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Journal of Functional Analysis 215 (2004) 341–365
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

On the ideal structure of some Banach algebras related to convolution operators on $L^p(G)$

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Received 9 September 2003; accepted 2 February 2004

Communicated by D. Sarason

Abstract

Let G be a locally compact group and let $p \in (1, \infty)$. Let $\mathscr{A}$ be any of the Banach spaces $C_{\delta,p}(G)$, $PF_p(G)$, $M_p(G)$, $AP_p(G)$, $WAP_p(G)$, $UC_p(G)$, $PM_p(G)$, of convolution operators on $L^p(G)$. It is shown that $PF_p(G)'$ can be isometrically embedded into $UC_p(G)'$. The structure of maximal regular ideals of $\mathscr{A}'$ (and of $MA_p(G)''$, $B_p(G)''$, $W_p(G)''$) is studied. Among other things it is shown that every maximal regular left (right, two sided) ideal in $\mathscr{A}'$ is either weak* dense or is the annihilator of a unique element in the spectrum of $A_p(G)$. Minimal ideals of $\mathscr{A}'$ is also studied. It is shown that a left ideal M in $\mathscr{A}'$ is minimal if and only if $M = \mathbf{C}\Psi$, where Ψ is either a right annihilator of $\mathscr{A}'$ or is a topologically x-invariant element (for some $x \in G$). Some results on minimal right ideals are also given.

Keywords: Generalized Fourier algebras; p-Convolution operators; Maximal regular ideals; Minimal ideals; Amenability

1. Introduction

Let G be a locally compact group and $p \in (1, \infty)$. Let

$$\lambda_p : L^1(G) \to \mathscr{L}(L^p(G)), \quad \lambda_p(f)(g) = f * g (f \in L^1(G), g \in L^p(G))$$

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doi:10.1016/j.jfa.2004.02.011

be the left regular representation of $L^1(G)$ on $L^p(G)$. As was shown by Dunkl and Ramirez [13], and Granirer [22], one may define various Banach spaces of convolution operators in $\mathscr{L}(L^p(G))$ including $PF_p(G)$, $M_p(G)$, $AP_p(G)$, $WAP_p(G)$, $UC_p(G)$, and $PM_p(G)$ (for the definitions of unexplained terms and terminologies in this introduction we refer the reader to Section 2). In the case of $p = 2$ and G abelian if $\hat{G}$ is the dual group of G, these spaces can be identified, via the transpose of the Fourier transform, to the usual Banach algebra of functions $\mathscr{C}_0(\hat{G})$ (continuous functions vanishing at infinity on the dual group $\hat{G}$), $\overline{B(\hat{G})}^{\|\cdot\|_\infty}$ (the norm closure of the Fourier–Stieltjes algebra in $L^\infty(\hat{G})$), $AP(\hat{G})$ (almost periodic functions), $WAP(\hat{G})$ (weakly almost periodic functions), $UC(\hat{G})$ (uniformly continuous functions), and $L^\infty(\hat{G})$. The space $PM_p(G)$ is the dual of the Banach algebra $A_p(G)$—the generalized Fourier algebra of G. Each of the above spaces of operators is a topologically introverted subspace of $PM_p(G) = A_p(G)'$, and as such, their dual space is a Banach algebra with the induced Arens product from $PM_p(G)' = A_p(G)''$. We also consider the closely related Banach algebras $MA_p(G)''$, $B_p(G)''$, and $W_p(G)''$. The purpose of this paper is to study the maximal regular ideals and the minimal ideals of these algebras in a unified manner. Our studies have been inspired by earlier works of Lau [29,30], Filali [17,18], Lau and Losert [31], Delaporte and Derighetti [10], Ghahramani and Lau [20], and Baker and Filali [3].

We start in the next section with some preliminaries and notations that will be used throughout the rest of the paper. In Section 3 we prove the existence of an isometric embedding E of the Banach algebra $W_p(G) = PF_p(G)'$ into $UC_p(G)'$ (Theorem 3.2). This allows us to obtain an embedding $v: W_p(G) \to X'$ into the dual of any topologically introverted subspace X of PM_p such that $PF_p(G) \subset X \subset UC_p(G)$. In addition to $PF_p(G)$ and $UC_p(G)$ other examples of such subspaces include $M_p(G)$, $WAP_p \cap UC_p$, $\overline{AP_p(G) + PF_p(G)}^{\|\cdot\|_{\mathrm{op}}}$ ($\|\cdot\|_{\mathrm{op}}$ is the operator norm on $\mathscr{L}(L^p(G))$), $\overline{AP_p(G) + M_p(G)}^{\|\cdot\|_{\mathrm{op}}}$, or $WAP_p(G)$ with G amenable in the last three cases. As an application we prove that X' can be written as the topological direct sum decomposition $X' = v(W_p(G)) \oplus PF_p(G)^\perp$. Such a decomposition was proved in [31, Proposition 4.2, p. 10] for $UC_2(G)$. It was proved earlier in [21, Lemma 1.1, p. 275] for $LUC(G)'$, and with a different method in [18, Lemma 2.2, p. 292] for the case of $LUC(G)'$ and $WAP(G)'$. Our result, interesting in its own right, is used in Section 4 to construct examples of weak* dense maximal regular ideals in X' which does not contain $A_p(G)$.

Let $\mathscr{A}$ be any of the Banach spaces $C_{\delta,p}(G)$, $PF_p(G)$, $M_p(G)$, $AP_p(G)$, $WAP_p(G)$, $UC_p(G)$, or $PM_p(G)$, and let $\mathscr{B}$ be any of the Banach algebras $MA_p(G)$, $B_p(G)$, or $W_p(G)$ (for the definitions see the next section). In Section 4, we extend some of the results on maximal regular ideals proved by Lau in [30] and by Ghahramani and Lau [20]. We show that if M is a maximal regular left (right, or two sided) ideal in $\mathscr{A}'$ (or in $\mathscr{B}''$) then either $M \cap A_p(G) = A_p(G)$, or $M \cap A_p(G) = I_x = \{u \in A_p(G)\colon u(x) = 0\}$ for a unique $x \in G$ (Corollary 4.6). In Theorem 4.8 we show that every maximal regular left (right, or two sided) ideal of $\mathscr{A}'$ (respectively, of $\mathscr{B}''$) is either weak* dense in $\mathscr{A}'$ (respectively, in $\mathscr{B}''$) or is of the form $M_x = \{\Psi \in \mathscr{A}' : \langle \Psi, \lambda_p(x) \rangle = 0\}$ for some $x \in G$ (respectively, $M_\phi = \{\Psi \in \mathscr{B}'' : \langle \Psi, \phi \rangle = 0\}$ for some non-zero continuous complex homomorphism ϕ on $\mathscr{B}$. The embedding of the previous section is used to construct examples of maximal regular ideals which are weak* dense in $\mathscr{A}'$; when G is not discrete such ideals can be chosen not to contain $A_p(G)$.

Section 5 is devoted to study the minimal ideals in $\mathscr{A}'$. In Theorem 5.8 we prove that a left ideal M in $\mathscr{A}'$ is minimal if and only if $M = \mathbf{C}\Psi$, where $\mathbf{C}$ is the field of complex numbers and Ψ is either a non-zero right annihilator of $\mathscr{A}'$ or a topologically x-invariant element for some $x \in G$; our result extends the main theorem proved in [17] to large classes of p-convolution operators.

We show that minimal ideals M exist in any of the Banach algebras $A_p(G)$, $W_p(G)$, $B_p(G)$, or $MA_p(G)$ if and only if G is discrete and $M = \mathbf{C}\chi_x$ for some $x \in G$, where χ_x is the characteristic function of the singleton $\{x\}$ (Corollary 5.6). Theorems 5.4 and 5.5 determine all the minimal idempotents in $\mathscr{A}'$, and all the minimal idempotents in the Banach algebras $A_p(G)$, $W_p(G)$, $B_p(G)$, and $MA_p(G)$. It should be remarked that neither the minimal left ideals nor the minimal idempotents are all known in any of the algebras $WAP(G)'$, $LUC(G)'$, or $L^1(G)''$ in case that G is not abelian and not compact [3].

2. Preliminaries and notations

All groups considered in this paper are locally compact, Hausdorff, topological groups (for short, locally compact groups) equipped with a Haar measure dx. If G is such a group, the spaces $L^p(G, dx)$ are simply denoted by $L^p(G)$. If f is a function on G and $a \in G$, we define ${}_af(x) = f(ax)$, $f_a(x) = f(xa)$, $\check{f}(x) = f(x^{-1})$, $f^*(x) = \overline{f(x^{-1})}/\Delta(x)$, for all $x \in G$. The support of a function f on G is defined as $Supp f = \overline{\{x \in G : f(x) \neq 0\}}$. If f and g are two functions on G, their convolution is defined by $f * g(x) = \int_G f(y)g(y^{-1}x)\,dx$ (wherever the integral is convergent). When G is abelian, $\hat{G}$ is the dual group of G consisting of all the continuous characters of G.

For $p \in (1, \infty)$ let $\mathscr{L}(L^p(G))$ be the space of all continuous linear operators on $L^p(G)$ (equipped with the usual operator norm $\|\cdot\|_{\mathrm{op}}$), and let $\lambda_p : M^1(G) \to \mathscr{L}(L^p(G))$ be the left regular representation of the measure algebra $M^1(G)$ on $\mathscr{L}(L^p(G))$; in other words, $\lambda_p(\mu)(g) = \mu * g$, where $\mu \in M^1(G), g \in L^p(G)$, and $\mu * g(x) = \int_G g(y^{-1}x)\,d\mu(y)$. Since there will be no risk of confusion, we let λ_p also represent the left regular representation of G on $\mathscr{L}(L^p(G))$, that is, $(\lambda_p(a)g)(x) = g(a^{-1}x)$, for every $a, x \in G$, $g \in L^p(G)$.

If B is a Banach algebra, we denote the dual of B by B', and we let $a \mapsto \tilde{a}$, $B \to B''$, to be the canonical embedding of B into its double dual. If $T \in B'$ and $a \in B$, the action of T on a is denoted by $T(a)$ or $\langle T, a \rangle$. Both B' and B'' are Banach left (and right) B-modules as follows: suppose $a, b \in B$, $T \in B'$, and $\Psi \in B''$, then by definition $\langle a \cdot T, b \rangle = \langle T, ba \rangle$, and $\langle T \cdot a, b \rangle = \langle T, ab \rangle$. Similarly the modules actions on

B'' are given by $\langle a \cdot \Psi, T \rangle = \langle \Psi, T \cdot a \rangle$, and $\langle \Psi \cdot a, T \rangle = \langle \Psi, a \cdot T \rangle$. When B is commutative, the left and right module actions coincide.

