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Faber Polynomials and Spectrum Localisation

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Abstract Let K be a compact connected subset of the complex plane, of non-void interior, and whose complement in the extended complex plane is connected. Denote by F_n the nth Faber polynomial associated with K. The aim of this paper is to find suitable Banach spaces of complex sequences, $\mathcal{R}$, such that statements of the following type hold true: if T is a bounded linear operator acting on the Banach space $\mathcal{X}$ such that $(\langle F_n(T)x, x^*\rangle)_{n\geq 0} \in \mathcal{R}$ for each pair $(x, x^*) \in \mathcal{X} \times \mathcal{X}^*$, then the spectrum of T is included in the interior of K. Generalisations of some results due to W. Mlak, N. Nikolski and J. van Neerven are, thus, obtained and several illustrative examples are given. An interesting feature of these generalisations is the influence of the geometry of K and the regularity of its boundary.

Keywords Operator theory · Spectrum · Faber polynomials

Mathematics Subject Classification 30B99 · 47A10

1 Introduction

Faber polynomials are now classical objects of study in complex analysis, function theory and approximation theory. In this paper we use Faber polynomials as a basic tool to study the following "spectrum localisation problem" in operator theory: identify conditions under which the spectrum of a Banach space bounded linear operator is included in the interior of a given compact set of the complex plane. Other instances

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of the use of Faber polynomials for various problems in operator theory can be found in [1–4, 17].

Computing or estimating the spectrum of a large matrix, or, more generally, of a bounded linear operator acting on a complex Banach space is an important problem in spectral theory, with possible applications in numerical analysis (see for instance [12]). The classical spectral radius formula [5]

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n},$$

going back to Arne Beurling and Israel Gelfand, allows us to find or estimate the radius $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ of the smallest closed disc in the plane centered in the origin and containing the spectrum $\sigma(T)$ of the given operator T. The condition $r(T) < 1$, corresponding to the exponential stability of T, is important for applications. To localise better the spectrum and to find inclusions of the form $\sigma(T) \subset K$ or $\sigma(T) \subset int(K)$ for some simply connected compact set K in the complex plane, the right tool to consider is the asymptotic behaviour of the sequence $(F_n(T))_{n\geq 0}$, where F_n is the Faber polynomial of degree n associated with K. We are interested in generalising, in this vein (replacing the closed unit disc by more general compact sets), several existing results in the literature [11, 13, 14, 18, 19]. An interesting feature of our study is the influence of the geometry of the compact set and the regularity of its boundary on the results obtained about the localisation of the spectrum.

The present paper is organized as follows. Section 2 is dedicated to the definition and basic properties of Faber polynomials. We present the influence of the regularity of the boundary of the compact set on the behaviour of these polynomials and present some known applications in operator theory. In Sects. 3 and 4, we present two approaches leading to different criteria for the spectrum to be included in the interior of a compact set. Some generalisations of the results of Mlak [11], Weiss [19], van Neerven [18] and Nikolski [13, 14] are given there. We give in Sect. 5 some explicit estimates based upon the preceding criteria.

2 Preliminaries on Faber Polynomials

2.1 Definition and Examples

We present here some classical results about Faber polynomials which will be used later. We follow the setting of [16]. The reader is referred to [16] or [8] for more details about Faber polynomials and related topics.

Let K be a compact connected subset of the complex plane $\mathbb{C}$, different from a singleton, and whose complement is connected. From the Riemann mapping theorem, we know that there exists a unique conformal map $\psi : \overline{\mathbb{D}}^c \longrightarrow K^c$ such that

$$\psi(\infty) = \infty \quad \text{and} \quad \psi'(\infty) > 0.$$

The map ψ has a Laurent expansion for $|w| > 1$ of the form

$$\psi(w) = \beta w + \beta_0 + \beta_1 w^{-1} + \cdots + \beta_k w^{-k} + \cdots$$

where $\beta > 0$ is the transfinite diameter, or (logarithmic) capacity, of K.

Let ϕ be the inverse function of ψ. The map $\phi : K^c \longrightarrow \overline{\mathbb{D}}^c$ has a Laurent expansion in a neighborhood of infinity of the form

$$\phi(z) = \frac{1}{\beta} z + b_0 + b_1 z^{-1} + \cdots + b_k z^{-k} + \cdots$$

For $n \in \mathbb{N}$, the polynomial part of the Laurent expansion of $\phi(z)^n$ is called the *Faber polynomial of order* n and is denoted by F_n. The Faber polynomial of order n has degree n and leading coefficient $1/\beta^n$. We consider the function ω_n defined by the following equation

$$\phi(z)^n = F_n(z) + \omega_n(z), \quad (z \in K^c).$$

Then, $z \mapsto \omega_n(z)$ is an analytic bounded function on K^c which tends to 0 at infinity.

For any $R > 1$, let Γ_R be the analytic Jordan curve $\{\psi(w) : |w| = R\}$. Denote by G_R its interior. If z is in G_R then we have

$$F_n(z) = \frac{1}{2i\pi} \int_{\Gamma_R} \frac{\phi(\zeta)^n}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2i\pi} \int_{|w|=R} \frac{w^n \psi'(w)}{\psi(w) - z} \mathrm{d}w.$$

One can deduce from this relation that the Faber polynomials satisfy the following asymptotic relations

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = |\phi(z)| \quad \text{for all } z \notin K \quad \text{and} \quad \lim_{n\to\infty} |F_n(z)|^{1/n} \leq 1 \quad \text{for all } z \in K.$$

It also implies that for any fixed z in G_R, $(F_n(z))_{n\geq 0}$ is the sequence of the Laurent coefficients of the map

$$w \mapsto \frac{\psi'(w)}{\psi(w) - z}, \quad |w| > R,$$

in the neighborhood of the point $w = \infty$. Therefore, the generating function of the Faber polynomials is given by

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \quad (|w| > R, z \in G_R). \tag{2.1}$$

The Faber polynomial of order n of the disk centered at z_0 and of radius R is given by the formula $F_n(z) = ((z - z_0)/R)^n$. The series of Faber polynomials of an analytic function in the neighborhood of the compact set K is a generalisation of the Taylor expansion of an analytic function in an open disc. More precisely, we have the following theorem (cf. [16, Theorem 3.2.2]).

Theorem 2.1 ([16]) *An analytic function f in a neighborhood of the compact set K can be uniquely expanded in a series of Faber polynomials with uniform convergence on a neigborhood of K. That is to say, there exist a neighborhood V of K and complex numbers $a_n = a_n(f)$ such that for every z in V, we have*

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z).$$

Example 2.2 Let $R > 1$ and K be the compact set delimited by the ellipse with the foci at the points 1 and -1 and semi-axes

$$a = \frac{1}{2}(R + 1/R) \quad \text{and} \quad b = \frac{1}{2}(R - 1/R).$$

The equation of this ellipse can be written in the form

$$z = \frac{1}{2}\left(Re^{i\theta} + \frac{1}{Re^{i\theta}}\right), \quad 0 \le \theta < 2\pi.$$

Hence, the exterior conformal map associated with K is given by

$$\psi(w) = \frac{1}{2}\left(Rw + \frac{1}{Rw}\right).$$

The corresponding Faber polynomials are

$$F_n(z) = \frac{2}{R^n} C_n(z), \quad (n \ge 1),$$

where C_n is the Chebychev polynomial of order n,

$$C_n(t) = \cos(n \arccos(t)), \quad (t \in [-1; 1]).$$

We also notice that, for any $n \ge 1$, we have

$$F_n(\psi(w)) = w^n + \frac{1}{R^{2n} w^n}.$$

2.2 Consequences of the Regularity of the Boundary Γ

If Γ, the boundary of K, is a Jordan curve, its regularity influences the behaviour of the map ψ and the asymptotic behaviour of the sequence of Faber polynomials $(F_n)_{n\ge 0}$ on Γ.

