Complex Analysis on lineally convex domains

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Lineally convex domains

Definition 1 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain. We say, that Ω is*
a) (weakly) lineally convex, *if for every $z \in \partial\Omega$ there is an affine complex hyperplane A_z such that $z \in A_z$ and $A_z \cap \Omega = \emptyset$.*
b) locally (weakly) lineally convex, *if for every $z \in \partial\Omega$ there is an open set U and an affine complex hyperplane A_z such that $z \in A_z$ and $A_z \cap \Omega \cap U = \emptyset$.*

Remark 2 *If a locally lineally convex domain Ω has a $\mathcal{C}^1$-smooth boundary, then it is (globally) lineally convex (L. Hörmander 1994);*

Lineal convexity can be characterized by the following differential inequality (Chr. Kiselman 1995):

Theorem 3 *Let $\Omega \Subset \mathbb{C}^n$ be a domain with $\mathcal{C}^2$-smooth boundary. Then Ω is lineally convex iff*

$$\mathcal{H}_{\varrho}^{\mathbb{R}}(x;s) = 2\left(\mathrm{Re}\,\mathcal{H}_{\varrho}(z;t) + \mathcal{L}_{\varrho}(z;t)\right) \geq 0 \tag{1}$$

holds for all $z \in \partial\Omega$ and for all $t \in T_z^h\partial\Omega$. Here $\mathcal{H}_{\varrho}^{\mathbb{R}}(z;s)$ is the real Hessian *of ϱ at z and $\mathcal{H}_{\varrho}(z;t)$ is the* complex Hessian *of ϱ. We have put $z_j = x_{2j-1} + i\,x_{2j}$; $t_j = s_{2j-1} + i\,s_{2j}$.*

The condition of the Theorem 3 is called the *Behnke-Peschl condition*. Its necessity in $\mathbb{C}^2$ was already discovered in 1935 by Behnke-Peschl, whereas only the positive definiteness of (1) on the holomorphic tangent space was shown to be sufficient.

Holomorphic support functions

$W = W(\partial\Omega)$ open neighborhood of $\partial\Omega$;
$r : W \to \mathbb{R}$ smooth defining function of Ω.
$\partial\Omega_\zeta := \{z : r(z) = r(\zeta)\}$ for $\zeta \in W$.
$t \in T^h_\zeta(\partial\Omega_\zeta)$ unit vector; put

$$z_{\zeta,t}(w) := \zeta - i\, w_1 \nu(\zeta) + w_2 t \quad \forall\, w = (w_1, w_2) \in \mathbb{C}^2 \tag{2}$$

(Parameterization of 2-dimensional $\mathbb{C}$-affine plane $A_{\zeta,t}$ spanned by $\nu(\zeta), t$, through ζ.)
Put $r_{\zeta,t}(w) := r(z_{\zeta,t}(w)) - r(\zeta)$,
$\Omega_{\zeta,t} = \{w \in \mathbb{C}^2 : r_{\zeta,t}(w) < 0\}$
Put for $j = 2, \ldots, 2m$

$$P^j_{\zeta,t}(w) := \sum_{k+l=j} \frac{1}{k!}\frac{1}{l!}\,\frac{\partial^j r_{\zeta,t}(0)}{\partial w_2^k \partial \overline{w}_2^l}\, w_2^k \overline{w}_2^l \tag{3}$$

(Notice, that the coefficients of $P^j_{\zeta,t}$ are $\mathcal{C}^\infty$ in (ζ, t).)

Definition 4 *For any polynomial*

$$\sum_{j=0}^{N} \sum_{|\alpha|+|\beta|=j} a_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta$$

on $\mathbb{C}^k$ *put*

$$\|P\| := \sum_{j=0}^{N} \sum_{|\alpha|+|\beta|=j} \left| a_{\alpha\overline{\beta}} \right| \tag{4}$$

Theorem 5 *(Diederich-Fornæss 2003) Let* $\widehat{c} > 0$ *be suitably chosen,* $\varepsilon > 0$ *small enough; then* $\exists\ S(z,\zeta) \in \mathcal{C}^\infty(\mathbb{C}^n \times W_0)$*, holomorphic polynomial in* z *of degree* $2m$ *for* $\zeta \in W_0$*, and* $\exists$ *radius* $d = d(\varepsilon)$*, s. th.*