The spectrum (or the space of characters) of a commutative normed algebra B is denoted by $\mathbf{X}(B)$, this is the set of all non-zero, continuous, homomorphisms from B into the complex numbers $\mathbf{C}$.

Let $PM_p(G) = \overline{\{\lambda_p(\mu) \colon \mu \in M^1(G)\}}^{\text{weak}^*}$, where the closure is with respect to the weak* topology $\sigma(\mathscr{L}(L^p(G)), L^p(G)\widehat{\otimes}L^{p'}(G))$ $(1/p + 1/p' = 1)$. The space $PM_p(G)$, called the space of p-pseudo-measures on G, is a Banach algebra with the usual operations of addition and product of operators in $\mathscr{L}(L^p(G))$. For $p = 2$, $PM_2(G) = VN(G)$ is the group von Neumann algebra of G. When G is abelian, the transpose of the Fourier transform is an isometric isomorphism between $VN(G)$ and $L^\infty(\hat{G})$. It is well known that $PM_p(G)$ can be identified with the dual of the generalized Fourier algebra $A_p(G)$ defined as the set of all functions $u \in \mathscr{C}_0(G)$, such that u has a series expansion $u = \sum_{i=1}^\infty g_i * \check{f_i}$ $(f_i \in L^p(G),\ g_i \in L^{p'}(G))$, with the property that $\sum_{i=1}^\infty ||f_i||_p\, ||g_i||_{p'} < \infty$ [25,26]. The norm of this algebra is defined as

$$||u||_{A_p} = \inf\left\{ \sum_{i=1}^\infty ||f_i||_p\, ||g_i||_{p'} \colon u = \sum_{i=1}^\infty g_i * \check{f_i},\ f_i \in L^p(G),\ g_i \in L^{p'}(G) \right\}.$$

When $\mu \in M^1(G)$, the dual action of $\lambda_p(\mu)$ on $A_p(G)$ is defined by $\langle \lambda_p(\mu), u \rangle = \int_G u(x)\, d\mu(x)$ $(u \in A_p(G))$. In particular $\langle \lambda_p(x), u \rangle = u(x)$ $(x \in G)$. If $T \in PM_p(G)$ we let *Supp* T to be the support of T defined as the set of all $x \in G$ such that for every neighborhood U of x there exists a function $u \in A_p(G)$ with *Supp* $u \subset U$ and $\langle T, u \rangle \neq 0$. With the usual operations of pointwise addition and multiplication, $A_p(G)$ is a commutative regular Banach algebra, and its subspace of elements with compact support is dense in $A_p(G)$ [26, Proposition 3, p. 101]. The map $G \to \mathbf{X}(A_p(G))$, $x \mapsto \varepsilon_x$ (where $\langle \varepsilon_x, u \rangle = u(x)$ for all $u \in A_p(G)$) is an isomorphism between the group G and the spectrum of $A_p(G)$ [26, Theorem 3, p. 102]. When $p = 2$, $A_2(G)$ coincides with the Fourier algebra $A(G)$, introduced by Eymard [14]. In case that G is abelian, the Fourier transform is an isometric algebra isomorphism from $L^1(\hat{G})$ onto $A_2(G)$. Convenient references for the elementary properties of $A_p(G)$ algebras are [26] and Eymard's Bourbaki seminar [15].

Let $MA_p(G)$ be the multiplier algebra of $A_p(G)$, that is, the set of all continuous functions v on G such that $vu \in A_p(G)$, for all $u \in A_p(G)$. With the multiplier norm $||v||_{MA_p} = \inf\{||vu||_{A_p} \colon u \in A_p(G), ||u||_{A_p} \leqslant 1\}$, $MA_p(G)$ is a Banach algebra containing $A_p(G)$. There is a natural $MA_p(G)$-module action on $PM_p(G)$ defined by $\langle v \cdot T, u \rangle = \langle T, uv \rangle (u \in A_p(G))$. If $v \in MA_p(G)$ and $\mu \in M^1(G)$ then $v \cdot \lambda_p(\mu) = \lambda_p(v \cdot \mu)$ (where $v \cdot \mu$ is the usual product of a continuous bounded function with a bounded measure).

Following [13,22], we define the following closed subspaces of $PM_p(G)$ (on $PM_p(G)$ the operator norm $||\cdot||_{\mathrm{op}}$ and the dual norm $||\cdot||_{PM_p}$ coincide):

$$C_{\delta,p}(G) = \overline{Span\,\{\lambda_p(x)\colon\ x\in G\}}^{||\cdot||_{\mathrm{op}}}, \quad PF_p(G) = \overline{\{\lambda_p(f)\colon f\in L^1(G)\}}^{||\cdot||_{\mathrm{op}}},$$

$$M_p(G) = \overline{\{\lambda_p(\mu)\colon\ \mu\in M^1(G)\}}^{||\cdot||_{\mathrm{op}}}, \quad UC_p(G) = \overline{A_p(G)\cdot PM_p(G)}^{||\cdot||_{\mathrm{op}}},$$

$$AP_p(G) = \{T\in PM_p(G)\colon\ A_p(G)\to PM_p(G), u\mapsto u\cdot T \text{ is compact}\},$$

$$WAP_p(G) = \{T\in PM_p(G)\colon\ A_p(G)\to PM_p(G), u\mapsto u\cdot T \text{ is weakly compact}\}.$$

We point out that for $p=2$, $C_{\delta,p}(G)$ is the same as the C^*-algebra $C^*_\delta(G)$ in [30] and elsewhere. When G is abelian and $p=2$, $C_{\delta,p}(G)$ is the space of continuous almost periodic functions on the dual group.

Of interest for us are also two other Banach algebras of continuous functions on G, $W_p(G)$ and $B_p(G)$. The first algebra has been introduced by Cowling [8] and is defined as follows. For K a compact subset of G let $I_K = \{u\in A_p(G)\colon\ u|_K = 0\}$ and let $A_p(K) = A_p(G)/I_K$, equipped with the quotient norm. A bounded continuous function w on G belongs to $W_p(G)$ and has norm at most C if and only if for every compact subset K of G, $w|_K\in A_p(K)$ and $||w|_K||_{A_p(K)}\leqslant C$. Cowling showed that $W_p(G)$ is the dual of the space $PF_p(G)$, with the dual action $\langle w, \lambda_p(f)\rangle = \int_G w(x)f(x)\,dx$. For G abelian $W_2(G)$ is isomorphic to $M^1(\hat{G})$, and for G amenable $W_2(G)$ is isomorphic to the Fourier–Stieltjes algebra of G as introduced by Eymard [14]. The Banach algebra $B_p(G)$ is introduced by Herz [27, p. 146]. We recall the definition. If X is a non-empty set (with discrete topology), then every $k\in\mathscr{C}_{00}(X\times X)$ is the kernel of a continuous linear operator T_k on $l^p(X)$ defined by $T_kf(x) = \sum_y k(x,y)f(y)$. The corresponding operator norm is denoted by $|||k|||_p$. The space $V_p(X)$ consists of all functions ϕ on $X\times X$ for which there is a constant $C>0$ with $|||\phi k|||_p\leqslant C|||k|||_p$ for every $k\in\mathscr{C}_{00}(X\times X)$. The smallest possible C is denoted by $||\phi||_{V_p(X)}$. One defines $B_p(G)$ to be the set of all continuous functions ϕ on G such that $M\phi\in V_p(G_d)$, where G_d is G equipped with the discrete topology, $M\phi(x,y) = \phi(xy^{-1})$, and $||\phi||_{B_p(G)} = ||M\phi||_{V_p(G_d)}$. When $p=2$ this is the space of completely bounded multipliers of $PF_2(G) = C^*_r(G)$ [9, p. 86]. In general, for an arbitrary locally compact group G, it follows from [27, Theorems 1, 2, p. 147, and Proposition 3, p. 154; 8, Theorem 5, p. 94] that $A_p(G)\subset W_p(G)\subset B_p(G)\subset MA_p(G)$, with decreasing norms $||\cdot||_{\sup}\leqslant||\cdot||_{MA_p}\leqslant||\cdot||_{B_p}\leqslant||\cdot||_{W_p}\leqslant||\cdot||_{A_p}$. When G is amenable, $MA_p(G)$, $W_p(G)$, and $B_p(G)$ are isometrically isomorphic [27].

When there is no risk of confusion, these spaces are denoted by $C_{\delta,p}$, PF_p, M_p, AP_p, WAP_p, UC_p, PM_p, MA_p, W_p, B_p and A_p, in the respective cases.

Suppose that B is a Banach algebra. A closed linear subspace Y of B' is called *left topologically invariant* if $T\cdot a\in Y$, for all $T\in Y, a\in B$. Suppose that Y is such a subspace of B'. Given $\Psi\in Y'$ and $T\in Y$, we define a continuous linear functional

$\Psi \odot T$ on B by $\langle \Psi \odot T, a \rangle = \langle \Psi, T \cdot a \rangle$ $(a \in B)$. If $\Psi \odot T \in Y$ for all choices of $\Psi \in Y'$ and $T \in Y$, then Y is called a *left topologically introverted* subspace of B' (one may similarly define a *right* topologically introverted subspace; the two notions coincide when B is commutative). Let Y be a left topologically introverted subspace of B'. As observed by Arens [2], one may turn Y' into a Banach algebra with the following product:

$$\langle \Gamma \Box \Psi, T \rangle = \langle \Gamma, \Psi \odot T \rangle (\Psi, \Gamma \in Y', T \in Y).$$

Since B' is certainly left topologically introverted, this procedure turns the double dual of any Banach algebra into a Banach algebra itself. The spaces $C_{\delta,p}$, PF_p, M_p, AP_p, WAP_p, and UC_p, are all left topologically introverted subspaces of PM_p.

A group G is called amenable if there exists a continuous linear functional $m \in L^\infty(G)'$ such that $||m|| = m(\mathbf{1}) = 1$, and $m({}_a f) = m(f)$ for all $f \in L^\infty(G)$ and all $a \in G$ ($\mathbf{1}$ is the constant function 1). Amenable groups include abelian groups, compact groups, and solvable groups. However, the free group on two generators (with discrete topology) is not amenable. For more information we refer the reader to [33,34]. It is well known that $A_p(G)$ has a bounded approximate identity if and only if G is amenable [26, Theorem 6, p. 120].

Throughout this paper we use $\mathscr{A}$ to denote any of the Banach spaces

$$C_{\delta,p}, PF_p, M_p, AP_p, WAP_p, UC_p, PM_p,$$

and we use $\mathscr{B}$ to denote any of the spaces MA_p, B_p, or W_p.

