Z>$ if

$$\overline{\nabla}_X <Y,Z> = <\overline{\nabla}_X Y, Z> + <Y, \overline{\nabla}_X Z>. \tag{3.13}$$

In Euclidean space, the components of the standard frame vectors are constant, and thus their rates of change in any direction vanish. Let e_i be arbitrary frame field with dual forms θ^i. The covariant derivatives of the frame vectors in the directions of a vector X, will in general yield new vectors. The new vectors must be linear combinations of the the basis vectors

$$\begin{aligned} \overline{\nabla}_X e_1 &= \omega^1{}_1(X)e_1 + \omega^2{}_1(X)e_2 + \omega^3{}_1(X)e_3 \\ \overline{\nabla}_X e_2 &= \omega^1{}_2(X)e_1 + \omega^2{}_2(X)e_2 + \omega^3{}_2(X)e_3 \\ \overline{\nabla}_X e_3 &= \omega^1{}_3(X)e_1 + \omega^2{}_3(X)e_2 + \omega^3{}_3(X)e_3 \end{aligned} \tag{3.14}$$

The coefficients can be more succinctly expressed using the compact index notation

$$\overline{\nabla}_X e_i = e_j \omega^j{}_i(X) \tag{3.15}$$

It follows immediately that

$$\omega^j{}_i(X) = \theta^j(\overline{\nabla}_X e_i). \tag{3.16}$$

Equivalently, one can take the inner product of both sides of equation (3.15) with e_k to get

$$\begin{aligned} < \overline{\nabla}_X e_i, e_k > &= < e_j \omega^j{}_i(X), e_k > \\ &= \omega^j{}_i(X) < e_j, e_k > \\ &= \omega^j{}_i(X) g_{jk} \end{aligned}$$

Hence,

$$< \overline{\nabla}_X e_i, e_k >= \omega_{ki}(X) \tag{3.17}$$

The left hand side of the last equation is the inner product of two vectors, so the expression represents an array of functions. Consequently, the right hand side also represents an array of functions. In addition, both expressions are linear on X, since by definition $\overline{\nabla}_X$ is linear on X. We conclude that the right hand side can be interpreted as a matrix in which, each entry is a 1-forms acting on the vector X to yield a function. The matrix valued quantity $\omega^i{}_j$ is called the **connection form** .

3.10 Definition Let $\overline{\nabla}_X$ be a Koszul connection and let $\{e_i\}$ be a frame. The **Christoffel** symbols associated with the connection in the given frame are the functions Γ^k_{ij} given by

$$\overline{\nabla}_{e_i} e_j = \Gamma^k_{ij} e_k \tag{3.18}$$

The Christoffel symbols are the coefficients which give the representation of the rate of change of the frame vectors in the direction of the frame vectors themselves. Many physicists therefore refer to the Christoffel symbols as the connection once again giving rise to possible confusion. The precise relation between the Christoffel symbols and the connection 1-forms is captured by the equations

$$\omega^k{}_i(e_j) = \Gamma^k_{ij}, \tag{3.19}$$

or equivalently

$$\omega^k_i = \Gamma^k_{ij} \theta^j \tag{3.20}$$

In a general frame in $\mathbf{R}^n$ there are n^2 entries in the connection 1-form and n^3 Christoffel symbols. The number of independent components is reduced if one assumes that the frame is orthonormal.

3.11 Proposition Let and e_i be an orthonormal frame and $\overline{\nabla}_X$ be a Koszul connection compatible with the metric . Then

$$\omega_{ji} = -\omega_{ij} \tag{3.21}$$

Proof: Since it is given that $< e_i, e_j >= \delta_{ij}$, we have

$$\begin{aligned} 0 &= \overline{\nabla}_X < e_i, e_j > \\ &= < \overline{\nabla}_X e_i, e_j > + < e_i, \overline{\nabla}_X e_j > \\ &= < \omega^k_{\ i} e_k, e_j > + < e_i, \omega^k_{\ j} e_k > \\ &= \omega^k_{\ i} < e_k, e_j > + \omega^k_{\ j} < e_i, e_k > \\ &= \omega^k_{\ i} g_{kj} + \omega^k_{\ j} g_{ik} \\ &= \omega_{ji} + \omega_{ij} \end{aligned}$$

thus proving that ω is indeed antisymmetric.