A function $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a *modulus of continuity* if it satisfies:

(1) ρ is increasing;
(2) $\lim_{t\to 0} \rho(t) = 0$;
(3) ρ is subadditive, i.e.

$$\rho(t + s) \le \rho(t) + \rho(s).$$

A modulus of continuity is said to satisfy *the Alper condition* if

$$\int_0^{\varepsilon} \frac{\rho(x)}{x} \log\left(\frac{1}{x}\right) \mathrm{d}x < \infty,$$

for some $\varepsilon > 0$.

For a fixed modulus of continuity ρ, a Jordan curve Γ is said to be of class C^{ρ} if it has a parametrization $\tau : [0; 1) \longrightarrow \Gamma$ that is differentiable, with $\tau'(x) \neq 0$ of all $x \in [0, 1)$ and τ' satisfying the continuity condition

$$|\tau'(x_1) - \tau'(x_2)| \leq \rho(|x_1 - x_2|), \quad x_1, x_2 \in [0; 1).$$

A Jordan arc is of class C^{ρ} if it is a subarc of a Jordan curve of class C^{ρ}.

The following theorem is a simplified version of [15, Theorem 1.1].

Theorem 2.3 ([15]) *Let Γ be a rectifiable Jordan curve. For $z_0 \in \Gamma$, assume that Γ has an exterior angle of opening $\alpha\pi$ at z_0, with $0 < \alpha \leq 2$, formed by C^{ρ} arcs, where ρ is a modulus of continuity satisfying the Alper condition. Then,*

$$F_n(z_0) = \alpha\phi(z_0)^n + o(1), \quad \textit{as } n \to \infty.$$

Remark 2.4 In the above result, the case $\alpha = 1$ corresponds to a point where the boundary form an arc of regularity C^{ρ}, the (excluded) case $\alpha = 0$ to an inside-pointing cusp and the case $\alpha = 2$ to an outside-pointing cusp.

It is often sufficient to consider $\rho(x) = x^{\beta}$ for some $\beta > 0$. The class C^{ρ} is then the Hölder class $C^{1,\beta}$.

The following definition will be used in the next sections as a basic geometric condition about the regularity of the boundary.

Definition 2.5 A rectifiable Jordan curve C will be called an *Alper curve with (possible) angles* if for all $\zeta \in C$ either C forms an arc of regularity C^{ρ} in a neighborhood of ζ, or C has an exterior angle of opening $\alpha\pi$ at ζ, $0 < \alpha \leq 2$, formed by C^{ρ} arcs, where ρ is a modulus of continuity satisfying the Alper condition.

We want to stress here that Alper curves with angles do not have inside-pointing cusps.

Remark 2.6 According to Theorem 2.3, if Γ is an Alper curve with (possible) angles, then for any $\zeta \in \Gamma$, the sequence $(|F_n(\zeta)|)_{n\geq 0}$ is eventually bounded below by a positive number.

We also record, for further use, the following classical result about analytic Jordan curves and an inequality of Kövari and Pommerenke [10].

Theorem 2.7 *Suppose that Γ is an analytic Jordan curve. Then ψ has an analytic univalent extension in $\overline{r\mathbb{D}}^c$ for some $r \in (0; 1)$.*

Theorem 2.8 ([10]) *If K is convex, then for all $z \in \Gamma$ and for all $n \geq 0$,*

$$|F_n(z) - \phi(z)^n| \leq 1.$$

2.3 Faber Polynomials and Operators on a Banach Space

Notation For a complex Banach space $\mathcal{X}$, we denote by $B(\mathcal{X})$ the Banach algebra of the bounded linear operators from $\mathcal{X}$ into itself and by $I = I_{\mathcal{X}}$ the identity on $\mathcal{X}$. The spectrum $\sigma(T)$ of $T \in B(\mathcal{X})$ is defined [5] by

$$\sigma(T) := \{\lambda \in \mathbb{C}; (T - \lambda I) \text{ is not invertible in } B(\mathcal{X})\}.$$

Lemma 2.9 *Let $\mathcal{X}$ be a Banach space. Let $T \in B(\mathcal{X})$ be such that $\sigma(T) \subset K$. Then, for $\lambda \in \mathbb{C}$, $|\lambda| > 1$, we have*

$$(\psi(\lambda)I - T)^{-1} = \psi'(\lambda)^{-1} \sum_{n=0}^{\infty} \lambda^{-n-1} F_n(T).$$

Proof Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Let $R = |\lambda|$. Define

$$f(z) = \frac{\psi'(\lambda)}{\psi(\lambda) - z}, \quad z \in G_R.$$

It follows from (2.1) that for every $z \in G_R$, we have

$$f(z) = \sum_{n=0}^{\infty} \lambda^{-n-1} F_n(z).$$

As f is an analytic function on G_R, we have from Theorem 2.1 the uniform convergence of the series in any compact subset of G_R.

Using $\sigma(T) \subset K$, the holomorphic functional calculus gives

$$(\psi(\lambda)I - T)^{-1} = \psi'(\lambda)^{-1} \sum_{n=0}^{\infty} \lambda^{-n-1} F_n(T).$$

□

In their article [1], A. Atzmon, A. Eremenko and M. Sodin have proved that for any compact set $K \subset \mathbb{C}$ with connected complement, there exists a sequence of polynomials $(P_n)_{n\geq 0}$ and a positive number r such that if T is a bounded operator on $\mathcal{X}$, its spectrum is included in K if and only if

$$\limsup_{n\to\infty} \|P_n(T)\|^{1/n} \leq r.$$

If K is connected, this can be expressed in terms of Faber polynomials.

Theorem 2.10 ([1]) *Let $\mathcal{X}$ be a Banach space and $T \in B(\mathcal{X})$. Then $\sigma(T) \subset K$ if and only if*

$$\limsup_{n\to\infty} \|F_n(T)\|^{1/n} \leq 1.$$

3 Condition on the Weak Faber Orbits

Let K be a simply connected compact subset of the complex plane with a non-empty interior. The aim of this section is to propose some conditions implying that $\sigma(T)$ is included in the interior of K. Such results have been proved in the case when K is the closed unit disc by Mlak [11], Weiss [19], van Neerven [18] and Nikolski [13,14].

Let $(F_n)_{n\geq 0}$ be the sequence of Faber polynomials associated with K.

Notation The set of finitely supported sequences will be denoted $\mathbb{C}^{(\mathbb{N})}$ and $(e_n)_{n\geq 0}$ will denote the canonical algebraic basis of $\mathbb{C}^{(\mathbb{N})}$. The characteristic function of a set A will be denoted by χ_A.

Definition 3.1 A Banach space $\mathcal{E} \subset \mathbb{C}^{\mathbb{N}}$ is said to be an *admissible sequence space* if it satisfies the following properties:

- if $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$ and $(b_n)_{n\geq 0} \in \mathcal{E}$, then $(a_n)_{n\geq 0} \in \mathcal{E}$ and $\|(a_n)_{n\geq 0}\|_{\mathcal{E}} \leq \|(b_n)_{n\geq 0}\|_{\mathcal{E}}$;
- for all $k \in \mathbb{N}$ there exists a sequence $(a_n)_{n\geq 0} \in \mathcal{E}$ such that $a_k \neq 0$.