3. An embedding of $W_p(G)$ into $UC_p(G)'$ and some of its consequences

If G is an *amenable* locally compact group it is proved by Granirer [23, Proposition 2.1, p. 160] and also by Delaporte and Derighetti [10, Proposition 7, p. 501] that $UC_p(G)'$ (as a Banach algebra) is isometrically isomorphic to $Hom_{A_p}(PM_p)$ (the space of all continuous A_p-module homomorphisms of PM_p). This provides an embedding $v: W_p \to UC_p(G)'$, defined by $v(w)(T) = w \cdot T$. In this section we improve this result by proving the existence of an *isometric* Banach algebra isomorphism from W_p into $UC_p{}'$ for an *arbitrary* locally compact group, using an alternative method inspired by the result of Lau and Losert [31, Proposition 4.2, p. 10] (for the case of $p = 2$). This approach allows to embed W_p isometrically into the dual of any other topologically introverted subspace X of PM_p such that $PF_p \subset X \subset UC_p$ (this include the cases UC_p, M_p, $WAP_p \cap UC_p$, $\overline{M_p + AP_p}^{||\cdot||_{op}}$ or WAP_p with G amenable in the last two cases, see for example [22, pp. 128–129]). We use the direct sum decomposition obtained from such embedding (Corollary 3.5) to construct examples of weak* dense maximal ideals in these algebras.

Suppose that $u \mapsto \tilde{u}$, $A_p(G) \to A_p(G)'' = PM_p{}'$, is the canonical embedding of $A_p(G)$ into its double dual. Let $w \in W_p = PF_p{}'$ be arbitrary, and let $\hat{w}$ be an extension of w to a continuous linear functional on PM_p such that $||w||_{W_p} = ||\hat{w}||_{PM_p{}'}$. Take (u_α) a net in $A_p(G)$ such that $||u_\alpha||_{A_p} \leqslant ||\hat{w}||_{PM_p'}$, and $\tilde{u}_\alpha \to \hat{w}$ in $\sigma(PM_p', PM_p)$ [11, Theorem V.4.5, p. 424]. It is not difficult to show that under such circumstances $||\tilde{u}_\alpha||_{PM_p'} \to ||\hat{w}||_{PM_p'}$. Now let R_α denote the restriction of $\tilde{u}_\alpha$ to the subspace UC_p of PM_p. Then, $||R_\alpha||_{UC_p'} \leqslant ||\tilde{u}_\alpha||_{PM_p'} \leqslant ||\hat{w}||_{PM_p'}$, and (R_α) is a Cauchy net (with respect to $\sigma(UC_p', UC_p)$-topology) in the closed ball of UC_p' with radius $||\hat{w}||_{PM_p'}$. It follows from Alaoglu's theorem [11, Theorem V.4.2, p. 424] that R_α converges to some element E_w in UC_p' with $||E_w||_{UC_p'} \leqslant ||\hat{w}||_{PM_p'}$.

Our notation E_w suggests that this element does not depend on the particular extension $\hat{w}$ of w; this is indeed true as we show in Theorem 3.2. But first we show that E_w does not depend on a particular choice of (u_α). More precisely, suppose that (v_β) is a net in $A_p(G)$ such that $||\tilde{v}_\beta||_{PM_p'} \leqslant ||\hat{w}||_{PM_p'}$ and $\tilde{v}_\beta \to \hat{w}$ in $\sigma(PM_p', PM_p)$. Let S_β be the restriction of $\tilde{v}_\beta$ to UC_p, and suppose that E_w' is the limit of S_β in $\sigma(UC_p', UC_p)$. Then for every $T \in UC_p$ we have $\langle E_w', T \rangle = \lim_\beta \langle S_\beta, T \rangle = \lim_\beta \langle \tilde{v}_\beta, T \rangle = \langle \hat{w}, T \rangle = \lim_\alpha \langle \tilde{u}_\alpha, T \rangle = \lim_\alpha \langle R_\alpha, T \rangle = \langle E_w, T \rangle$, as we wanted to show.

Now suppose that $E : W_p(G) \to UC_p(G)'$, $w \mapsto E_w$ is the embedding constructed above. In order to show that this embedding is linear we need the following lemma.

Lemma 3.1. *Suppose that r_1, r_2 are positive, and suppose that the sets $S_{r_1} = \{v \in W_p\colon ||v||_{W_p} = r_1\}$ and $B_{r_2} = \{T \in PM_p\colon ||T||_{PM_p} \leqslant r_2\}$ are equipped with their respective induced* weak* *topologies. Let $S_{r_1} \times B_{r_2}$ have the product topology, and PM_p have its* weak* *topology. Then the map $S_{r_1} \times B_{r_2} \to PM_p$, $(v, T) \mapsto v \cdot T$ is continuous.*

Proof. By a result of Fendler [16, Theorem of p. 131], on S_{r_1} the induced weak* topology and the $A_p(G)$-multiplier topology coincide (in the latter topology, $v_\beta \to v$ if $||v_\beta u - vu||_{A_p} \to 0$ for all $u \in A_p(G)$). Suppose that $(v_\beta, T_\beta)_\beta$ is a net in $S_{r_1} \times B_{r_2}$ converging to $(v, T) \in S_{r_1} \times B_{r_2}$. Take $u \in A_p(G)$, we show that $\langle v_\beta \cdot T_\beta, u \rangle \to \langle v \cdot T, u \rangle$. We write

$$\langle v_\beta \cdot T_\beta, u \rangle - \langle v \cdot T, u \rangle = \langle T_\beta, v_\beta u \rangle - \langle T, vu \rangle$$

$$= \langle T_\beta, (v_\beta - v)u \rangle + \langle T_\beta - T, vu \rangle.$$

The second term tends to zero since $T_\beta \to T$ in the weak* topology, and the first term tends to zero since

$$|\langle T_\beta, (v_\beta - v)u \rangle| \leqslant ||T_\beta||_{PM_p} ||(v_\beta - v)u||_{A_p} \to 0. \qquad \square$$

Theorem 3.2. *The embedding $E : W_p(G) \to UC_p(G)'$, $w \mapsto E_w$ has the following properties:*

1. *$E_w|_{PF_p} = w$. In particular, if $f \in L^1(G)$, then $\langle E_w, \lambda_p(f) \rangle = \int_G w(t) f(t)\, dt$.*

2. *E is isometric.*
3. *For every* $w \in W_p$, $v \in A_p(G)$, $T \in PM_p$, *we have*

$$\langle E_w, v \cdot T \rangle = \langle T, vw \rangle.$$

Consequently E_w *does not depend on the particular isometric extension* $\hat{w}$; *and if* $u \in A_p(G)$, $E_u = u$.
4. *E is an algebra homomorphism.*
5. E_w *is the unique norm preserving extension of w to a continuous linear functional on* UC_p.

Proof. (1) For $T \in PF_p$, $\langle E_w, T \rangle = \lim_\alpha \langle R_\alpha, T \rangle = \lim_\alpha \langle \tilde{u}_\alpha, T \rangle = \langle \hat{w}, T \rangle = \langle w, T \rangle$.

(2) Since $PF_p \subset UC_p$, it follows that $||E_w||_{UC_p'} \geqslant ||w||_{W_p}$. Now suppose that $T \in UC_p$, then $|\langle E_w, T \rangle| = \lim_\alpha |\langle R_\alpha, T \rangle| = \lim_\alpha |\langle \tilde{u}_\alpha, T \rangle| \leqslant \lim_\alpha ||\tilde{u}_\alpha||_{PM_p'} ||T||_{PM_p} = ||\hat{w}||_{PM_p'} ||T||_{PM_p} = ||w||_{W_p} ||T||_{PM_p'}$ (recall that $||\tilde{u}_\alpha||_{PM_p'} \to ||\hat{w}||_{PM_p'}$). Thus $||E_w||_{UC_p'} \leqslant ||w||_{W_p}$ and therefore $||E_w||_{UC_p'} = ||w||_{W_p}$.

(3) The assumption $\tilde{u}_\alpha \to \hat{w}$ in $\sigma(PM_p', PM_p)$ implies that for every $T \in PF_p$, $\langle u_\alpha, T \rangle = \langle \tilde{u}_\alpha, T \rangle \to \langle \hat{w}, T \rangle = \langle w, T \rangle$. Taking into account that $||u_\alpha||_{A_p} = ||\tilde{u}_\alpha||_{PM_p'} \to ||\hat{w}||_{PM_p'} = ||w||_{W_p}$, we obtain $u_\alpha / ||u_\alpha||_{A_p} \to w / ||w||_{W_p}$ in $\sigma(W_p, PF_p)$. Now for every $v \in A_p(G)$ and $T \in PM_p$ we have $\langle E_w, v \cdot T \rangle = \lim_\alpha \langle \tilde{u}_\alpha, v \cdot T \rangle = \lim_\alpha \langle v \cdot T, u_\alpha \rangle = \lim_\alpha \langle T, v u_\alpha \rangle = \lim_\alpha \langle u_\alpha \cdot T, v \rangle = \lim_\alpha ||u_\alpha||_{A_p} \langle u_\alpha / ||u_\alpha||_{A_p} \cdot T, v \rangle = ||w||_{W_p} \langle w / ||w||_{W_p} \cdot T, v \rangle = \langle w \cdot T, v \rangle = \langle T, vw \rangle$ (where the above lemma is used to obtain the sixth equality). In particular, since $A_p \cdot PM_p$ is norm dense in UC_p it follows that E_w does not depend on the particular extension $\hat{w}$.

(4) Suppose that $w_1, w_2 \in W_p$, and $c \in \mathbf{C}$. Since $A_p \cdot PM_p$ is dense in UC_p, it suffices to show the equality $E_{(w_1 + c w_2)} = E_{w_1} + c E_{w_2}$ on this dense subset. In fact, for every $u \in A_p(G)$ and $T \in PM_p$, we have $\langle E_{(w_1 + c w_2)}, u \cdot T \rangle = \langle T, u(w_1 + c w_2) \rangle = \langle T, u w_1 \rangle + c \langle T, u w_2 \rangle = \langle E_{w_1}, u \cdot T \rangle + c \langle E_{w_2}, u \cdot T \rangle$. This shows that E is linear. The proof that E preserves the product is similar.