3.12 Corollary The Christoffel symbols of a Koszul connection in an orthonormal frame are antisymmetric on the lower indices; that is

$$\Gamma^k_{\ ji} = -\Gamma^k_{\ ij}. \tag{3.22}$$

We conclude that in an orthonormal frame in $\mathbf{R}^n$ the number of independent coefficients of the connection 1-form is $(1/2)n(n-1)$ since by antisymmetry, the diagonal entries are zero, and one only needs to count the number of entries in the upper triangular part of the $n \times n$ matrix ω_{ij} Similarly, the number of independent Christoffel symbols gets reduced to $(1/2)n^2(n-1)$. In the case of an orthonormal frame in $\mathbf{R}^3$, where g_{ij} is diagonal, $\omega^i_{\ j}$ is also antisymmetric, so the connection equations become

$$\overline{\nabla}_X \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega^1_{\ 2}(X) & \omega^1_{\ 3}(X) \\ -\omega^1_{\ 2}(X) & 0 & \omega^2_{\ 3}(X) \\ -\omega^1_{\ 3}(X) & -\omega^2_{\ 3}(X) & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{3.23}$$

Comparing the Frenet frame equation(1.27), we notice the obvious similarity to the general frame equations above. Clearly, the Frenet frame is a special case in which the basis vectors have been adapted to a curve resulting in a simpler connection in which some of the coefficients vanish. A further simplification occurs in the Frenet frame since here the equations represent the rate of change of the frame only along the direction of the curve rather than an arbitrary direction vector X.

3.4 Cartan Equations

Perhaps, the most important contribution to the development of Differential Geometry is the work of Cartan culminating into famous equations of structure which we discuss in this chapter.

First Structure Equation

3.13 Theorem Let $\{e_i\}$ be a frame with connection $\omega^{\ i}_j$ and dual coframe θ^i. Then

$$\Theta^i \equiv d\theta^i + \omega^i_{\ j} \wedge \theta^j = 0 \tag{3.24}$$

Proof: Let

$$e_i = \partial_j A^j_{\ i}.$$

be a frame, and let θ^i be the corresponding coframe. Since $\theta^i(e_j)$, we have

$$\theta^i = (A^{-1})^i_{\ j} dx^j.$$

Let X be an arbitrary vector field. Then

$$\begin{aligned} \overline{\nabla}_X e_i &= \overline{\nabla}_X(\partial_j A^j_{\ i}) \\ e_j \omega^j_{\ i}(X) &= \partial_j X(A^j_{\ i}) \\ &= \partial_j d(A^j_{\ i})(X) \\ &= e_k (A^{-1})^k_{\ j} d(A^j_{\ i})(X) \\ \omega^k_{\ i}(X) &= (A^{-1})^k_{\ j} d(A^j_{\ i})(X). \end{aligned}$$

Hence,

$$\omega^k_{\ i} = (A^{-1})^k_{\ j} d(A^j_{\ i}),$$

or in matrix notation

$$\omega = A^{-1} dA \tag{3.25}$$

On the other hand, taking the exterior derivative of θ^i, we find

$$\begin{aligned} d\theta^i &= d(A^{-1})^i_{\ j} \wedge dx^j \\ &= d(A^{-1})^i_{\ j} \wedge A^j_{\ k}\theta^k \\ d\theta &= d(A^{-1})A \wedge \theta. \end{aligned}$$

But, since $A^{-1}A = I$, we have $d(A^{-1})A = -A^{-1}dA = -\omega$, hence

$$d\theta = -\omega \wedge \theta. \tag{3.26}$$

In other words

$$d\theta^i + \omega^i_{\ j} \wedge \theta^j = 0$$

Second Structure Equation

Let θ^i be a coframe in $\mathbf{R}^n$ with connection $\omega^i_{\ j}$. Taking the exterior derivative of the first equation of structure and recalling the properties (2.34) we get

$$\begin{aligned} d(d\theta^i) + d(\omega^i_{\ j} \wedge \theta^j) &= 0 \\ d\omega^i_{\ j} \wedge \theta^j - \omega^i_{\ j} \wedge d\theta^j &= 0, \end{aligned}$$