Proposition 3.2 *Let $\mathcal{E}$ be an admissible sequence space. Then $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{E}$.*

Proof Let $k \in \mathbb{N}$. There exists a sequence $(a_n)_{n\geq 0} \in \mathcal{E}$ such that $a_k \neq 0$ by the second condition of admissibility. Then $|\chi_{\{k\}}(n)| \leq |a_k|^{-1}|a_n|$ for all $n \in \mathbb{N}$. It follows then from the first condition of admissibility that $\chi_{\{k\}}$ belongs to $\mathcal{E}$. By taking finite linear combinations, we get that the finitely supported sequences belong to $\mathcal{E}$. □

The following definition extends the corresponding unit-disc analogue from [14].

Definition 3.3 Let $\mathcal{E} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space and $T \in B(\mathcal{X})$. The operator T is said to be of *weak type $\mathcal{E}$* if for every pair $(x; x^*) \in \mathcal{X} \times \mathcal{X}^*$, we have

$$\big(\langle F_n(T)x; x^*\rangle\big)_{n\geq 0} \in \mathcal{E}.$$

Theorem 3.5, which is the main result of this section, is a generalisation of the following result of J. van Neerven [18].

Theorem 3.4 ([18]) *Let $\mathcal{X}$ be a Banach space, $T \in B(\mathcal{X})$ and $\mathcal{E}$ an admissible sequence space such that for any $\zeta \in \mathbb{T}$,*

$$\|(1, \zeta, \zeta^2, \ldots, \zeta^N, 0, 0, \ldots)\|_{\mathcal{E}} \underset{N\to\infty}{\longrightarrow} \infty.$$

If T is of weak type $\mathcal{E}$, then

$$r(T) < 1.$$

Theorem 3.5 *Let $\mathcal{X}$ be a Banach space, $T \in B(\mathcal{X})$ and $\mathcal{E}$ an admissible sequence space such that for any $\zeta \in \partial K$,*

$$\|(F_0(\zeta), \ldots, F_N(\zeta), 0, 0, \ldots)\|_{\mathcal{E}} \underset{N\to\infty}{\longrightarrow} \infty.$$

If T is of weak type $\mathcal{E}$, then

$$\sigma(T) \subset int(K).$$

Lemma 3.6 *Let $\mathcal{E} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space such that for any $k \in \mathbb{N}$, the map $(a_n)_{n\geq 0} \mapsto a_k$ is bounded. Suppose that $T \in B(\mathcal{X})$ is a weak type $\mathcal{E}$ operator. Then,*

$$\sup\left\{\left\|\left(\langle F_n(T)x; x^*\rangle\right)_{n\geq 0}\right\|_{\mathcal{E}}; \|x^*\| \leq 1, \|x\| \leq 1\right\} < \infty.$$

Proof Let us prove that for any $x \in \mathcal{X}$, the map

$$\begin{aligned} f_x &: \mathcal{X}^* \to \mathcal{E} \\ x^* &\mapsto \left(\langle F_n(T)x; x^*\rangle\right)_{n\geq 0} \end{aligned}$$

is bounded.

Indeed, f_x is well defined, because T is a weak type $\mathcal{E}$ operator. Let $(x_n^*)_{n\geq 0} \subset \mathcal{X}^*$, $x^* \in \mathcal{X}^*$ and $r \in \mathcal{E}$ be such that

$$x_n^* \to x^* \quad \text{and} \quad f_x(x_n^*) \overset{\mathcal{E}}{\longrightarrow} r.$$

We infer from the boundedness of the maps $(a_n)_{n\geq 0} \mapsto a_k$ that for any $k \in \mathbb{N}$,

$$\left[f_x(x_n^*)\right]_k \underset{n\to\infty}{\longrightarrow} r_k,$$

where $[f_x(x_n^*)]_k$ denotes the kth element of the sequence $f_x(x_n^*)$. For any $k \in \mathbb{N}$,

$$[f_x(x_n^*)]_k = \langle F_k(T)x; x_n^*\rangle \underset{n\to\infty}{\longrightarrow} \langle F_k(T)x; x^*\rangle = [f_x(x^*)]_k.$$

Thus $f_x(x^*) = r$ and, according to the closed graph theorem [5], f_x is bounded. In particular, for any $x \in \mathcal{X}$,

$$\sup\left\{\left\|\left(\langle F_n(T)x; x^*\rangle\right)_{n\geq 0}\right\|_{\mathcal{E}}; \|x^*\| \leq 1\right\} < \infty. \tag{3.1}$$

For any $x^* \in \mathcal{X}^*$ let

$$\begin{aligned} g_{x^*} &: \mathcal{X} \to \mathcal{E} \\ x &\mapsto f_x(x^*). \end{aligned}$$

Then, in the same way as for f_x, we can prove that, for every x^*, the map g_{x^*} is bounded. Using (3.1) we obtain that

$$\sup_{\|x^*\|\leq 1} \|g_{x^*}(x)\| < \infty,$$

for an arbitrary $x \in \mathcal{X}$. Using the uniform boundedness principle [5], we infer that $\{g_{x^*}\}_{\|x^*\|\leq 1}$ is a bounded set. Thus,

$$\sup \left\{ \left\| \left(\langle F_n(T)x; x^* \rangle \right)_{n\geq 0} \right\|_{\mathcal{E}}; \|x^*\| \leq 1, \|x\| \leq 1 \right\} < \infty,$$

which completes the proof. □

Lemma 3.7 *Let $T \in B(\mathcal{X})$ be a linear operator acting on the Banach space $\mathcal{X}$. For all $\lambda \in \partial\sigma(T)$, all $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exist $x_N \in \mathcal{X}$ and $x_N^* \in \mathcal{X}^*$ satisfying $\|x_N\| = 1$, $\|x_N^*\| = 1$ and*

$$|\langle F_n(T)x_N; x_N^* \rangle| > |F_n(\lambda)| - \varepsilon, \quad n = 0, 1, \ldots, N.$$

Proof Let $\lambda \in \partial\sigma(T)$, $\varepsilon > 0$ and $N \in \mathbb{N}$. As $\lambda \in \partial\sigma(T)$, we have that λ belongs to $\sigma_{app}(T)$, the approximate point spectrum of T (cf. [5, Chapter 7.6]). By definition, this means that there exists a sequence $(y_n)_{n\geq 0}$ in $\mathcal{X}$ such that $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|Ty_n - \lambda y_n\| \underset{n\to\infty}{\longrightarrow} 0.$$

Thus, for each fixed $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \|F_k(T)y_n - F_k(\lambda)y_n\| = 0.$$

Let n_1 be such that

$$\|F_k(T)y_{n_1} - F_k(\lambda)y_{n_1}\| < \varepsilon, \quad \forall k = 0, 1, \ldots, N.$$

Let $x_N = y_{n_1}$ and $x_N^* \in \mathcal{X}^*$ be such that $\|x_N^*\| = 1$ and $\langle x_N^*; x_N \rangle = 1$. Then for $k \in \{0, 1, \ldots, N\}$, we have

$$\begin{aligned} |\langle F_k(T)x_N; x_N^* \rangle| &\geq |\langle F_k(\lambda)x_N; x_N^* \rangle| - |\langle F_k(T)x_N - F_k(\lambda)x_N; x_N^* \rangle| \\ &> |F_k(\lambda)| - \varepsilon. \end{aligned}$$

□

Proof of Theorem 3.5 From Lemma 3.6, we know that there exists $M \geq 0$ such that for any $x \in \mathcal{X}$, $x^* \in \mathcal{X}^*$, with $\|x\| = 1$ and $\|x^*\| = 1$, we have

$$\left\| \left(\langle F_n(T)x; x^* \rangle \right)_{n\geq 0} \right\|_{\mathcal{E}} \leq M.$$