(5) Suppose that $\kappa(w)$ is another extension of w to a continuous linear functional on UC_p such that $||\kappa(w)||_{UC_p'} = ||w||_{W_p}$. Let $\hat{\kappa}(w)$ be a norm preserving extension of $\kappa(w)$ to a continuous linear functional on PM_p; in other words $\hat{\kappa}(w)$ is a norm preserving extension of w to a continuous linear functional on PM_p. Let (v_β) be a net in $A_p(G)$ such that $||v_\beta||_{A_p} = ||\tilde{v}_\beta||_{PM_p'} \leqslant ||\hat{\kappa}(w)||_{PM_p'}$ and $\tilde{v}_\beta \to \hat{\kappa}(w)$ in $\sigma(PM_p', PM_p)$. Then for every $v \in A_p(G)$, $T \in UC_p$, $\langle \kappa(w), v \cdot T \rangle = \langle \hat{\kappa}(w), v \cdot T \rangle = \lim_\beta \langle \tilde{v}_\beta, v \cdot T \rangle = \lim_\beta \langle \tilde{v}_\beta |_{UC_p}, v \cdot T \rangle = \langle T, vw \rangle = \langle E_w, v \cdot T \rangle$ (the fourth equality is obtained by a similar procedure to (3)). Since $\kappa(w)$ and E_w coincide on the dense subset $A_p \cdot PM_p$, they are equal on UC_p. □

For a topologically introverted subspace X of PM_p such that $PF_p \subset X \subset UC_p$ we have the following corollaries:

Corollary 3.3. *The map* $\nu : W_p \to X'$, $\nu(w) = E_w|_X$ *is an isometric algebra homomorphism.*

We recall that PM_p has a natural MA_p Banach module structure which is defined as follows: if $v \in MA_p$, $T \in PM_p$, and $u \in A_p$, then $\langle v \cdot T, u \rangle = \langle T, vu \rangle$. The inequalities $\| \cdot \|_{MA_p} \leqslant \| \cdot \|_{B_p} \leqslant \| \cdot \|_{W_p} \leqslant \| \cdot \|_{A_p}$ imply that PM_p can be viewed as a Banach module with respect to any of the Banach algebras A_p, W_p, and B_p as well. In our next result we state several properties of the embedding $\nu : W_p \to X'$.

Proposition 3.4. *Let* $w \in W_p$, $T \in X$, *and* $\Psi \in X'$.

(1) $\nu(w) \odot T = w \cdot T$.
(2) $\nu(w)$ *is in the center of* X'.
(3) *Assuming* X' *is equipped with its weak* topology, the map* $X' \to X'$, $\Psi \mapsto \nu(w) \square \Psi$ *is continuous.*
(4) *If* M_w *is the continuous linear operator on* X *defined by* $M_w(T) = w \cdot T$, *then its transpose is given by* ${}^t M_w(\Psi) = \nu(w) \square \Psi$.

In order to prove (2) one may first prove the result for $X = UC_p$ using the density of $A_p \cdot PM_p$ in UC_p, and then complete the proof for the general case noting that X' can be identified with a quotient of $UC_p{}'$. The proof of the other assertions are easy, and we omit the details.

As another corollary of Theorem 3.2 we obtain now the following decomposition result which extends earlier results of Ghahramani et al. [21, Lemma 1.1, p. 275] and Filali [18, Lemma 2.2, p. 292]:

Corollary 3.5. *Let* $PF_p^\perp = \{\Gamma \in X' : \Gamma(PF_p) = \{0\}\}$. *Then* $PF_p^\perp$ *is a weak* closed two sided ideal of* X' *and* $X' = \nu(W_p) \oplus PF_p^\perp$ (*where* $\oplus$ *represents topological direct sum of Banach spaces*).

Proof. Note that if $PF_p = X$ (for example, when G is discrete $PF_p = X = UC_p$) then $PF_p^\perp = \{0\}$ is a trivial ideal and there remains nothing to prove. So we assume that $PF_p \neq X$. The fact that $PF_p^\perp$ is a weak* closed subspace of X' follows easily from the definitions. To show the decomposition note first that W_p being the dual of PF_p implies that $\nu(W_p) \cap PF_p^\perp = \{0\}$. If $\Psi \in X'$ and if $w = \Psi|_{PF_p}$, then $\Psi = \nu(w) + (\Psi - \nu(w))$ and $\Psi - \nu(w) \in PF_p^\perp$ which gives the required decomposition.

Next to verify that $PF_p^\perp$ is a two sided ideal, suppose that $\Psi \in X'$, $\Gamma \in PF_p^\perp$, $T \in PF_p$, and $u \in A_p(G)$ are arbitrary. Then $\langle \Gamma \odot T, u \rangle = \langle \Gamma, u \cdot T \rangle = 0$ (since PF_p is topologically invariant). This implies that $\Gamma \odot T = 0$ and hence $\Psi \square \Gamma \in PF_p^\perp$. To prove that $PF_p^\perp$ is a right ideal we use the decomposition $X' = \nu(W_p) \oplus PF_p^\perp$ to write

$\Psi = \Psi_1 + \Psi_2$ $(\Psi_1 \in \nu(W_p),\ \Psi_2 \in PF_p^\perp)$. Since $\nu(W_p)$ is in the center of X' we can write

$$\Gamma \square \Psi = \Gamma \square \Psi_1 + \Gamma \square \Psi_2 = \Psi_1 \square \Gamma + \Gamma \square \Psi_2.$$

Both terms on the right-hand side belong to $PF_p^\perp$ since $PF_p^\perp$ is a left ideal. Thus $\Gamma \square \Psi \in PF_p^\perp$ which completes the proof. □

One may ask if an embedding as in Corollary 3.3 exists when W_p is replaced by MA_p or B_p. The existence of such embedding is equivalent to the amenability of G. More precisely,

Theorem 3.6. *A group G is amenable if and only if there exists an embedding ρ of $MA_p(G)$ (respectively, $B_p(G)$) into X' such that $\langle \rho(v), \lambda_p(f) \rangle = \int_G v(t)f(t)\,dt$ $(v \in MA_p$ (respectively, $v \in B_p)$ and $f \in L^1(G))$.*

Proof. If G is amenable, then $MA_p(G) = B_p(G) = W_p(G)$ [27, Theorems 1 and 2, p. 147] and so Theorem 3.2(1) and Corollary 3.3 provide the required result. Conversely if $\mathbf{1}$ is the constant function 1 on G then $\langle \rho(\mathbf{1}), \lambda_p(f) \rangle = \int_G f(t)\,dt$ which means that $\mathbf{1} \in PF_p{}' = W_p$. By a result of Cowling [8, see the proof of Theorem 5, p. 94], G must be amenable. □

As one might expect the embedding $E : W_p \to UC_p{}'$ is surjective only when $PF_p = UC_p$.

Theorem 3.7. *The embedding E of W_p into $UC_p{}'$ is onto if and only if G is discrete.*

Proof. When G is discrete by Granirer [22, Proposition 15, p. 129], $PF_p = UC_p$ and hence $W_p = UC_p{}'$, that is, E is onto.

Conversely, suppose that $E(W_p) = UC_p{}'$. If G is not discrete, then by Granirer [24, Theorem 6, p. 3401] there are at least two different topological invariant means Ψ_1 and Ψ_2 on PM_p. Let Γ_1 and Γ_2 be the topological invariant means on UC_p, obtained by restricting Ψ_1 and Ψ_2 to UC_p. On one hand, from the definition of UC_p and the assumption that $\Psi_1 \neq \Psi_2$, it easily follows that $\Gamma_1 \neq \Gamma_2$. On the other hand,

$$\Gamma_1 = \Gamma_2(\lambda_p(e))\Gamma_1 = \Gamma_2 \square \Gamma_1 = \Gamma_1 \square \Gamma_2 = \Gamma_1(\lambda_p(e))\Gamma_2 = \Gamma_2$$

(the commutativity of $UC_p{}'$ follows from $E(W_p) = UC_p{}'$). The contradiction implies that G must be discrete. □

4. Maximal regular ideals

Recall that we use $\mathscr{A}$ to denote any of the Banach spaces

$$C_{\delta,p}, PF_p, M_p, AP_p, WAP_p, UC_p, PM_p,$$

and we use $\mathscr{B}$ to denote any of the Banach algebras MA_p, B_p, or W_p.

In [17, Theorem 3.2, p. 573], Filali showed that if B is a commutative Banach algebra, and if M is a maximal regular left (right, or two sided) ideal of B'' (the double dual of B), then M is either weak* dense in B'' or there is a unique multiplicative linear functional $\phi \in B'$ such that

$$M = \{\Psi \in B'' \colon \Psi(\phi) = 0\}.$$

Special cases of the above result when B is the Fourier algebra $A(G)$, or when B is a strongly regular Banach algebra in $\mathscr{C}_0(G)$, have been proved with different methods by Lau [30, Corollary 6.4, p. 59] and by Ghahramani and Lau [20, Corollary 4.5, p. 186].

We remark that the result of Filali is not in general true if B is not commutative. For example if $B = \mathbb{M}_2$ is the algebra of all 2×2 matrices over the complex numbers, then the Banach algebra B'' can be identified with B itself, and

$$J = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \colon x, y \in \mathbf{C} \right\}$$

is a weak* closed, maximal regular left ideal in B'' which is not the annihilator of a multiplicative linear functional on B.

We start our study of maximal regular ideals in $\mathscr{A}'$ and $\mathscr{B}''$ with a few preliminary lemmas.

Lemma 4.1. *The space $\mathscr{A}$ is a topologically introverted subspace of $PM_p(G)$.*

Proof. This assertion is easy to prove directly for WAP_p and UC_p. For the other cases the result follows from [32, Lemma 1.2, p. 178]. $\square$

In view of the above lemma, $\mathscr{A}'$ is a Banach algebra with the Arens product $\square$ defined in Section 2.

