JOURNAL OF APPROXIMATION THEORY 42, 239–244 (1984)

On the Eneström–Kakeya Theorem

K. K. DEWAN* AND N. K. GOVIL

Department of Mathematics, Indian Institute of Technology, New Delhi 110 016, India, and Department of Mathematics, Auburn University, Auburn, Alabama 36849, U.S.A.

Communicated by Oved Shisha

Received August 31, 1981; revised January 26, 1984

A classical result of Eneström and Kakeya (if $a_n \geqslant a_{n-1} \geqslant a_{n-2} \cdots \geqslant a_1 > 0$, then, for $|z| > 1$, $\sum_{k=0}^{n} a_k z^k \neq 0$) is extended to polynomials whose coefficients are monotonic but not necessarily positive.

1. INTRODUCTION AND STATEMENT OF RESULTS

The following result is well known.

THEOREM A (Eneström–Kakeya). *If*

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0, \tag{1.1}$$

then, for $|z| > 1$, $\sum_{k=0}^{n} a_k z^k \neq 0$.

Theorem A has been extended in various ways (cf. [1, 2, 5]). Joyal, Labelle and Rahman [4, Theorem 3] proved

THEOREM B. *If* $p(z) = \sum_{k=0}^{n} a_k z^k$ *is a polynomial of (exact) degree* n $(n \geqslant 1)$ *such that*

$$a_n \geqslant a_{n-1} \geqslant a_{n-2} \geqslant \cdots \geqslant a_1 \geqslant a_0, \tag{1.2}$$

then $p(z)$ *has all its zeros in the circle.*

$$|z| \leqslant \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{1.3}$$

*Current Address: Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India.

0021-9045/84 $3.00

In this paper we show that the disk given by (1.3) can be replaced by an annulus with a smaller outer radius. More precisely, we prove the following

THEOREM. *Let $p(z)=\sum_{k=0}^{n} a_k z^k$ be a polynomial of (exact) degree n $(n \geqslant 1)$ such that*

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0.$$

Then $p(z)$ has all its zeros in the annulus (perhaps degenerate)

$$R_2 \leqslant |z| \leqslant R_1.$$

Here

$$R_1 = \frac{c}{2}\left(\frac{1}{|a_n|}-\frac{1}{M_1}\right)+\left\{\frac{c^2}{4}\left(\frac{1}{|a_n|}-\frac{1}{M_1}\right)^2+\frac{M_1}{|a_n|}\right\}^{1/2}$$

and

$$R_2 = \frac{1}{2M_2^2}\left[-R_1^2 b(M_2-|a_0|)+\{R_1^4 b^2(M_2-|a_0|)^2+4|a_0|R_1^2 M_2^3\}^{1/2}\right],$$

where

$$\begin{aligned} M_1 &= a_n - a_0 + |a_0|, \\ M_2 &= R_1^n(|a_n| R_1 + a_n - a_0), \\ c &= a_n - a_{n-1}, \\ b &= a_1 - a_0. \end{aligned} \tag{1.4}$$

Moreover

$$0 \leqslant R_2 \leqslant 1 \leqslant R_1 \leqslant \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{1.5}$$

If $a_0 > 0$, the last theorem yields Theorem A.

To prove (1.5), observe that the inequality $R_1 \geqslant 1$ follows from the definition of R_1 noting that $c \geqslant 0$ and $M_1 \geqslant |a_n|$. The inequality $0 \leqslant R_2 \leqslant 1$ follows from the definition of R_2 and the inequality $|a_0| \leqslant (M_2/R_1^2)\cdot(M_2+R_1^2 b)/(M_2+b)$, which is a consequence of (3.7). Thus in order to establish (1.5), it remains only to show that $R_1 \leqslant (a_n - a_0 + |a_0|)/|a_n|$, and for this, note that