Suppose that $\sigma(T) \nsubseteq int(K)$, then there exists $\lambda \in \partial\sigma(T)\backslash int(K)$. From Lemma 3.7, we get that for every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $x_N \in \mathcal{X}$ and $x_N^* \in \mathcal{X}^*$ satisfying $\|x_N\| \leq 1$, $\|x_N^*\| \leq 1$ and

$$|\langle F_n(T)x_N; x_N^*\rangle| > |F_n(\lambda)| - \varepsilon, \quad n = 0, 1, \dots, N.$$

Therefore, $|\langle F_n(T)x_N; x_N^*\rangle| \geq |F_n(\lambda)|\chi_{\{0,\dots,N\}}(n) - \varepsilon\chi_{\{0,\dots,N\}}(n)$ for all n. Thus,

$$\left\| \left(\langle F_n(T)x_N; x_N^*\rangle\right)_{n\geq 0} \right\|_{\mathcal{E}} \geq \|(F_0(\zeta), \dots, F_N(\zeta), 0, 0, \dots)\|_{\mathcal{E}} - \varepsilon\|\chi_{\{0,\dots,N\}}\|_{\mathcal{E}}.$$

We obtain that for any $\varepsilon > 0$ and $N \in \mathbb{N}$,

$$\|(F_0(\zeta), \dots, F_N(\zeta), 0, 0, \dots)\|_{\mathcal{E}} - \varepsilon\|\chi_{\{0,\dots,N\}}\|_{\mathcal{E}} \leq M.$$

Hence, for every $N \in \mathbb{N}$, we have

$$\|(F_0(\zeta), \dots, F_N(\zeta), 0, 0, \dots)\|_{\mathcal{E}} \leq M,$$

which gives the desired contradiction if $\lambda \in \partial K$. If $\lambda \notin K$, then

$$|F_n(\lambda)|^{1/n} \underset{n\to\infty}{\longrightarrow} R > 1.$$

But we know that for any $\zeta \in \partial K$, $\limsup_{n\to\infty} |F_n(\zeta)|^{1/n} \leq 1$, and

$$\|(F_0(\zeta), \dots, F_N(\zeta), 0, 0, \dots)\|_{\mathcal{E}} \underset{N\to\infty}{\longrightarrow} \infty.$$

Thus, if $\lambda \notin K$, we get from the first condition of admissibility that

$$\|(F_0(\zeta), \dots, F_N(\zeta), 0, 0, \dots)\|_{\mathcal{E}} \underset{N\to\infty}{\longrightarrow} \infty,$$

which gives a contradiction. □

We now give some illustrating examples; we start with one based on weighted l^p spaces.

Corollary 3.8 *Let $1 \leq p < \infty$ and let $(\omega_n)_{n\geq 0}$ be a sequence of non-negative numbers. We suppose that for any $\zeta \in \partial K$,*

$$\sum_{n=0}^{\infty} \omega_n |F_n(\zeta)|^p = \infty.$$

Let $\mathcal{X}$ be a Banach space. Let $T \in B(\mathcal{X})$ be such that for every $(x, x^*) \in \mathcal{X} \times \mathcal{X}^*$ we have

$$\sum_{n=0}^{\infty} \omega_n |\langle F_n(T)x; x^*\rangle|^p < \infty.$$

Then $\sigma(T) \subset int(K)$.

Corollary 3.9 *Let* $(\omega_n)_{n\geq 0}$ *be a sequence of non-negative numbers. We suppose that for any* $\zeta \in \partial K$,

$$\limsup_{n\to\infty} \omega_n |F_n(\zeta)| = \infty.$$

Let $\mathcal{X}$ *be a Banach space. Let* $T \in B(\mathcal{X})$ *be such that for every* $(x, x^*) \in \mathcal{X} \times \mathcal{X}^*$ *we have*

$$\sup_{n\geq 0} \omega_n |\langle F_n(T)x; x^*\rangle| < \infty.$$

Then $\sigma(T) \subset int(K)$.

Remark 3.10 According to Remark 2.6, if ∂K is an Alper curve with (possible) angles, the conditions on the sequence $(\omega_n)_{n\geq 0}$ can be replaced by

$$\sum_{n=0}^{\infty} \omega_n = \infty,$$

in Corollary 3.8 and by

$$\limsup_{n\to\infty} \omega_n = \infty$$

in Corollary 3.9.

Proof of Corollaries 3.8 and 3.9 We have that

$$\lim_{|z|\to\infty} |\phi(z)| = \infty.$$

Therefore, there exists a $R > 1$ such that

$$\sigma(T) \cap \{z : |\phi(z)| \geq R\} = \varnothing.$$

Thus, we get from Theorem 2.10 that

$$\limsup_{n\to\infty} \|F_n(T)\|^{1/n} < R.$$

Let $(\tilde{\omega}_n)_{n\geq 0}$ be the positive sequence defined by

$$\tilde{\omega}_n = \begin{cases} \omega_n & \text{if } \omega_n \neq 0, \\ R^{-np} & \text{if } \omega_n = 0. \end{cases}$$

For every $(x, x^*) \in \mathcal{X} \times \mathcal{X}^*$, we have

$$\sum_{n=0}^{\infty} \tilde{\omega}_n |\langle F_n(T)x; x^* \rangle|^p < \infty.$$

If $p < \infty$, then the space

$$l^p(\tilde{\omega}) := \left\{ (a_n)_{n\geq 0}; \sum_{n=0}^{\infty} \tilde{\omega}_n |a_n|^p < \infty \right\}$$

with the norm:

$$\left\| (a_n)_{n\geq 0} \right\|_{l^p(\tilde{\omega})} = \left(\sum_{n=0}^{\infty} \tilde{\omega}_n |a_n|^p \right)^{1/p}$$

is an admissible sequence space and the condition

$$\|(F_0(\zeta), \ldots, F_N(\zeta), 0, 0, \ldots)\|_{l^p(\tilde{\omega})} \underset{N\to\infty}{\longrightarrow} \infty$$

is equivalent to

$$\sum_{n=0}^{\infty} \tilde{\omega}_n |F_n(\zeta)|^p = \infty.$$

If $p = \infty$, then the space

$$l^\infty(\tilde{\omega}) = \left\{ (a_n)_{n\geq 0}; \sup_{n\geq 0} \tilde{\omega}_n |a_n| < \infty \right\}$$

with the norm

$$\left\| (a_n)_{n\geq 0} \right\|_{l^\infty(\tilde{\omega})} = \sup_{n\geq 0} \tilde{\omega}_n |a_n|$$

is an admissible sequence space and the condition

$$\|(F_0(\zeta), \ldots, F_N(\zeta), 0, 0, \ldots)\|_{l^p(\tilde{\omega})} \underset{N\to\infty}{\longrightarrow} \infty$$

is equivalent to

$$\limsup_{n\to\infty} \tilde{\omega}_n |F_n(\zeta)| = \infty.$$

It is then sufficient to apply Theorem 3.5 with the space $\mathcal{E} = l^p(\tilde{\omega})$ to conclude. □

We give now another application of Theorem 3.5 based on Orlicz spaces.

Definition 3.11 A function $\Phi : [0;\infty) \longrightarrow [0;\infty)$ is a *Young function* if it is convex and it satisfies

$$\frac{\Phi(x)}{x} \underset{x\to 0}{\longrightarrow} 0 \quad \text{and} \quad \frac{\Phi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty.$$

For any complex sequence $(x_n)_{n\geq 0}$, we set

$$M_\Phi(x) = \sum_{n=0}^{\infty} \Phi(|x_n|).$$

The *Orlicz space* L^Φ is the space of all sequences $(x_n)_{n\geq 0}$ such that there exists $k > 0$ such that $M_\Phi(kx) < \infty$.