Lemma 4.2. *The Banach algebra $A_p(G)$ can be identified with a subalgebra in the center of each of the algebras $\mathscr{A}'$ and $\mathscr{B}''$.*

Proof. For the case of the Banach algebra $\mathscr{B}''$ our claim follows from the fact that $A_p(G)$ is a subalgebra of $\mathscr{B}$. For $\mathscr{A} = PF_p$, our claim follows from [8, Theorem 4, p. 92]. In the case of $\mathscr{A} = PM_p = A_p(G)'$, our claim is clear. When $\mathscr{A}$ is equal to $C_{\delta,p}$, M_p, AP_p, WAP_p, or UC_p, then $\mathscr{A}$ is a norm closed subspace of PM_p containing

$\{\lambda_p(x)\colon x\in G\}$. Since $\mathscr{A}' = PM_p'/(\mathscr{A}^\perp)$, where $\mathscr{A}^\perp = \{\Psi\in PM_p'\colon \Psi(\mathscr{A}) = \{0\}\}$, and since $\mathscr{A}$ contains $\{\lambda_p(x)\colon x\in G\}$, it follows that the canonical continuous linear map defined by $A_p(G)\to PM_p'\to\mathscr{A}'$ is an injection from $A_p(G)$ into $\mathscr{A}'$. For simplicity we denote the image of this map by $A_p(G)$, and the image of an element $u\in A_p(G)$ is denoted by u itself. Note also that if $u\in A_p(G)$, $T\in\mathscr{A}$, then $|\langle u,T\rangle| = |\langle T,u\rangle|\leqslant ||T||_{PM_p}||u||_{A_p}$, so $||u||_{\mathscr{A}'}\leqslant||u||_{A_p}$ as we expect. In all cases the verification that $A_p(G)$ is a subalgebra in the center of $\mathscr{A}'$ is a routine. $\square$

Remarks 4.3. (1) The space $A_p(G)$ is weak* dense in $\mathscr{A}'$. This is clear for $\mathscr{A} = PM_p$. For the other cases, suppose that $\Gamma+\mathscr{A}^\perp\in\mathscr{A}'$ where $\Gamma\in PM_p'$, and $\mathscr{A}^\perp = \{\Psi\in PM_p'\colon \Psi(\mathscr{A}) = \{0\}\}$. Let (v_β) be a net in $A_p(G)$ such that $v_\beta\to\Gamma$ in $\sigma(PM_p', PM_p)$. Then for every $T\in\mathscr{A}$,

$$\langle v_\beta+\mathscr{A}^\perp, T\rangle = \langle v_\beta, T\rangle\to\langle\Gamma,T\rangle = \langle\Gamma+\mathscr{A}^\perp, T\rangle,$$

that is, $v_\beta+\mathscr{A}^\perp\to\Gamma+\mathscr{A}^\perp$ in the weak* topology of $\mathscr{A}'$.

If X is a topologically introverted subspace of PM_p such that $PF_p\subset X\subset UC_p$, then a similar argument shows that $v(A_p(G))$ is weak* dense in X'.

(2) For $x\in G$, let ε_x be the homomorphism on $\mathscr{B}$ defined by $\varepsilon_x(u) = \langle\varepsilon,u\rangle = u(x)$. As may be known, it can be readily seen that if $\phi\in\mathbf{X}(\mathscr{B})$ and if $\phi\neq\varepsilon_x$ for every $x\in G$, then $\phi(A_p(G)) = \{0\}$ (to see this, notice that if $\phi(A_p(G))\neq\{0\}$, then $\phi|_{A_p(G)}\in\mathbf{X}(A_p(G))$ and hence $\phi|_{A_p(G)} = \varepsilon_x$ for some $x\in G$. Now if $v\in\mathscr{B}$ is arbitrary and if $u\in A_p(G)$ is such that $u(x) = 1$, then $vu\in A_p(G)$ and

$$v(x) = v(x)u(x) = \phi(uv) = \phi(u)\phi(v) = u(x)\phi(v) = \phi(v),$$

that is, $\phi(v) = v(x)$ for all $v\in\mathscr{B}$, which is a contradiction).

(3) Since $\mathscr{A}$ is topologically invariant, there exists a natural $A_p(G)$-module action on $\mathscr{A}'$ defined by $\langle u\cdot\Psi, T\rangle = \langle\Psi, u\cdot T\rangle$ $(\Psi\in\mathscr{A}',\ T\in\mathscr{A},\ u\in A_p(G))$. It is easy to verify that $u\square\Psi = u\cdot\Psi$.

Lemma 4.4. *Suppose that B is a commutative Banach algebra and A is a dense subalgebra of B. Let $\mathbf{X}(B)|_A = \{\phi|_A\colon \phi\in\mathbf{X}(B)\}$. Then $\mathbf{X}(B)|_A = \mathbf{X}(A)$.*

Proof. It is clear that $\mathbf{X}(B)|_A\subset\mathbf{X}(A)$. Now suppose that $\phi\in\mathbf{X}(A)$ and let $\hat{\phi}$ be the unique continuous linear extension of ϕ to the whole of B. Let $a,b\in B$ and let $(a_m)_m$, and $(b_n)_n$ be two sequences in A converging to a and b, respectively. Then the net $(a_mb_n)_{(m,n)}$ converges to ab and we have

$$\hat{\phi}(ab) = \lim_{(m,n)}\phi(a_mb_n) = \lim_{(m,n)}\phi(a_m)\phi(b_n) = \hat{\phi}(a)\hat{\phi}(b). \qquad\square$$

Proposition 4.5. *Let B be a Banach algebra and let A be a subalgebra contained in the center of B. If M is a maximal regular left (right, or two sided) ideal of B then either* $M \cap A = A$ *or* $M \cap A = I_\phi = \{a \in A\colon \phi(a) = 0\}$ *for some unique* $\phi \in \mathbf{X}(A)$.

Proof. If M does not contain A it would not contain $\bar{A}$ (the closure of A in the norm topology) either and therefore by Filali [17, Lemma 3.1, p. 573], $M \cap \bar{A}$ is a maximal regular ideal in $\bar{A}$. Hence for some unique $\hat{\phi} \in \mathbf{X}(\bar{A})$, $M \cap \bar{A} = \{a \in \bar{A}\colon \hat{\phi}(a) = 0\}$. If we set $\phi = \hat{\phi}|_A$, then by the above lemma $\phi \in \mathbf{X}(A)$, and of course $M \cap A = \{a \in A\colon \phi(a) = 0\}$. □

As a corollary of Proposition 4.5 and Lemma 4.2 we obtain

Corollary 4.6. *If M is a maximal regular left (right, or two sided) ideal in* $\mathscr{A}'$, *(or in* $\mathscr{B}''$*), then either* $M \cap A_p(G) = A_p(G)$, *or* $M \cap A_p(G) = I_x = \{u \in A_p(G)\colon u(x) = 0\}$ *for a unique* $x \in G$.

The above corollary was proved for $\mathscr{A} = PM_2$ by Lau [30, Theorem 4.3, p. 58] and extended to PM_p by Ghahramani and Lau [20, Theroem 4.2, p. 185]. Their methods are entirely different from ours, it relies on the abstract Tauberian theorem [28, Theorem 39.27, p. 499] while ours is based on the fact that $A_p(G)$ is commutative and uses a result due to Allan [1, Corollary 2.4, p. 195].

Corollary 4.7. *If M is a maximal regular left (right, or two sided) ideal in* $\mathscr{A}'$, *and if there exists a non-zero* $T \in \mathscr{A}$ *such that* $M \subset \{\Psi \in \mathscr{A}'\colon \langle \Psi, T \rangle = 0\}$, *then for some* $x \in G$, $M = M_x = \{\Psi \in \mathscr{A}'\colon \langle \Psi, \lambda_p(x) \rangle = 0\}$.

We remark that $\lambda_p(x) \notin PF_p$ unless G is discrete [20, Lemma 4.9, p. 188]. This necessitate an interpretation of the action $\langle \Psi, \lambda_p(x) \rangle$ when $\mathscr{A} = PF_p$. Since Ψ is a continuous function in this case, we simply define $\langle \Psi, \lambda_p(x) \rangle$ as $\Psi(x)$.

Proof. By Corollary 4.6, for some $x \in G$, $I_x \subset M$, and hence $\langle u, T \rangle = 0$, for all $u \in I_x$. Now suppose that y is an element of G different from x such that $y \in Supp\, T$. If U is a compact neighborhood of y not containing x, then there exists $u \in A_p(G)$ such that $Supp\, u \subset U$ and $\langle u, T \rangle \neq 0$ (see for example [26, p. 101]). This contradicts $I_x \subset M$. Therefore $Supp\, T = \{x\}$, and so $T = c \cdot \lambda_p(x)$ (for some $c \in \mathbf{C}$). Thus $M \subset M_x = \{\Psi \in \mathscr{A}'\colon \langle \Psi, \lambda_p(x) \rangle = 0\}$. Since M is maximal and is contained in the ideal M_x it follows that $M = M_x$. □

Theorem 4.8. (1) *Every maximal regular left (right, or two sided) ideal of* $\mathscr{A}'$ *is either weak* dense in* $\mathscr{A}'$ *or is of the form*

$$M_x = \{\Psi \in \mathscr{A}'\colon \langle \Psi, \lambda_p(x) \rangle = 0\},$$

for some $x \in G$.

(2) *Every maximal regular left (right, or two sided) ideal of* $\mathscr{B}''$ *is either weak** *dense in* $\mathscr{B}''$ *or is of the form*

$$M_\phi = \{\Psi \in \mathscr{B}'' : \langle \Psi, \phi \rangle = 0\},$$

for some $\phi \in \mathbf{X}(\mathscr{B})$.

Proof. (1) Suppose that I is a maximal regular left (right, or two sided) ideal of $\mathscr{A}'$. We prove that the weak* closure $\bar{I}$ of I is a closed regular left (right, or two sided) ideal of $\mathscr{A}'$. If $\Psi \in \bar{I}$, and $\Gamma \in \mathscr{A}'$, then one can find a net (u_α) in I and a net (v_β) in $A_p(G)$ such that in weak* topology, $\Psi = \lim_\alpha u_\alpha$, and $\Gamma = \lim_\beta v_\beta$ (Remark 4.3(1)). Using the continuity of the product on the left side, we have

$$\Gamma \Box \Psi = \lim_\beta v_\beta \Box \Psi = \lim_\beta \Psi \Box v_\beta = \lim_\beta \lim_\alpha u_\alpha v_\beta = \lim_\beta \lim_\alpha v_\beta u_\alpha \in \bar{I}.$$

(Similarly, in the case that I is a right ideal $\Psi \Box \Gamma = \lim_\alpha u_\alpha \Box \Gamma \in \bar{I}$.) Since I is a maximal regular left (right, or two sided) ideal, it follows that either $\bar{I} = \mathscr{A}'$ (and so I is weak* dense in $\mathscr{A}'$), or $\bar{I} = I$. Suppose that $I = \bar{I}$. By Corollary 4.6, $I \cap A_p(G) = I_x = \{u \in A_p(G) : u(x) = 0\}$ for some $x \in G$. An argument similar to the proof of [6, Theorem 5.3, p. 865] shows that $\overline{I_x} = M_x = \{\Psi \in \mathscr{A}' : \langle \Psi, \lambda_p(x) \rangle = 0\}$. So $M_x \subset \overline{I \cap A_p(G)}$. Since M_x is a maximal regular ideal in $\mathscr{A}'$ and $\overline{I \cap A_p} \subset I$, we have $I = M_x$.