$$\frac{M_1}{|a_n|} \geqslant \frac{c}{2}\left(\frac{1}{|a_n|}-\frac{1}{M_1}\right)+\left\{\frac{c^2}{4}\left(\frac{1}{|a_n|}-\frac{1}{M_1}\right)^2+\frac{M_1}{|a_n|}\right\}^{1/2}$$

if

$$2M_1^2 \geqslant c(M_1 - |a_n|) + \{c^2(M_1 - |a_n|)^2 + 4M_1^3 |a_n|\}^{1/2},$$

which is true if

$$(M_1 - c)(M_1 - |a_n|) \geqslant 0. \tag{1.6}$$

Since (1.6) holds, the inequality $R_1 \leqslant (a_n - a_0 + |a_0|)/|a_n|$ follows and the proof of (1.5) is complete.

The results obtained by our Theorem are at least as good as those obtained by Theorem B, but in some cases the results obtained by our Theorem are very much better than those obtained by Theorem B. To illustrate this we consider the following examples.

EXAMPLE 1.

$$p(z) = 6z^4 + 4z^3 + 3z^2 + 2z - 100.$$

Theorem B gives that $p(z)$ has all its zeros in $|z| \leqslant 34.3334$, while our Theorem gives that $p(z)$ has all its zeros in $0 \cdot 1297 \leqslant |z| \leqslant 6.0236$.

EXAMPLE 2.

$$p(z) = \frac{1}{2} z^5 + \frac{1}{2} z^4 + \frac{2}{5} z^3 + \frac{3}{10} z^2 + \frac{1}{5} z - (10)^3.$$

Theorem B gives that $p(z)$ has all its zeros in $|z| \leqslant 4001$ and by our Theorem, all the zeros of $p(z)$ are contained in $|z| \leqslant 63.2535$.

2. LEMMAS

LEMMA 1. *If $p(z)$ is analytic inside and on the unit circle, $|p(z)| \leqslant M$ ($M > 0$) on $|z| = 1$, and $p(0) = a$, then*

$$|p(z)| \leqslant M \frac{M|z| + |a|}{|a||z| + M} \tag{2.1}$$

for $|z| < 1$.

Lemma 1 is a well-known generalization of Schwarz's lemma.

From a lemma due to Govil, Rahman and Schmeisser [3, p. 325], one can easily prove

LEMMA 2. *If $p(z)$ is analytic in $|z| \leqslant R$, $p(0)=0$, $p'(0)=b$, and $|p(z)| \leqslant M$ for $|z|=R$, then, for $|z| \leqslant R$,*

$$|p(z)| \leqslant \frac{M|z|}{R^2} \frac{M|z|+R^2|b|}{M+|z||b|}. \tag{2.2}$$

3. PROOF OF THE THEOREM

Consider

$$\begin{aligned} g(z) &= (1-z)\,p(z) \\ &= -a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1})\, z^k + a_0 \\ &= -a_n z^{n+1} + P(z), \qquad \text{say.} \end{aligned} \tag{3.1}$$

If by $Q(z)$ we denote the polynomial $z^n P(1/z)$, then

$$Q(z) = \sum_{k=1}^{n} (a_k - a_{k-1})\, z^{n-k} + a_0 z^n.$$

For $|z|=1$, we have

$$|Q(z)| \leqslant \sum_{k=1}^{n} (a_k - a_{k-1}) + |a_0| = M_1.$$

Applying Lemma 1 to the function $Q(z)$, we get for $|z| \leqslant 1$

$$\left| z^n P\left(\frac{1}{z}\right) \right| = |Q(z)| \leqslant M_1 \frac{M_1|z| + (a_n - a_{n-1})}{(a_n - a_{n-1})|z| + M_1}. \tag{3.2}$$

If $|z|>1$, then (3.2) yields

$$|P(z)| \leqslant M_1 |z|^n \frac{M_1 + (a_n - a_{n-1})|z|}{(a_n - a_{n-1}) + M_1|z|}. \tag{3.3}$$