Theorem 3.12 *Let Φ be a Young function, the Orlicz space L^ϕ with the norm*

$$\|(x_n)_{n\geq 0}\|_{L^\Phi} = \inf\left\{k;\, M_\Phi\left(\frac{1}{k}x\right) \leq 1\right\}$$

is a Banach space and if $\Phi(t) > 0$ for all $t > 0$, then

$$\|\chi_{\{0,\ldots,n-1\}}\|_{L^\Phi} \underset{n\to\infty}{\longrightarrow} \infty.$$

For the proof we use the following useful result.

Lemma 3.13 ([18]) *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be non-decreasing with $\varphi(t) > 0$ for all $t > 0$. Then, there exists a Young function Φ such that the Orlicz space L^Φ contains every sequence $(a_n)_{n\geq 0}$ such that*

$$\sum_{n\geq 0} \varphi(|a_n|) < \infty,$$

and satisfies

$$\|\chi_{\{0,\ldots,n-1\}}\|_{L^\Phi} \underset{n\to\infty}{\longrightarrow} \infty.$$

Proof Replacing φ by a multiple of φ, we may assume that $\varphi(1) = 1$. Let Φ be defined by

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s.$$

Then, Φ is a Young function and if $(a_n)_{n\geq 0}$ is such that

$$\sum_{n\geq 0} \varphi(|a_n|) < \infty,$$

then the set of indexes n such that $|a_n| > 1$ is finite and if $|a_n| \leq 1$, then $\Phi(|a_n|) \leq \varphi(|a_n|)$. Thus we get

$$\sum_{n\geq 0} \Phi(|a_n|) < \infty,$$

so $(a_n)_{n\geq 0} \in L^\Phi$.

But Φ is strictly positive and increasing on $(0; \infty)$, therefore, we obtain

$$\|\chi_{\{0,\dots,n-1\}}\|_{L^\Phi} = \inf\left\{k; \sum_{k=0}^{n-1} \Phi(1/k) \leq 1\right\} \underset{n\to\infty}{\longrightarrow} \infty.$$

□

As Orlicz spaces of sequences are admissible sequence spaces, we obtain the following result.

Corollary 3.14 *We suppose that $(|F_n(\zeta)|)_{n\geq 0}$ is eventually bounded below by a positive number for every $\zeta \in \partial K$. Let $T \in B(\mathcal{X})$ and let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-decreasing function with $\varphi(t) > 0$ for all $t > 0$. If for every $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$*

$$\sum_{n=0}^{\infty} \varphi\left(|\langle F_n(T)x; x^*\rangle|\right) < \infty,$$

then

$$\sigma(T) \subset int(K).$$

Remark 3.15 According to Remark 2.6, the above condition about the sequence $(|F_n(\zeta)|)_{n\geq 0}$ is automatically satisfied whenever ∂K is an Alper curve with (possible) angles.

Definition 3.16 The function φ is said to satisfy the Δ_2-condition at 0 if there exist $\varepsilon > 0$ and $K > 0$ such that for any $x \in [0; \varepsilon]$, we have $\varphi(t/2) \geq K\varphi(t)$.

Corollary 3.17 *We suppose that $(|F_n(\zeta)|)_{n\geq 0}$ is eventually bounded below by a positive number for every $\zeta \in \partial K$. Let $T \in B(\mathcal{X})$, let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-decreasing function satisfying the Δ_2-condition at 0 with $\varphi(t) > 0$ for $t > 0$, and let $(\omega_n)_{n\geq 0}$ be a sequence of non-negative numbers such that*

$$\sum_{n=0}^{\infty} \omega_n \varphi(\omega_n) = \infty.$$

If for every $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$,

$$\sum_{n=0}^{\infty} \varphi\left(\omega_n |\langle F_n(T)x; x^*\rangle|\right) < \infty,$$

then

$$\sigma(T) \subset int(K).$$

Proof Adapting the proof of Corollary 3.8, we may suppose ω_n to be positive for every n. Replacing if necessary ω_n by $\min(\omega_n, \varepsilon)$, we may also assume that $\omega_n \in (0; \varepsilon]$ for each n. Let Φ be as in Lemma 3.13 and let $\mathcal{E}$ be the space

$$\mathcal{E} = \left\{(x_n)_{n\geq 0}; (\omega_n x_n)_{n\geq 0} \in L^{\Phi}\right\},$$

with the norm

$$\|(x_n)_{n\geq 0}\|_{\mathcal{E}} = \|(\omega_n x_n)_{n\geq 0}\|_{L^{\Phi}}.$$

The Δ_2 condition gives that for any $t \in (0; \varepsilon]$,

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s \geq \frac{t}{2}\varphi(t/2) \geq \frac{Kt}{2}\varphi(t).$$

Let $k > 1$, and m be the integer part of $(\ln(k)/\ln(2) + 1)$. Then

$$\sum_{j=0}^{n-1} \Phi(\omega_j/k) \geq \frac{K^m}{2^m} \sum_{j=0}^{n-1} \omega_j \varphi(\omega_j).$$

This proves that

$$\|\chi_{\{0,\ldots,n-1\}}\|_{\mathcal{E}} \underset{n\to\infty}{\longrightarrow} \infty.$$

An application of Theorem 3.5 completes the proof. □

Corollary 3.18 *We suppose that $(|F_n(\zeta)|)_{n\geq 0}$ is eventually bounded below by a positive number for every $\zeta \in \partial K$. Let $T \in B(\mathcal{X})$. Suppose there exists a positive function φ on $[0; 1)$ such that*

$$\sqrt{1-r}\,\varphi(r) \underset{r\to 1}{\longrightarrow} 0$$

and such that

$$\frac{1}{\varphi(r)} \|(r^n \langle F_n(T)x; x^*\rangle)_{n\geq 0}\|_2 \underset{r\to 1}{\longrightarrow} 0,$$

for every $x \in \mathcal{X}$ and every $x^* \in \mathcal{X}^*$. Then

$$\sigma(T) \subset int(K).$$

Lemma 3.19 *Let φ be a positive function on $[0;1)$ such that $\varphi(r)$ tends to infinity and $\sqrt{1-r}\,\varphi(r)$ tends to 0 as r tends to 1. Then the space $\mathcal{E}$ defined by*

$$\mathcal{E} = \left\{(x_n)_{n\geq 0};\ \frac{1}{\varphi(r)}\|(r^n x_n)_{n\geq 0}\|_2 \underset{r\to 1}{\longrightarrow} 0\right\},$$

with the norm

$$\|(x_n)_{n\geq 0}\|_{\mathcal{E}} = \sup_{r\in(0;1)} \frac{1}{\varphi(r)}\|(r^n x_n)_{n\geq 0}\|_2,$$

is an admissible Banach function space over $\mathbb{N}$ such that

$$\|\chi_{\{0,\ldots,n\}}\|_{\mathcal{E}} \underset{n\to\infty}{\longrightarrow} \infty.$$

Proof It is easy to see that $\mathcal{E}$ is an admissible sequence space. Furthermore, we have

$$\|\chi_{\{0,\ldots,n-1\}}\|_{\mathcal{E}} = \sup_{r\in(0;1)} \frac{1}{\varphi(r)}\|(1, r, r^2, \ldots, r^{n-1}, 0, 0, \ldots)\|_2,$$

and

$$\|(1, r, r^2, \ldots, r^{n-1}, 0, 0, \ldots)\|_2^2 = \sum_{k=0}^{n-1} r^{2k} = \frac{1-r^{2n}}{1-r^2}.$$