(2) Our result for the case of $\mathscr{B}''$ follows from [17, Theorem 3.2, p. 573]. □

Remark 4.9. The above proof shows that maximal regular left ideals in $\mathscr{A}'$ that are not right ideals, are weak* dense in $\mathscr{A}'$ (since those that are not weak* dense are shown to be two sided ideals). A similar statement holds true for right ideals.

Theorem 4.10. *Let* X *be a topologically introverted subspace of* PM_p *which is contained in* UC_p *and contains* PF_p *properly, and let* $PF_p^\perp = \{\Psi \in X' : \Psi(PF_p) = \{0\}\}$. *Let* M *be a maximal regular ideal of* W_p. *Then* $v(M) \oplus PF_p^\perp$ *is a maximal regular two sided ideal of* X'. *Conversely, if* N *is a maximal regular two sided ideal of* X' *and* $PF_p^\perp \subset N$, *then* $N = v(M) \oplus PF_p^\perp$ *for some maximal regular ideal* M *of* W_p.

Proof. Since $PF_p^\perp$ is a two sided ideal in X' (Corollary 3.5), given $v(m) + \Gamma \in v(M) \oplus PF_p^\perp$ and $v(w) + \Psi \in X'$, we have

$$(v(m) + \Gamma) \Box (v(w) + \Psi) = v(mw) + \Gamma \Box v(w) + (v(m) + \Gamma) \Box \Psi \in v(M) \oplus PF_p^\perp.$$

Similarly $(v(w) + \Psi) \Box (v(m) + \Gamma) \in v(M) \oplus PF_p^\perp$. If w_0 is a unit of W_p modulo M (that is, $ww_0 - w \in M$ $(w \in W_p)$), then $v(w_0)$ is a unit of X' modulo $v(M) \oplus PF_p^\perp$. So it remains to show that $v(M) \oplus PF_p^\perp$ is maximal. Suppose that N is a maximal regular ideal of X' containing $v(M) \oplus PF_p^\perp$. Since $v(W_p)$ is in the center of X'

(Proposition 3.4), by Proposition 4.5, $N \cap v(W_p)$ is a maximal regular ideal of $v(W_p)$ containing $v(M)$, and thus $N \cap v(W_p) = v(M)$. Moreover if $\Psi = v(w) + \Gamma \in N$ ($w \in W_p$, $\Gamma \in PF_p^{\perp}$), the assumption $PF_p^{\perp} \subset N$ implies that $v(w) = \Psi - \Gamma \in N \cap v(W_p) = v(M)$. In other words, $N \subset v(M) \oplus PF_p^{\perp}$ and thus $N = v(M) \oplus PF_p^{\perp}$.

The converse of our theorem follows by a similar argument and we omit the details. □

Example 4.11. Let X be a topologically introverted subspace of PM_p which is contained in UC_p and contains PF_p properly (so in particular G is not discrete). As an application of Theorem 4.10 and the decomposition $X' = v(W_p) \oplus PF_p^{\perp}$ (Corollary 3.5), we give two examples of weak* dense maximal regular two sided ideals in X' which seem to be the only known examples of this type. Let M be a maximal regular ideal of W_p containing $A_p(G)$ (such ideals are the kernels of those characters on W_p that are not evaluations at a point $x \in G$). Then $v(M) \oplus PF_p^{\perp}$ is a maximal regular two sided ideal of X'. Since $v(A_p(G))$ is weak* dense in X' (Remark 4.3) so is $v(M) \oplus PF_p^{\perp}$.

For our second example let us assume that there exists an $x \in G$ such that $\lambda_p(x) \in X$ (this is the case in all of our examples of X, see our introduction to Section 3). Let $M = \{w \in W_p\colon\ w(x) = 0\}$. Then by Theorem 4.10 $v(M) \oplus PF_p^{\perp}$ is a maximal regular two sided ideal of X'. An argument similar to [6, Theorem 5.3, p. 865] shows that $\overline{v(I_x)} = M_x = \{\Psi \in X'\colon\ \langle \Psi, \lambda_p(x) \rangle = 0\}$ (the closure is in weak* topology). Since both M_x and $\overline{v(M)}$ are two sided ideals in X' with M_x being maximal, we have $\overline{v(M)} = M_x$ and

$$\overline{v(M) \oplus PF_p^{\perp}} \supset \overline{v(M)} \oplus PF_p^{\perp} = M_x \oplus PF_p^{\perp} = X'$$

(last equality follows from maximality of M_x). We conclude that $v(M) \oplus PF_p^{\perp}$ is weak* dense.

When $\mathscr{A} = PF_p$, $A_p(G)$ is an ideal of $\mathscr{A}' = W_p$. As the next lemma shows, when $\mathscr{A} \neq PF_p$, $A_p(G)$ is an ideal of $\mathscr{A}'$ only when G is discrete.

Lemma 4.12. (1) *If G is discrete, then $A_p(G)$ is an ideal in $\mathscr{A}'$. The converse is true when $\mathscr{A} \neq PF_p$.*

(2) *If G is amenable and discrete, then $A_p(G)$ is an ideal in $\mathscr{B}''$. Conversely, if $A_p(G)$ is an ideal in $\mathscr{B}''$, then G is discrete.*

Proof. The proof of (1) is similar to that given by Lau [30, Theorem 3.7, p. 57] (see also [19, Lemma 3.3, p. 220]) once we recall that G discrete implies $C_{\delta,p} = PF_p = UC_p \subset AP_p \subset WAP_p \subset PM_p$ [22, Proposition 15, p. 129].

We give a proof for (2). Suppose that G is amenable and discrete. From [8, Theorem 5, p. 94; 27, Theorems 1 and 2, p. 147] it follows that for amenable groups not only $MA_p = \mathscr{B}$ (with the equality of norms), but also $||\cdot||_{A_p} = ||\cdot||_{MA_p}$ on A_p. Consequently by Conway [7, Theorem V.2.3, p. 129], $A_p{}' = PM_p \cong MA_p{}'/A_p^{\perp}$, where $A_p^{\perp} = \{T \in MA_p{}' \colon T(A_p) = \{0\}\}$, and by Conway [7, Theorem V.2.2, p. 129]

$$A_p'' = PM_p{}' \cong (MA_p{}'/A_p^{\perp})' \cong (A_p^{\perp})^{\perp} = \{\Gamma \in MA_p'' \colon \Gamma(A_p^{\perp}) = \{0\}\} \subset MA_p''.$$

Next we observe that if $w \in A_p$, $\Psi \in MA_p''$, and $T \in A_p^{\perp} \subset MA_p{}'$, then $w \odot T = w \cdot T = 0$ (in $MA_p{}'$), and hence $\langle \Psi \Box w, T \rangle = 0$. Thus $\Psi \Box w \in (A_p^{\perp})^{\perp} \cong A_p''$ ($\Psi \in MA_p'', w \in A_p$). Now suppose that $\Psi \in MA_p''$, $u \in A_p \cap \mathscr{C}_{00}$, and $w \in A_p$ is such that $w|_{Supp\ u} = 1$. Then

$$\Psi \Box u = \Psi \Box (wu) = \Psi \Box (w \Box u) = (\Psi \Box w) \Box u.$$

So if $T \in MA_p{}'$ is arbitrary and if $[T] \in PM_p \cong MA_p{}'/A_p^{\perp}$ is the corresponding element, then

$$\begin{aligned}
\langle \Psi \Box u, T \rangle &= \langle (\Psi \Box w) \Box u, T \rangle \\
&= \langle (\Psi \Box w) \Box u, [T] \rangle \quad \text{(the product in } A_p'' \text{ coincides with} \\
&\qquad\qquad\qquad\qquad\qquad \text{the product in } MA_p'') \\
&= \langle \Psi \Box w, u \odot [T] \rangle \\
&= \langle \Psi \Box w, u \cdot [T] \rangle \\
&= \langle v, u \cdot [T] \rangle \quad (\text{where } v = \Psi \Box w|_{UC_p} \in MA_p = PF_p{}') \\
&= \langle v, u \odot [T] \rangle \\
&= \langle vu, [T] \rangle \quad (v \Box u = vu \text{ in } MA_p \subset MA_p'') \\
&= \langle uv, T \rangle \quad (vu \in A_p).
\end{aligned}$$

This proves that $\Psi \Box u = vu \in A_p$, for every $u \in A_p \cap \mathscr{C}_{00}$. Now an argument based on the density of $A_p \cap \mathscr{C}_{00}$ in A_p will complete the proof that A_p is an ideal in $\mathscr{B}''$.

The proof of the converse is similar to the case (1) above. □

Corollary 4.13. (1) *If G is discrete then every maximal regular left (right, or two sided) ideal of $\mathscr{A}'$ either contains $A_p(G)$ or is of the form $M_x = \{\Psi \in \mathscr{A}' \colon \langle \Psi, \lambda_p(x) \rangle = 0\}$ for some $x \in G$. The converse holds if $\mathscr{A}$ contains PF_p properly.*

(2) *Suppose that G is amenable. Then G is discrete if and only if each maximal regular left (right, or two sided) ideal of $\mathscr{B}''$ either contains $A_p(G)$ or is of the form $N_x = \{\Psi \in \mathscr{B}'' \colon \Psi(\varepsilon_x) = 0\}$ for some $x \in G$.*

Proof. (1) Let G be discrete and let M be a maximal regular left (right, or two sided) ideal of $\mathscr{A}'$ which does not contain $A_p(G)$. Then by Filali [17, Lemma 3.1, p. 573], $M \cap A_p(G)$ is a maximal regular ideal in $A_p(G)$, and hence for some $x \in G$, $M \cap A_p(G) = \{u \in A_p(G)\colon u(x) = 0\}$. Let $v \in A_p(G)$ be such that $v(x) \neq 0$ and let $\Psi \in M$. Since G is discrete, by Lemma 4.12, $A_p(G)$ is an ideal of $\mathscr{A}'$ and therefore $v \Box \Psi \in M \cap A_p(G)$. Hence

$$0 = (v \Box \Psi)(x) = \langle \tilde{\lambda}_p(x), v \Box \Psi \rangle = \langle \tilde{\lambda}_p(x), v \rangle \langle \tilde{\lambda}_p(x), \Psi \rangle = v(x) \langle \Psi, \lambda_p(x) \rangle$$

($\tilde{\lambda}_p(x)$ is the canonical image of $\lambda_p(x)$ in PM_p''). But $v(x) \neq 0$, so $\Psi \in M_x$, that is, $M \subset M_x$. From the maximality of M it follows that $M = M_x$.