Thus for $|z| = R > 1$,

$$\begin{aligned} |g(z)| &\geqslant |-a_n z^{n+1} + P(z)| \\ &\geqslant |a_n| R^{n+1} - |P(z)| \\ &\geqslant |a_n| R^{n+1} - M_1 R^n \frac{M_1 + R(a_n - a_{n-1})}{M_1 R + (a_n - a_{n-1})} \qquad \text{(by (3.3))} \end{aligned}$$

$$= |a_n| R^{n+1} - M_1 R^n \frac{M_1 + cR}{M_1 R + c} \qquad \text{(by (1.4))}$$

$$= \frac{R^n}{M_1 R + c} [M_1 |a_n| R^2 - cR(M_1 - |a_n|) - M_1^2]$$

$$> 0$$

if

$$R > \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$
$$= R_1.$$

Therefore $p(z)$ has all its zeros in

$$|z| \leqslant R_1. \tag{3.4}$$

Next we show that $p(z)$ has no zeros in $|z| < R_2$. We have by (3.1)

$$g(z) = a_0 + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k - a_n z^{n+1}$$
$$= a_0 + f(z), \qquad \text{say.} \tag{3.5}$$

Clearly, if $|z| \leqslant R_1$, then

$$|f(z)| \leqslant |a_n| R_1^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1}) R_1^k$$
$$\leqslant |a_n| R_1^{n+1} + R_1^n \sum_{k=1}^{n} (a_k - a_{k-1})$$
$$= |a_n| R_1^{n+1} + R_1^n (a_n - a_0)$$
$$= R_1^n (|a_n| R_1 + a_n - a_0)$$
$$= M_2. \tag{3.6}$$

Further, since $f(0) = 0$ and $f'(0) = a_1 - a_0 = b$, by Lemma 2 we have

$$|f(z)| \leqslant \frac{M_2 |z|}{R_1^2} \frac{M_2 |z| + R_1^2 b}{M_2 + |z| b} \tag{3.7}$$

for $|z| \leqslant R_1$.

Combining (3.5) and (3.7), we get, for $|z| \leqslant R_1$,

$$|g(z)| \geqslant |a_0| - \frac{M_2|z|}{R_1^2} \frac{M_2|z| + R_1^2 b}{M_2 + |z| b}$$

$$= -\frac{1}{R_1^2(M_2 + |z| b)} [|z|^2 M_2^2 + R_1^2 b |z|(M_2 - |a_0|) - |a_0| R_1^2 M_2]$$

$$> 0,$$

if

$$|z| < \frac{-R_1^2 b(M_2 - |a_0|) + \{R_1^4 b^2 (M_2 - |a_0|)^2 + 4 |a_0| R_1^2 M_2^3\}^{1/2}}{2M_2^2}$$

$$= R_2$$

(since $M_2 - |a_0| = M_2 - |f(1)| \geqslant 0$ by (3.6)), which implies that $p(z)$ has no zeros in

$$|z| < R_2, \tag{3.8}$$

and the theorem follows.

Acknowledgment

The authors are grateful to the referee for his valuable suggestions.

References

1. G. T. Cargo and O. Shisha, Zeros of polynomials and fractional order differences of their coefficients, *J. Math. Anal. Appl.* **7** (1963), 176–182.
2. N. K. Govil and Q. I. Rahman, On the Eneström–Kakeya Theorem, *Tôhoku Math. J.* **20** (1968), 126–136.
3. N. K. Govil, Q. I. Rahman, and G. Schmeisser, On the derivative of a polynomial, *Illinois J. Math.* **23** (1979), 319–329.
4. A. Joyal, G. Labelle, and Q. I. Rahman, On the location of zeros of polynomials, *Canad. Math. Bull.* **10** (1967), 53–63.
5. P. V. Krishnaiah, On Kakeya's Theorem, *J. London Math. Soc.* **30** (1955), 314–319.