Therefore,

$$\begin{aligned}\frac{1}{\varphi(r)}\|(1, r, r^2, \ldots, r^{n-1}, 0, 0, \ldots)\|_2 &= \frac{1}{\varphi(r)}\frac{\sqrt{1-r^n}}{\sqrt{1-r}}\frac{\sqrt{1+r^n}}{\sqrt{1+r}}\\ &\geq \frac{1}{2}\frac{\sqrt{1-r^n}}{\varphi(r)\sqrt{1-r}}.\end{aligned}$$

Let $M > 0$. As $\sqrt{1-r}\varphi(r)$ tends to 0 as r tends to 1, there exists $r_0 < 1$ such that

$$\frac{1}{\varphi(r_0)\sqrt{1-r_0}} > 4M.$$

Let N be such that $\sqrt{1-r_0^N} > 1/2$. We get that for each $n \geq N$, we have

$$\|\chi_{\{0,\ldots,n-1\}}\|_{\mathcal{E}} \geq M.$$

The proof is complete. □

Proof of Corollary 3.18 Replacing, if necessary, $\varphi(r)$ by $\max(\varphi(r); (1-r)^{-1/4})$, we may assume that $\varphi(r) \underset{r\to 1}{\longrightarrow} \infty$. We can then apply Lemma 3.19 and Theorem 3.5 to obtain the inclusion $\sigma(T) \subset int(K)$. □

4 Condition on the Weak Resolvent

The purpose of this section is, as in the previous one, to find conditions assuring spectral inclusion in the interior of a given simply connected compact set K. We suppose here that the spectrum of the bounded linear operator under consideration is included in the compact set K. Nikolski [13] called conditions of this type Tauberian conditions for the spectral radius. The main reason for interest in this approach is that it is well adapted to the conditions expressed in terms of the weak resolvent, that is to say on the functions

$$z \notin K \mapsto \langle (zI - T)^{-1}x; x^*\rangle,$$

where $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$.

We recall that $(F_n)_{n\geq 0}$ denotes the sequence of Faber polynomials associated with K and $(e_n)_{n\geq 0}$ denotes the canonical algebraic basis of $\mathbb{C}^{(\mathbb{N})}$.

Let $\mathcal{R} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space such that

(1) for any $k \in \mathbb{N}$, the map $(a_n)_{n\geq 0} \mapsto a_k$ is bounded;
(2) the set of finitely supported sequences $\mathbb{C}^{(\mathbb{N})}$ is dense in $\mathcal{R}$.

As $\mathbb{C}^{(\mathbb{N})}$ is dense in $\mathcal{R}$, each element φ in $\mathcal{R}^*$ is characterized by the sequence $(\varphi_n)_{n\geq 0}$, where

$$\varphi_n = \langle e_n; \varphi\rangle_{\mathcal{R}}.$$

This allows us to identify $\mathcal{R}^*$ with a subset of $\mathbb{C}^{\mathbb{N}}$. Via this identification, we can translate the condition (1) as " $\mathbb{C}^{(\mathbb{N})}$ is included in $\mathcal{R}^*$ ".

Proposition 4.1 *If $\varphi \in \mathcal{R}^*$ and $f \in \mathbb{C}^{(\mathbb{N})}$, the duality mapping is given by*

$$\langle f; \varphi\rangle_{\mathcal{R}} = \sum_{n=0}^{\infty} f_n \varphi_n.$$

If $f \in \mathcal{R}$ and $\varphi \in \mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$, this equality still holds.

Proof Let $\varphi \in \mathcal{R}^*$ be finitely supported and let n_0 be such that $\langle e_n; \varphi\rangle = 0$ whenever $n \geq n_0$. From (2), we know that there exists a sequence $(g_k)_{k\geq 0}$ of elements in $\mathbb{C}^{(\mathbb{N})}$ such that

$$g_k \underset{k\to\infty}{\longrightarrow} f \quad \text{in} \quad \mathcal{R}.$$

To prevent any confusion, we will denote by $[g_k]_n$ the nth element of the sequence g_k. Hence,

$$\langle f;\varphi\rangle_{\mathcal{R}} = \lim_{k\to\infty}\langle g_k;\varphi\rangle_{\mathcal{R}}.$$

We also have $\langle g_k;\varphi\rangle_{\mathcal{R}} = \sum_{n=0}^{n_0}[g_k]_n\varphi_n$. Therefore, using the hypothesis (1), we get that $[g_k]_n \underset{k\to\infty}{\longrightarrow} f_n$, for every n.

We obtain

$$\langle f;\varphi\rangle_{\mathcal{R}} = \lim_{k\to\infty}\langle g_k;\varphi\rangle_{\mathcal{R}} = \sum_{n=0}^{\infty} f_n\varphi_n.$$

□

The crucial condition needed for our Tauberian conditions is the following one.

(3) For any $\zeta \in \partial K$, the map V_ζ, defined in $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$ by

$$V_\zeta : \varphi \mapsto \sum_{n\geq 0}\varphi_n F_n(\zeta)$$

is not bounded.

Theorem 4.2 *Let $\mathcal{R} \subset \mathbb{C}^{\mathbb{N}}$ be a Banach space satisfying conditions (1) to (3) for K. Let $T \in B(\mathcal{X})$ be a weak type $\mathcal{R}$ operator with $\sigma(T) \subset K$. Then, in fact,*

$$\sigma(T) \subset int(K).$$

Proof Lemma 3.6 implies that

$$C := \sup\left\{\left\|\left(\langle F_n(T)x;x^*\rangle\right)_{n\geq 0}\right\|_{\mathcal{R}};\ \|x^*\|\leq 1,\ \|x\|\leq 1\right\} < \infty.$$

Using Proposition 4.1 we get, for every $\varphi \in \mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$,

$$\begin{aligned}\left\|\sum_{n\geq 0}\varphi_n F_n(T)\right\| &= \sup\left\{\left|\sum_{n\geq 0}\varphi_n\langle F_n(T)x;x^*\rangle\right|;\ \|x\|\leq 1,\ \|x^*\|\leq 1\right\}\\ &\leq \sup\left\{\left\|\left(\langle F_n(T)x;x^*\rangle\right)_{n\geq 0}\right\|_{\mathcal{R}};\ \|x\|\leq 1,\ \|x^*\|\leq 1\right\}\|\varphi\|_{\mathcal{R}^*}\\ &\leq C\|\varphi\|_{\mathcal{R}^*}.\end{aligned}$$

Recall that, according to the spectral mapping theorem [5], we have $\sigma(P(A)) = P(\sigma(A))$ for every polynomial P. Hence, if $\zeta \in \sigma(T)$, we have

$$\left|\sum_{n\geq 0}\varphi_n F_n(\zeta)\right| \leq r\left(\sum_{n\geq 0}\varphi_n F_n(T)\right) \leq \left\|\sum_{n\geq 0}\varphi_n F_n(T)\right\| \leq C\|\varphi\|_{\mathcal{R}^*},$$

where $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$ is the spectral radius of T. We deduce from this inequality that for any $\zeta \in \sigma(T)$, the map V_ζ is bounded in $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$.

This gives us

$$\partial K \cap \sigma(T) = \varnothing.$$

□

Remark 4.3 A weaker version of Corollary 3.8 can be deduced from Theorem 4.2 (one has to suppose that $\sigma(T) \subset K$). Note that Corollary 3.9 cannot be deduced from Theorem 4.2.

The main reason for interest in the approach of this section is that we can get a generalisation of the following result due to Mlak [11].

Theorem 4.4 ([11]) *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Suppose that $r(T) \le 1$ and that for any $(x, y) \in \mathcal{H} \times \mathcal{H}$,*

$$z \mapsto \langle (zI - T)^{-1}x; y\rangle \in H^1(\mathbb{D}).$$

Then $r(T) < 1$.