For the converse, suppose that G is not discrete, we obtain a maximal regular two sided ideal of $\mathscr{A}'$ that neither contains $A_p(G)$ nor is of the form M_x. Let first $\mathscr{A}$ be such that $M_p \subset \mathscr{A} \subset UC_p$ (this includes the cases UC_p, M_p, $WAP_p \cap UC_p$, $\overline{AP_p + M_p}$ (closure in the norm topology), or WAP_p with G amenable in the last two cases). Then applying Theorem 4.10, we see that $v(M) \oplus PF_p^{\perp}$ is the required ideal if we choose M as a maximal regular ideal of W_p not containing $A_p(G)$, that is, $M = \{w \in W_p\colon w(x) = 0\}$ for some $x \in G$.

When $\mathscr{A} = PM_p$, $\overline{AP_p + M_p}$ or WAP_p with G arbitrary, we consider the map ${}^t\imath : \mathscr{A}' \to M_p'$ (the transpose of the inclusion map $\imath : M_p \to \mathscr{A}$), and let $J = v(M) \oplus PF_p^{\perp}$ be the maximal ideal given above in M_p'. Then $({}^t\imath)^{-1}(J)$ is the required ideal in $\mathscr{A}'$.

(2) The proof of the necessity is similar to part (1). The sufficiency part of the statement can be proved by an argument similar to [30, Theorem 4.8, p. 59]. □

Remarks 4.14. (1) For $\mathscr{A} = PF_p$ the first statement in part (1) of Corollary 4.13, is true even if G is not discrete. This follows from the fact that $A_p(G)$ is always an ideal of $PF_p' = W_p$. We do not know if the converse statement in part (1) holds for $C_{\delta,p}$ or AP_p in general. However if G is a compact group such that for each positive integer n the number of n-dimensional irreducible unitary representations of G is finite (such as $SU(2)$ [28, Theorem 29.27, p. 136]), then as shown by Dunkl and Ramirez [12, Theorem 11, p. 529], $PF_2 \subset AP_2$. This implies that for such groups, the converse holds also for AP_2 with the same method of proof.

(2) Corollary 4.13 was proved in [20, Theorem 4.11, p. 189] for $\mathscr{A} = PM_p$.

As our final result in this section we state a theorem that gives conditions under which *every* maximal regular left ideal of $\mathscr{A}'$ contains $I_x = \{u \in A_p(G)\colon u(x) = 0\}$ for some *unique* $x \in G$. This result extends a result of Ghahramani and Lau [20, Theorem 4.7, p. 187] which deals with a strongly regular Banach algebra A in $\mathscr{C}_0(X)$, where X is the spectrum of A. Using our Corollary 4.6 and Remarks 4.3, one can give a proof similar to that given by Ghahramani and Lau.

Theorem 4.15. *For an amenable group G the following are equivalent.*

(1) *The Banach algebra* $\mathscr{A}'$ *has an identity contained in the ideal* N *generated by* $A_p(G)$.
(2) *Every maximal regular left ideal* M *of* $\mathscr{A}'$ *contains* I_x *for some unique* $x \in G$.
(3) *The ideal* N *in* $\mathscr{A}'$ *generated by* $A_p(G)$ *contains a right identity of* $\mathscr{A}'$.

5. Minimal ideals

We start by generalizing the notion of topological invariance on $\mathscr{A}$ to that of topological x-invariance on $\mathscr{A}$, where $x \in G$. We then determine all the minimal idempotents and all the minimal left ideals in $\mathscr{A}'$.

Definition 5.1. An element Ψ in $\mathscr{A}'$ is *topologically x-invariant* for some $x \in G$ when

$$\langle \Psi, u \cdot T \rangle = u(x) \langle \Psi, T \rangle \quad \text{for all } u \in A_p(G) \text{ and } T \in \mathscr{A}.$$

In other words, $u \square \Psi = u \cdot \Psi = u(x)\Psi$, for all $u \in A_p(G)$. When $x = e$, Ψ is called *topologically invariant*. Since $PM_p{}'$ always has a topologically invariant element [22, Theorem 5, p. 123], so has $\mathscr{A}'$ by restriction.

Proposition 5.2. *An element* Ψ *in* $\mathscr{A}'$ *is topologically x-invariant for some* $x \in G$ *if and only if* Ψ_x *defined on* $\mathscr{A}$ *by*

$$\langle \Psi_x, T \rangle = \langle \Psi, \lambda_p(x) T \rangle$$

is topologically invariant.

Proof. It is easy to verify that $\lambda_p(x)\mathscr{A} = \mathscr{A}$, so in particular the action of Ψ on $\lambda_p(x)T$ is well defined. Suppose that Ψ is topologically x-invariant for some $x \in G$. Note first, that for every $u, v \in A_p(G)$ and $T \in \mathscr{A}$, we have

$$\begin{aligned}
\langle \lambda_p(x)(u \cdot T), v \rangle &= \langle u \cdot T, \lambda_p(x^{-1}) v \rangle \\
&= \langle u \cdot T, {}_x v \rangle \\
&= \langle T, u({}_x v) \rangle \\
&= \langle T, {}_x({}_{x^{-1}} u v) \rangle \\
&= \langle \lambda_p(x) T, {}_{x^{-1}} u v \rangle \\
&= \langle {}_{x^{-1}} u \cdot (\lambda_p(x) T), v \rangle .
\end{aligned}$$

Thus $\lambda_p(x)(u \cdot T) = {}_{x^{-1}}u \cdot (\lambda_p(x)T)$. Consequently,

$$\begin{aligned}\langle \Psi_x, u \cdot T \rangle &= \langle \Psi, \lambda_p(x)(u \cdot T) \rangle \\ &= \langle \Psi, {}_{x^{-1}}u \cdot (\lambda_p(x)T) \rangle \\ &= {}_{x^{-1}}u(x) \langle \Psi, \lambda_p(x)T \rangle \quad \text{(since } \Psi \text{ is topologically } x\text{-invariant)} \\ &= u(e) \langle \Psi_x, T \rangle,\end{aligned}$$

showing that Ψ_x is topologically invariant.

The converse follows in a similar fashion. $\square$

Recall that an element $\Psi \in \mathscr{A}'$ is a *right annihilator of* $\mathscr{A}'$ when $\mathscr{A}' \square \Psi = \{0\}$. As may be known, an element $\Psi \in \mathscr{A}'$ is a right annihilator of $\mathscr{A}'$ if and only if it is zero on $A_p(G) \cdot \mathscr{A}$. For, suppose that $\Psi(A_p(G) \cdot \mathscr{A}) = \{0\}$, then for every $u \in A_p(G)$, $\langle u \cdot \Psi, T \rangle = \langle \Psi, u \cdot T \rangle = 0$ $(T \in \mathscr{A})$. Accordingly $\Phi \square \Psi = \lim_\alpha u_\alpha \square \Psi = 0$ if $\Phi \in \mathscr{A}'$ and (u_α) is a net in $A_p(G)$ converging to Φ in $\sigma(\mathscr{A}', \mathscr{A})$ (Remarks 4.3(1) and (3)). The converse is clear.

Proposition 5.3. *If $\Psi \in \mathscr{A}'$ is either topologically x-invariant for some $x \in G$ or a non-zero right annihilator of $\mathscr{A}'$, then the left ideal generated by Ψ in $\mathscr{A}'$ is of dimension one, and hence is minimal.*

Proof. Suppose that $\Psi \in \mathscr{A}'$ is topologically x-invariant for some $x \in G$. Note first, that for every $T \in \mathscr{A}$ and $u \in A_p(G)$, we have

$$\langle \Psi \odot T, u \rangle = \langle \Psi, u \cdot T \rangle = u(x) \langle \Psi, T \rangle = \langle \Psi(T)\lambda_p(x), u \rangle,$$

and so $\Psi \odot T = \Psi(T)\lambda_p(x)$. It follows that for every $\Phi \in \mathscr{A}'$ and $T \in \mathscr{A}$, we have

$$\begin{aligned}\langle \Phi \square \Psi, T \rangle &= \langle \Phi, \Psi \odot T \rangle = \langle \Phi, \Psi(T)\lambda_p(x) \rangle = \Phi(\lambda_p(x))\Psi(T) \\ &= \Phi(\lambda_p(x)) \langle \Psi, T \rangle.\end{aligned}$$

In other words,

$$\Phi \square \Psi = \Phi(\lambda_p(x))\Psi, \tag{$*$}$$

showing that the left ideal generated by Ψ in $\mathscr{A}'$ is $\mathbf{C}\Psi$, as required.

When Ψ is a right annihilator of $\mathscr{A}'$, the assertion is clear. $\square$

Recall that in a Banach algebra B, *a minimal idempotent* is a non-zero element μ satisfying $\mu^2 = \mu$ and $\mu B \mu = \mathbf{C}\mu$. In the group algebra $L^1(G)$, the minimal idempotents are the coefficients of integrable representations of G, see [4]. Since $L^1(G)$ is an ideal of the measure algebra $M^1(G)$, the minimal idempotents of $M^1(G)$ are in fact in $L^1(G)$ and so they are also coefficients of integrable representations.

The same is true for the minimal idempotents of $L^1(G)''$ when G is compact since $L^1(G)$ is also an ideal of $L^1(G)''$ in this case. When G is not compact the minimal idempotents are not fully known in any of the algebras $L^1(G)''$, $LUC(G)'$ and $WAP(G)'$ even for the case of G abelian and discrete, see [3]. However for the algebras considered in this paper, the minimal idempotents are completely determined by the following theorem.

Theorem 5.4. *An element Ψ in $\mathscr{A}'$ is a minimal idempotent if and only if it is topologically x-invariant for some $x \in G$ with $\Psi(\lambda_p(x)) = 1$.*

Proof. Let Ψ be topologically x-invariant with $\Psi(\lambda_p(x)) = 1$, and let Φ be any element in $\mathscr{A}'$. Then by $(*)$, $\Psi \Box \Psi = \Psi(\lambda_p(x))\Psi = \Psi$, and $\Psi \Box \Phi \Box \Psi = \Phi(\lambda_p(x))\Psi(\lambda_p(x))\Psi = \Phi(\lambda_p(x))\Psi$, as required.