1. $\forall \zeta \in W_0 \Rightarrow \widehat{S}(\zeta,\zeta) = 0$,
2. $\forall t \in T^{10}_\zeta \partial\Omega_\zeta$, $\|t\| = 1$ *put*

   $S_{\zeta,t}(w) := \widehat{S}(z_{\zeta,t}(w),\zeta)$ *on* $|w| < d(\varepsilon) \Rightarrow$

   $$\operatorname{Re} S_{\zeta,t}(w) \leq r_{\zeta,t}(w) h_{\zeta,t}(w) - \varepsilon \widehat{c} \sum_{j=2}^{2m} \left\| P^j_{\zeta,t} \right\| |w_2|^j$$

   where $h_{\zeta,t}(w) \in \mathcal{C}^\infty$ *is of the form* $h_{\zeta,t}(w) = h(\zeta - iw_1 n_\zeta + w_2 t, \zeta)$ *with a strictly positive* $\mathcal{C}^\infty$ *function* h *on* $\{z \in \mathbb{C}^n : |z| < d(\varepsilon)\} \times W_0$.

The real hypersurface $\{z : \text{Re } \widehat{S}(z) = 0\}$ bends away from $\partial\Omega$ in all directions with the correct order.

Anisotropic behavior: an example

$B = B(0;1) \subset \mathbb{C}^n$, $n \geq 2$
$z^d := (0, \ldots, 1-d) \in B$

One has

$\Delta_d^{x_n} :=$

$\{z^{(1)}(\zeta) = (0, \ldots, 0, 1-d+\zeta) : \zeta \in \Delta(0;d)\} \subset B$

$\Delta_{\sqrt{d}}^{x_1} :=$

$\{z^{(2)}(\zeta) = (\zeta, 0, \ldots, 0, 1-d) : \zeta \in \Delta(0, \sqrt{d})\} \subset B$

The Cauchy integral formula immediately gives:

$$\begin{aligned} \left|\frac{\partial f}{\partial z_n}(z^d)\right| &\leq C\ \frac{1}{d} \\ \left|\frac{\partial f}{\partial z_1}(z^d)\right| &\leq C\ \frac{1}{\sqrt{d}} \end{aligned} \tag{5}$$

Pseudogeometries on lineally convex domains The systematic use of pseudogeometries in estimating singular integrals goes back to the work of E. Stein. For linearly convex domains of finite type the following definitions were first given by J. McNeal in 1994. They were carried over to lineally convex domains of finite type by M. Conrad in his thesis (Wuppertal 2002).

Definition 6 *Let $v \in \mathbb{C}^n$, $\|v\| = 1$, $\zeta \in W_0$ Scaling factor of order ε in the direction v at ζ:*

$\tau(\zeta, v, \varepsilon) :=$

$\max\{c : |r(\zeta + \lambda v) - r(\zeta)| < \varepsilon \, \forall \lambda \in \mathbb{C}, |\lambda| < c\}$

By induction:

$$v_1 := \frac{\nabla r(\zeta)}{|\nabla r(\zeta)|}$$

Inductive step: minimizing $\tau(\zeta, v, \varepsilon)$ $\forall v \in \langle v_1, \ldots, v_{k-1} \rangle^\perp$, $\|v\| = 1$ gives v_k

Together: $(v_1, \ldots, v_n)$ ε-*extremal basis* at ζ.
Corresponding coordinates at ζ:

$$(z_{k,\zeta,\varepsilon})_{k=1}^{n}$$

Put

$$\tau_k(\zeta, \varepsilon) := \tau(\zeta, v_k, \varepsilon)$$

Definition 7 Distinguished polydisc *at* ζ *with respect to* ε*:*

$$P_\varepsilon(\zeta) := \{z \in \mathbb{C}^n : \left|z_{k,\zeta,\varepsilon}\right| \leq \tau_k(\zeta,\varepsilon)\,\forall k\}$$