To extend Mlak's result, we need a generalisation of Hardy spaces of the disc to more general domains. We follow the setting of [6].

Definition 4.5 ([6]) For $p \in [1; \infty)$, a function f analytic in $\mathbb{C}\backslash K$ is said to be of class $E^p(\mathbb{C}\backslash K)$ if there exists a sequence of rectifiable Jordan curves $C_1, C_2, \ldots$ in $\mathbb{C}\backslash K$ such that C_n eventually surrounds each compact subset of $\mathbb{C}\backslash K$, and such that

$$\sup_n \int_{C_n} |f(z)|^p |\mathrm{d}z| < \infty.$$

The following characterization ([6, Theorem 10.1]) of the functions in $E^p(\mathbb{C}\backslash K)$ is well adapted here as it uses the conformal representation ϕ.

Theorem 4.6 ([6]) *An analytic function f is of class $E^p(\mathbb{C}\backslash K)$ if and only if,*

$$\sup_{r\in(1;2)} \int_{|\phi(z)|=r} |f(z)|^p |\mathrm{d}z| < \infty.$$

We can now state the generalisation of Mlak's result [11] as a corollary to Theorem 4.2.

Corollary 4.7 *Suppose that $\lim_n |F_n(\zeta)| = 0$ for no $\zeta \in \partial K$. Let $T \in B(\mathcal{X})$ be such that $\sigma(T) \subset K$.*

If for any $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$, we have

$$z \mapsto \langle (zI - T)^{-1}x; x^*\rangle \in E^1(\mathbb{C}\backslash K),$$

then $\sigma(T) \subset int(K)$.

Remark 4.8 According to Remark 2.6, if ∂K is an Alper curve with (possible) angles, the condition on the sequence $(|F_n(\zeta)|)_{n\geq 0}$ is automatically satisfied.

Lemma 4.9 *Suppose that* $\lim_n |F_n(\zeta)| = 0$ *for no* $\zeta \in \partial K$. *Then the space*

$$h_p := \left\{ (x_n)_{n\geq 0};\ f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(\mathbb{D}) \right\},$$

with the norm

$$\|(x_n)_{n\geq 0}\| = \left\| \sum_{n=0}^{\infty} x_n z^n \right\|_{H^p(\mathbb{D})}$$

satisfies the conditions (1) to (3).

Proof It is easy to check that the conditions (1) and (2) are satisfied. Let us prove that the condition (3) is also satisfied.

Suppose on the contrary that condition (3) does not hold. Then, there exists $\zeta \in \partial K$ such that V_ζ is bounded in $\mathbb{C}^{(\mathbb{N})} \subset \mathcal{R}^*$. Let $a_n = F_n(\zeta)$ for every n.

According to the Hahn–Banach theorem, there exists a bounded extension of V_ζ to $(h^p)^*$, which we still denote as V_ζ. Hence, there exists $M \geq 0$ such that for every $x^* = (x_n^*)_{n\geq 0} \in (h^p)^*$,

$$|V_\zeta\left(x^*\right)| \leq M\|(x^*)\|_{(h^p)^*}.$$

But $\|f(r.)\|_{H^p} \leq \|f\|_{H^p}$ for every $r \in (0;1)$ and every $f \in H^p$. Thus, for every $(x_n^*)_{n\geq 0} \in (h^p)^*$,

$$\begin{aligned}
\|(r^n x_n^*)_{n\geq 0}\|_{(h^p)^*} &= \sup\left\{\left|\left\langle (r^n x_n^*)_{n\geq 0}; (x_n)_{n\geq 0}\right\rangle\right|;\ \|(x_n)_{n\geq 0}\|_{h^p} \leq 1\right\} \\
&= \sup\left\{\left|\left\langle (x_n^*)_{n\geq 0}; (r^n x_n)_{n\geq 0}\right\rangle\right|;\ \|(x_n)_{n\geq 0}\|_{h^p} \leq 1\right\} \\
&\leq \|(x_n^*)_{n\geq 0}\|_{(h^p)^*}.
\end{aligned}$$

So, for every $r \in (0;1)$ and for every $(x_n^*)_{n\geq 0} \in (h^p)^*$,

$$\left|\sum_{n=0}^{\infty} x_n^* r^n a_n\right| \leq M\|(x_n^*)_{n\geq 0}\|_{(h^p)^*}.$$

Consequently, $\|(r^n a_n)_{n\geq 0}\|_{h^p} \leq M$ for each $r \in (0;1)$, and thus, $(a_n)_{n\geq 0}$ is in h^p. We have $f(e^{i\theta}) = \sum a_n e^{in\theta} \in H^p(\mathbb{T})$ and $a_n = F_n(\zeta) \underset{n\to\infty}{\longrightarrow} 0$. This contradiction completes the proof. □

Proof of Corollary 4.7 We get from Lemma 4.9 and Theorem 4.2 that it is sufficient to prove that for any $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$, we have

$$(\langle F_n(T)x; x^*\rangle)_{n\geq 0} \in h^1.$$

From Lemma 2.9, we get that for any $r > 1$,

$$\int_{|\phi(z)|=r} |\langle (zI - T)^{-1}x; x^*\rangle||\mathrm{d}z| = \int_{z\in r\mathbb{T}} |\langle (\psi(z)I - T)^{-1}x; x^*\rangle||\psi'(z)||\mathrm{d}z|$$
$$= \int_{z\in r\mathbb{T}} \left|\sum_{n=0}^{\infty} z^{-n-1}\langle F_n(T)x; x^*\rangle\right| |\mathrm{d}z|$$
$$= \int_{z\in r^{-1}\mathbb{T}} \left|\sum_{n=0}^{\infty} z^{n}\langle F_n(T)x; x^*\rangle\right| r|\mathrm{d}z|.$$

This proves that

$$z \mapsto \sum_{n=0}^{\infty} z^n \langle F_n(T)x; x^*\rangle \in H^1(\mathbb{D}).$$

□

Remark 4.10 The space h^1 is not an admissible sequence space, thus Corollary 4.7 cannot be deduced from Theorem 3.5.

5 Explicit Estimates

5.1 General Case

We suppose in this section that Γ is an analytic Jordan curve. Let $\mathcal{E}$ be a Banach function space over $\mathbb{N}$ containing $(r^n)_{n\geq 0}$ for any $r \in (0; 1)$ and satisfying the condition

$$\|\chi_{\{0,\ldots,n-1\}}\|_{\mathcal{E}} \underset{n\to\infty}{\longrightarrow} \infty.$$

Let $T \in B(\mathcal{X})$ be a weak type $\mathcal{E}$ operator.

As we have supposed Γ to be analytic, we get from Theorem 2.7 that the map $\psi : \overline{\mathbb{D}}^c \longrightarrow K^c$ has an analytic extension to $\overline{r_0\mathbb{D}}^c$ for some $r_0 < 1$.

Notation For any $r \in (r_0; 1)$, let Γ_r be the analytic Jordan curve defined by

$$\Gamma_r = \{\psi(z); |z| = r\}$$

and let K_r be the simply connected compact set delimited by Γ_r.