Substituting recursively from the first equation of structure, we get

$$\begin{aligned} d\omega^i_{\ j} \wedge \theta^j - \omega^i_{\ j} \wedge (-\omega^j_{\ k} \wedge \theta^k) &= 0 \\ d\omega^i_{\ j} \wedge \theta^j + \omega^i_{\ k} \wedge \omega^k_{\ j} \wedge \theta^j &= 0 \\ (d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j}) \wedge \theta^j &= 0 \\ d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} &= 0. \end{aligned}$$

We now introduce the following

3.14 Definition The curvature Ω of a connection ω is the matrix valued 2-form

$$\Omega^i_{\ j} \equiv d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} \tag{3.27}$$

3.15 Theorem Let θ be a coframe with connection ω in $\mathbf{R}^n$. The the curvature form vanishes

$$\Omega = d\omega + \omega \wedge \omega = 0 \tag{3.28}$$

Proof: Given the there is a non-singular matrix A such that $\theta = A^{-1}dx$ and $\omega = A^{-1}dA$, have

$$d\omega = d(A^{-1}) \wedge dA$$

On the other hand,

$$\begin{aligned} \omega \wedge \omega &= (A^{-1}dA) \wedge (A^{-1}dA) \\ &= -d(A^{-1})A \wedge A^{-1}dA \\ &= -d(A^{-1})(AA^{-1}) \wedge dA \\ &= -d(A^{-1}) \wedge dA. \end{aligned}$$

Therefore, $d\Omega = -\omega \wedge \omega$.

Change of Basis

We briefly explore the behavior of the quantities Θ^i and $\Omega^i{}_j$ under a change of basis.

Let e_i be frame with dual forms θ^i, and let $\overline{e}_i$ another frame related to the first frame by an invertible transformation.

$$\overline{e}_i = e_j B^j{}_i, \tag{3.29}$$

which we will write in matrix notation as $\overline{e} = eB$. Referring back to the definition of connections (3.15), we introduce the **covariant differential** $\overline{\nabla}$ by the formula

$$\begin{aligned} \overline{\nabla} e_i &= e_j \otimes \omega^j{}_i \\ &= e_j \omega^j{}_i \\ \overline{\nabla} e &= e\,\omega \end{aligned} \tag{3.30}$$

where once again, we have simplified the equation by using matrix notation. This definition is elegant because it does not explicitly show the dependence on X in the connection (3.15). The idea of switching from derivatives to differentials is familiar from basic calculus, but we should point out that in the present context, the situation is more subtle. The operator $\overline{\nabla}$ here maps a vector field to a matrix-valued tensor of rank $T^{1,1}$. Another way to view the covariant differential, is to think of $\overline{\nabla}$ as an operator such that if e is a frame, and X a vector field, then $\overline{\nabla} e(X) = \overline{\nabla}_X e$. If f is a function, then $\overline{\nabla} f(X) = \overline{\nabla}_X f = df(X)$, so that $\overline{\nabla} f = df$. In other words, $\overline{\nabla}$ behaves like a covariant derivative on vectors, but like a differential on functions. We require $\overline{\nabla}$ to behave like a derivation on tensor products

$$\overline{\nabla}(T_1 \otimes T_2) = \overline{\nabla} T_1 \otimes T_2 + T_1 \otimes \overline{\nabla} T_2 \tag{3.31}$$

Taking the exterior differential of (3.29) and using (3.30) recursively, we get

$$\begin{aligned} \overline{\nabla}\overline{e} &= \overline{\nabla} e \otimes B + e \otimes \overline{\nabla} B \\ &= (\overline{\nabla} e)B + e(dB) \\ &= e\,\omega B + e(dB) \\ &= \overline{e}B^{-1}\omega B + \overline{e}B^{-1}dB \\ &= \overline{e}[B^{-1}\omega B + B^{-1}dB] \\ &= \overline{e}\,\overline{\omega} \end{aligned}$$

provided that the connection $\overline{\omega}$ in the new frame $\overline{e}$ is related to the connection ω by the transformation law

$$\overline{\omega} = B^{-1}\omega B + B^{-1}dB. \tag{3.32}$$

It should be noted than if e is the standard frame $e_i = \partial_i$ in $\mathbf{R}^n$, then $\overline{\nabla} e = 0$, so that $\omega = 0$. In this case, the formula above reduces to $\overline{\omega} = B^{-1}dB$, showing that the transformation rule is consistent with equation (3.25).