Remark 8 *For* $A > 0$ *the distinguished polydisc blown up by the factor* A

$$A\,P_\varepsilon(\zeta) := \{z \in \mathbb{C}^n : |z_{k,\zeta,\varepsilon}| \leq A\,\tau_k(\zeta,\varepsilon)\,\forall k\}$$

A ***pseudodistance*** adapted to the anisotropic geometry of the domains:

Definition 9 *Let* $z, \zeta \in W_0$*. Put*

$$d(z,\zeta) := \inf\{\varepsilon : z \in P_\varepsilon(\zeta)\}$$

Essential properties of the pseudodistance:

1. **Scaling factors and defining function**
w orthonormal coordinate system at z, v_j unit vector in the w_j-direction. Then
$$\left|\frac{\partial^{|\alpha+\beta|} r(z)}{\partial w^\alpha \partial \bar{w}^\beta}\right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, v_j, \varepsilon)^{\alpha_j+\beta_j}}$$
$\forall\ \alpha$, β with $|\alpha+\beta| \geq 1$.

2. If $v = \sum_{j=1}^n a_j v_j$, where $(v_1, \ldots, v_n)$ is the ε-extremal basis at ζ, then we have
$$\frac{1}{\tau(\zeta, v, \varepsilon)} \approx \sum_{j=1}^n \frac{|a_j|}{\tau_j(\zeta, \varepsilon)}.$$

3. **Scaling property**
$\forall\ K\ \exists\ c(K)$, $C(K)$ (only depending on K) such that
$$\begin{aligned} P_{c(K)\varepsilon}(\zeta) \subset &\ K P_\varepsilon(\zeta) &\subset P_{C(K)\varepsilon}(\zeta), \\ c(K)P_\varepsilon(\zeta) \subset &\ P_{K\varepsilon}(\zeta) &\subset C(K)P_\varepsilon(\zeta). \end{aligned}$$
for ζ near $\partial\Omega$ and all $\varepsilon > 0$ small enough.

4. **Comparability**
   For every $z \in P_\varepsilon(\zeta)$ we have $\tau(\zeta, v, \varepsilon) \approx \tau(z, v, \varepsilon)$.

5. We have $\tau_1(\zeta, \varepsilon) \approx \varepsilon$ and if v is a unit vector in complex tangential direction we also have $\varepsilon^{\frac{1}{2}} \lesssim \tau(\zeta, v, \varepsilon)$.

6. **Weak symmetry property**
   The pseudodistance $d(z, \zeta)$ satisfies the properties
$$d(z, \zeta) \approx d(\zeta, z) \tag{6}$$

7. **Weak triangle inequality**
$$d(z, \zeta) \lesssim d(z, w) + d(w, \zeta). \tag{7}$$

8. If $\pi(z)$ is the projection of a point z to the boundary $\partial\Omega$ then $d(z, \pi(z)) \approx |r(z)|$

9. $z \in P_\varepsilon(\zeta)$ implies $d(z, \zeta) \leq \varepsilon$

10. $z \notin P_\varepsilon(\zeta)$ implies $d(z, \zeta) \gtrsim \varepsilon$ (not $\geq \varepsilon$);

Pseudometric estimates

The *scaling property* allows to choose a $C > 0$ for which we can define the polyannuli

$$P^i_\varepsilon(\zeta) := CP_{2^{-i}\varepsilon}(\zeta) \setminus \tfrac{1}{2}P_{2^{-i}\varepsilon}(\zeta)$$

Two coverings

$$\bigcup_{i=0}^{\infty} P^i_\varepsilon(\zeta) \supset P_\varepsilon(\zeta)\setminus\{\zeta\}, \bigcup_{i=0}^{i_0(\varepsilon)} P^i_1(\zeta) \supset P_1(\zeta)\setminus P_\varepsilon(\zeta). \tag{8}$$

where $i_0(\varepsilon) < -\log_2(c\varepsilon)$

The basic estimates are

Lemma 10 *$U = U(\zeta_0)$ small enough; $z \in \Omega \cap U$; ε small enough. Then*

$$\begin{array}{lll} |S(z,\zeta)| & \gtrsim & \varepsilon \qquad \forall \zeta \in \partial\Omega \cap P^0_\varepsilon(\pi(z)) \\ |S(z,\zeta)| & \gtrsim & |\varrho(z)| \quad \forall \zeta \in \partial\Omega \cap P_{|\varrho(z)|}(\pi(z)) \end{array} \tag{9}$$