We know from Theorem 3.5 that the spectrum of T is included in $int(K)$. Considering the possible applications in numerical analysis, it is the aim of this section to give an estimate of the "shrinking radius" $r \in (r_0; 1)$ such $\sigma(T) \subset K_r$. In the case of the disc, this was done by N. Nikolski in [14]. Our estimate of the “shrinking radius” is given in terms of the constant $C(T, \mathcal{E})$ defined by

$$C(T,\mathcal{E}) = \sup\left\{\left\|\left(\langle F_n(T)x; x^*\rangle\right)_{n\geq 0}\right\|_{\mathcal{E}}; \|x^*\| \leq 1, \|x\| \leq 1\right\}.$$

Notation For $r \in (r_0; 1)$ we denote

$$p(r) = \inf_{r\leq|\phi(\zeta)|<1} \|(F_n(\zeta))_{n\geq 0}\|_{\mathcal{E}}$$

and, for $M > 0$

$$p^{-1}(M) = \begin{cases} r_0 & \text{if } \{r \in (r_0; 1); p(r) \leq M\} = \varnothing, \\ \sup\{r \in (r_0; 1); p(r) \leq M\} & \text{otherwise.} \end{cases}$$

Theorem 5.1 *Let T be a weak type $\mathcal{E}$ operator for the compact set K and set $r = p^{-1}(C(T, \mathcal{E}))$. Then*

$$\sigma(T) \subset K_r.$$

Proof We argue by contradiction. Suppose that $\sigma(T) \not\subseteq K_r$; then there exists $\lambda \in \partial\sigma(T)\backslash K_r$. From Lemma 3.7, we get that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exist $x_N \in \mathcal{X}$ and $x_N^* \in \mathcal{X}^*$ satisfying $\|x_N\| = 1$, $\|x_N^*\| = 1$ and

$$|\langle x_N^*; F_k(T)x_N\rangle| > |F_k(\lambda)| - \varepsilon, \quad k \in \{0, 1, \dots, N\}.$$

By definition of $C(T, \mathcal{E})$, we have

$$C(T,\mathcal{E}) \geq \left\|\left(\langle F_n(T)x_N; x_N^*\rangle\right)_{n\geq 0}\right\|_{\mathcal{E}} \geq \|(F_0(\zeta), \dots, F_N(\zeta), 0, 0, \dots) - \varepsilon\chi_{\{0,\dots,N\}}\|_{\mathcal{E}}.$$

Thus, for every $N \in \mathbb{N}$,

$$\|(F_0(\lambda), F_1(\lambda), \dots, F_N(\lambda), 0, 0, \dots)\|_{\mathcal{E}} \leq C(T, \mathcal{E}).$$

But $F_n(\lambda) \sim \phi(\lambda)^n$, $n \to \infty$, and $|\phi(\lambda)| < 1$. Therefore $(F_n(\lambda))_{n\geq 0} \in \mathcal{E}$ and

$$\left\|(F_n(\lambda))_{n\geq 0}\right\|_{\mathcal{E}} \leq C(T, \mathcal{E}).$$

Using $\lambda \notin K_r$, and taking into account the definition of r as $r = p^{-1}(C(T, \mathcal{E}))$, we have

$$\left\|(F_n(\lambda))_{n\geq 0}\right\|_{\mathcal{E}} > C(T, \mathcal{E}).$$

This contradiction completes the proof. □

5.2 Concrete Examples

In case of some concrete compacts subsets, the previously described quantities can be explicitly estimated. We begin with the case of a compact set K such that K_{r_0} is convex. The following proof has been suggested to the author by a referee.

Corollary 5.2 *Suppose* $r < 1$ *and* $1 \leq p < \infty$. *Let* T *be a weak type* l^p *operator for a compact set* K *delimited by an analytic Jordan curve. Suppose that* K_{r_0} *is convex and*

$$C(T, l^p) \leq \left(\frac{1}{1-r^p}\right)^{1/p} - \left(\frac{1}{1-r_0^p}\right)^{1/p}.$$

Then

$$\sigma(T) \subset K_r.$$

Proof One can get, from the inequality of Kövari and Pommerenke [10] (cf. Theorem 2.8) and a change of variable, that for any $z \in \Gamma_{r_0}$ and any $n \in \mathbb{N}$, we have

$$|F_n(z) - \phi(z)^n| \leq r_0^n.$$

As $F_n - \phi^n$ is an analytic function on $\overline{\mathbb{C}} \backslash K_0$ and vanishes at ∞, the maximum principle gives us that for any $z \notin K_{r_0}$,

$$|F_n(z) - \phi(z)^n| \leq r_0^n.$$

Let $w \in \mathbb{C}$ be such that $|w| > r_0$. Then

$$\begin{aligned}\left\|(F_n(\psi(w)))_{n\geq 0}\right\|_{l^p} &= \left\|\left(w^n + (F_n(\psi(w)) - w^n)\right)_{n\geq 0}\right\|_{l^p} \\ &\geq \|(w^n)_{n\geq 0}\|_{l^p} - \left\|\left(r_0^n\right)_{n\geq 0}\right\|_{l^p}.\end{aligned}$$

Thus, for every $t \in (r_0; 1)$, we have

$$\begin{aligned}p(t) &\geq \left(\sum_{n=0}^{\infty} t^{np}\right)^{1/p} - \left(\sum_{n=0}^{\infty} (r_0)^{np}\right)^{1/p} \\ &\geq \left(\frac{1}{1-t^p}\right)^{1/p} - \left(\frac{1}{1-(r_0)^p}\right)^{1/p}.\end{aligned}$$

The conclusion follows now from Theorem 5.1. □

We now turn to the more specific case of an ellipse. Let K be the compact set delimited by the ellipse with the foci -1 and 1, and semi-axes

$$a = \frac{1}{2}\left(R + \frac{1}{R}\right) \quad \text{and} \quad b = \frac{1}{2}\left(R - \frac{1}{R}\right),$$

with $R > 1$. Then (cf. Sect. 2) the function ψ is given by

$$\psi(w) = \frac{1}{2}\left(Rw + \frac{1}{Rw}\right)$$

and the sequence of Faber polynomials is given by

$$F_n(z) = \frac{2}{R^n} C_n(z), \quad n \geq 1,$$

where $(C_n)_{n\in\mathbb{N}}$ is the sequence of Chebychev polynomials. We also have

$$F_n(\psi(w)) = w^n + \frac{1}{R^{2n}w^n} \quad (n \geq 1).$$

Corollary 5.3 *Let K be the elliptic compact set described above. Suppose $r < 1$ and $1 \leq p < \infty$. Let $T \in B(\mathcal{X})$ be a weak type l^p operator for K. Suppose that*

$$C(T, l^p) \leq \left(\frac{1}{1 - r^p}\right)^{1/p} - \left(\frac{1}{1 - \left(\frac{1}{rR^2}\right)^p}\right)^{1/p}.$$

Then

$$\sigma(T) \subset K_r.$$

Remark 5.4 This estimate is better than the one given in Corollary 5.2 as $r_0 = 1/R > 1/rR^2$ for $r > r_0$.

Proof It is easy to verify that $r_0 = 1/R$. Let $w \in \mathbb{C}$ be such that $|w| > 1/R$. Then

$$\begin{aligned}\left\|(F_n(\psi(w)))_{n\geq 0}\right\|_{l^p} &= \left\|\left(w^n + \frac{1}{R^{2n}w^n}\right)_{n\geq 0}\right\|_{l^p} \\ &\geq \|(w^n)_{n\geq 0}\|_{l^p} - \left\|\left(\frac{1}{R^{2n}w^n}\right)_{n\geq 0}\right\|_{l^p}.\end{aligned}$$

Thus, for every $t \in (1/R; 1)$,

$$p(t) \geq \left(\sum_{n=0}^{\infty} t^{np}\right)^{1/p} - \left(\sum_{n=0}^{\infty} \left(\frac{1}{tR^2}\right)^{np}\right)^{1/p}$$

$$\geq \left(\frac{1}{1-t^p}\right)^{1/p} - \left(\frac{1}{1-\left(\frac{1}{tR^2}\right)^p}\right)^{1/p}.$$

The conclusion follows now from Theorem 5.1. □

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