One has a natural decomposition of S of the form

$$S(z,\zeta) = \sum_{j=1}^{n} Q_j(z,\zeta)(z_j - \zeta_j)$$

The Cauchy-Fantappié form is defined by

$$W(z,\zeta) := \sum_i \frac{Q_i(z,\zeta)}{S(z,\zeta)}\, d\zeta_i.$$

Fix a point $\zeta_0 \in \partial\Omega$.
$w = w(z)$ (resp. $\eta = \eta(\zeta)$): ε-extremal coordinates of z (resp. ζ) at ζ_0;
inverses $z = z(w)$ (resp. $\zeta = \zeta(\eta)$)

$$Q_j^*(w,\eta) := Q_j(z(w),\zeta(\eta))$$

Lemma 11 $\forall$ w, η *small enough, one has*

$$\begin{aligned}
|Q_k^*(w,\eta)| &\lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)} \\
\left|\frac{\partial}{\partial w_i} Q_k^*(w,\eta)\right| &\lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)} \\
\left|\frac{\partial}{\partial \eta_j} Q_k^*(w,\eta)\right| &\lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)} \\
\left|\frac{\partial^2}{\partial w_i \partial \eta_j} Q_k^*(w,\eta)\right| &\lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)\tau_i(\zeta_0,\varepsilon)\tau_j(\zeta_0,\varepsilon)}
\end{aligned}$$

Solving $\overline{\partial}$ with estimates

Method: Berndtsson-Andersson type integral kernels using support function S and estimates from above.

Always: $\Omega \Subset \mathbb{C}^n$ lineally convex, $\mathcal{C}^\infty$-smooth, type $m < \infty$.
For $1 \leq p \leq \infty$ define
$L^p_{(0,q)}(\Omega) := \{f\ (0,q)-$form on $\Omega,$ coefficients $\in L^p(\Omega)\}$
$\Lambda^\alpha_{(0,q)}(\Omega) :=$
$\{f\ (0,q)$ – form on Ω, coefficients $\in \Lambda^\alpha(\Omega)\}$
where
$\Lambda^\alpha(\Omega) :=$
$\{f \in L^\infty(\Omega) : \sup_{z_0 \neq z_1} \{\frac{|f(z_1 - f(z_0)|}{|z_1 - z_0|^\alpha} < \infty\}$

A typical isotropic result

Theorem 12 *(Di-Fischer-Fornæss 1999/2003)* *There are bounded linear $\overline{\partial}$-solving operators*

$$T_q : L^\infty_{(0,q+1)}(\Omega) \to \Lambda^{1/m}_{(0,q)}(\Omega)$$

There are many refinements of this to

- L^p-norms (B. Fischer 2001),
- in terms of multitype (T. Hefer 2001),
- to $\mathcal{C}^k$-Hölder spaces (W. Alexandre 2002),

(all proved only for smooth *linearly* convex domains of finite type).

Remark 13 *Although the data and the resulting objects in these statements are all characterized by isotropic conditions, the proofs are necessarily* non-isotropic.

An anisotropic result

Theorem 14 *(Di-Fischer 2003) Let* $d(z,\zeta)$ *be the pseudometric on* Ω*. Put*

$$\Lambda^{(d,1/m)}_{(0,q)}(\Omega) := \{f \in L^{\infty}_{(0,q)}(\Omega) : \|f\|_{(d,1/m)} := \max_{z_0 \neq z_1} \frac{|f(z_0)-f(z_1)|}{\max\{|d(z_0)-d(z_1)|^{\frac{1}{m}}, |z_0-z_1|^{1-\varepsilon}\}} < \infty\}$$

Then $\exists$ *bounded linear* $\overline{\partial}$*-solving operator*

$$T_q : L^{\infty}_{(0,q+1)} \to \Lambda^{(d,1/m)}_{(0,q)}(\Omega)$$

Remark 15 *It can be expected that all results from slide no. 12 have analogous non-isotropic versions on lineally convex domains of finite type.*