Conversely, suppose that $\Psi \in \mathscr{A}'$ is a minimal idempotent. Then, for every $u \in A_p(G)$ there exists some $F(u) \in \mathbf{C}$ such that

$$u \cdot \Psi = u \Box \Psi = u \Box \Psi^2 = \Psi \Box u \Box \Psi = F(u)\Psi.$$

Since $\Psi \Box (uv) \Box \Psi = F(uv)\Psi$, and

$$\Psi \Box (uv) \Box \Psi = (u \Box \Psi) \Box (v \Box \Psi) = (F(u)\Psi) \Box (F(v)\Psi) = F(u)F(v)\Psi,$$

we see that $F(uv) = F(u)F(v)$ and so F is a homomorphism on the algebra $A_p(G)$. Next we check that F is not trivial on $A_p(G)$. Suppose otherwise, that is, $F(u) = 0$ for all $u \in A_p(G)$. Let (u_α) be a net in $A_p(G)$ converging to Ψ in $\sigma(\mathscr{A}', \mathscr{A})$. Then

$$\Psi = \Psi \Box \Psi \Box \Psi = \lim_\alpha u_\alpha \Box \Psi \Box \Psi = \lim_\alpha \Psi \Box u_\alpha \Box \Psi = \lim_\alpha F(u_\alpha)\Psi = 0,$$

which is impossible. So F is in the spectrum of $A_p(G)$, that is for some $x \in G$, $F(u) = u(x)$ $(u \in A_p(G))$ [26, Theorem 3, p. 102]. Accordingly, $u \cdot \Psi = u(x)\Psi$, showing that Ψ is topologically x-invariant. Furthermore, by $(*)$, $\Psi \Box \Psi = \Psi(\lambda_p(x))\Psi = \Psi$, showing that $\Psi(\lambda_p(x)) = 1$ and completing the proof. $\Box$

Theorem 5.5. *Let B be any of the algebras $A_p(G)$, $W_p(G)$, $B_p(G)$, or $MA_p(G)$. Then minimal idempotents exist in B if and only if G is discrete. In such a case the minimal idempotents are given by the characteristic functions χ_x of single point sets $\{x\}$ $(x \in G)$.*

Proof. Let w be a minimal idempotent in B. Then from the commutativity of B, $vw = F(v)w$ $(F(v) \in \mathbf{C})$ for each $v \in B$. As in the proof of Theorem 5.4, F is a homomorphism on B. Since by definition $w \neq 0$, we must have $F|_{A_p} \neq 0$. It follows from Remarks 4.3(2), that $F = \varepsilon_x$ for some $x \in G$. In other words, $vw = v(x)w$ $(v \in B)$, and in particular $w(x) = 1$. Now we check that $w(y) = 0$ for all $y \in G$, $y \neq x$. Pick $v \in B$ such that $v(y) \neq v(x)$ (since $A_p(G)$ separates the points of G, so does B). This yields the following identity $v(y)w(y) = (vw)(y) = v(x)w(y)$, which holds only if $w(y) = 0$. Thus G must be discrete and $w = \chi_x$.

The converse is clear. □

We recall that an algebra B is *semi-prime* if $\{0\}$ is the only two sided ideal J of B with $J^2 = \{0\}$. As known (see for example [5, Proposition IV.30.6, p. 155]), for semi-prime algebras, minimal idempotents determine all the minimal left (and right) ideals. Since the algebras $A_p(G)$, $W_p(G)$, $B_p(G)$, and $MA_p(G)$ are semi-prime, we have the following corollary.

Corollary 5.6. *Let B be any of the algebras $A_p(G)$, $W_p(G)$, $B_p(G)$, or $MA_p(G)$. Then minimal ideals M exist in B if and only if G is discrete and $M = \mathbf{C}\chi_x$ for some $x \in G$.*

Proof. Let M be a minimal ideal of B, and let by Bonsall and Duncan [5, Proposition IV.30.6, p. 155] w be a minimal idempotent in B generating M, that is, $M = Bw$. Thus G must be discrete and $w = \chi_x$ for some $x \in G$. Consequently, $M = B\chi_x = \mathbf{C}\chi_x$, as required.

The converse is clear. □

The algebra $\mathscr{A}'$ is in general not semi-prime as we will see in Proposition 5.7. Notice that the proof given below in statement (i) of Proposition 5.7 provides also new elements in the radical of $\mathscr{A}'$. However, using different techniques, we are able to show in Theorem 5.8 that indeed the non-zero right annihilators of $\mathscr{A}'$ (when they exist) and the topologically x-invariant elements of $\mathscr{A}'$ $(x \in G)$ determine completely the minimal left ideals of $\mathscr{A}'$ and also the minimal right ideals when $\mathscr{A} = AP_p$ or $\mathscr{A} = WAP_p$ (Remark 5.9).

Proposition 5.7. (1) *If G is not discrete, then the algebras ${UC_p}'$ and ${PM_p}'$ are not semi-prime.*

(2) *If G is not compact, then the algebra ${PM_2}'$ is not semi-prime.*

Proof. (1) In this proof we assume that $\mathscr{A}$ stands for either PM_p or UC_p. For each $x \in G$, let

$$M_x = \{\Psi \in \mathscr{A}' \colon \Psi \text{ is topologically } x\text{-invariant and } \Psi(\lambda_p(x)) = 0\}.$$

Note that the elements of M_x are zero at every $\lambda_p(y)$, $(y \in G)$: for if $y \neq x$ and if $u \in A_p(G)$ is such that $u(x) \neq u(y)$, then $\langle u \cdot \lambda_p(y), v \rangle = \langle \lambda_p(y), uv \rangle = u(y) \langle \lambda_p(y), v \rangle$, that is, $u \cdot \lambda_p(y) = u(y)\lambda_p(y)$; then

$$u(x)\langle \Psi, \lambda_p(y) \rangle = \langle \Psi, u \cdot \lambda_p(y) \rangle = \langle \Psi, u(y)\lambda_p(y) \rangle = u(y)\langle \Psi, \lambda_p(y) \rangle,$$

which is clearly true only if $\langle \Psi, \lambda_p(y) \rangle = 0$, as required. We check now that M_x is not trivial. We start with $x = e$. By Granirer [24, Theorem 6, p. 3401] there are at least two distinct topologically invariant elements Ψ_1 and Ψ_2 on PM_p. The restrictions of Ψ_1 and Ψ_2 to UC_p are topologically invariant and stay distinct;

otherwise, for every $u \in A_p(G)$ and $T \in PM_p$, we have

$$u(e)\langle \Psi_1, T\rangle = \langle \Psi_1, u\cdot T\rangle = \langle \Psi_2, u\cdot T\rangle = u(e)\langle \Psi_2, T\rangle,$$

and so $\Psi_1 = \Psi_2$ on PM_p. Now with a slight abuse of language let Ψ_1 and Ψ_2 be two topologically invariant elements of $\mathscr{A}'$. Let $\Psi = \Psi_1 - \Psi_2$. Then $\Psi \neq 0$ and $\Psi(\lambda_p(e)) = 0$ showing that $M_e \neq \{0\}$. To see that $M_x \neq \{0\}$ for every $x \in G$, we just need to observe that $\Psi_{x^{-1}} \in M_x$ if $\Psi \in M_e$ (see Proposition 5.2). Now let $\mathscr{R}$ be the linear span of $\cup_x M_x$ and $\overline{\mathscr{R}}$ its norm closure in $\mathscr{A}'$. We claim that $(\mathscr{A}' \square \Psi \square \mathscr{A}')^2 = \{0\}$ for every $\Psi \in \overline{\mathscr{R}}$. It suffices to show that $\Psi \square \Phi \square \Psi = 0$ for every $\Phi \in \mathscr{A}'$. Let us first assume that $\Psi = \sum_{i=1}^n a_i \Psi_i$ where $a_i \in \mathbf{C}$ and $\Psi_i \in M_{x_i}$. Then, by $(*)$

$$\begin{aligned}\Psi \square \Phi \square \Psi &= \Psi \square \left(\sum_{i=1}^n a_i \Phi(\lambda_p(x_i))\Psi_i\right)\\ &= \sum_{i=1}^n a_i \Phi(\lambda_p(x_i))\Psi(\lambda_p(x_i))\Psi_i\\ &= \sum_{i=1}^n a_i \Phi(\lambda_p(x_i))\left(\sum_{j=1}^n a_j \Psi_j(\lambda_p(x_i))\right)\Psi_i\\ &= 0.\end{aligned}$$

Now let $\Psi \in \overline{\mathscr{R}}$ be arbitrary. For $\varepsilon > 0$, pick Ψ' in $\mathscr{R}$ such that $||\Psi - \Psi'|| < \varepsilon$. Then, for every $\Phi \in \mathscr{A}'$,

$$\begin{aligned}||\Psi \square \Phi \square \Psi|| &= ||\Psi \square \Phi \square \Psi - \Psi' \square \Phi \square \Psi'||\\ &\leqslant ||\Psi \square \Phi \square \Psi - \Psi \square \Phi \square \Psi'|| + ||\Psi \square \Phi \square \Psi' - \Psi' \square \Phi \square \Psi'||\\ &\leqslant ||\Psi \square \Phi|| ||\Psi - \Psi'|| + ||\Psi - \Psi'|| ||\Phi \square \Psi'||\\ &< ||\Phi||(2||\Psi|| + 1)\varepsilon.\end{aligned}$$

Thus $\Psi \square \Phi \square \Psi = 0$, as required.

For statement (2), see the proof of [30, Theorem 3.4, p. 56]. $\square$

Theorem 5.8. *A left ideal M in $\mathscr{A}'$ is minimal if and only if $M = \mathbf{C}\Psi$, where Ψ is either a non-zero right annihilator of $\mathscr{A}'$ or a topologically x-invariant element for some $x \in G$.*

Proof. The sufficiency part of the theorem was proved earlier in Proposition 5.3. We prove the necessity. Let M be a minimal left ideal of $\mathscr{A}'$ and let Ψ be a non-zero element in M. If $\Phi \square \Psi = 0$ for every $\Phi \in \mathscr{A}'$, then Ψ is a right annihilator of $\mathscr{A}'$ and $M = \mathbf{C}\Psi$. Otherwise, consider

$$\Psi^{\perp} = \{\Phi \in \mathscr{A}' \colon \Phi \square \Psi = 0\}.$$

By Filali [17, Lemma 3.4, p. 574], $\Psi^{\perp}$ is a maximal regular left ideal of $\mathscr{A}'$. Since $\mathscr{A}' \square \Psi = M$, let us pick $\Theta \in \mathscr{A}'$ such that $\Theta \square \Psi = \Psi$. Such a Θ is then a right identity of $\mathscr{A}'$ modulo $\Psi^{\perp}$. Since the map $\Phi \mapsto \Phi \square \Phi'$ is weak* continuous for each $\Phi' \in \mathscr{A}'$, we see that $\Psi^{\perp}$ is weak* closed. Therefore, by Theorem 4.8, there exists $x \in G$ such that

$$\Psi^{\perp} = \{\Phi \in \mathscr{A}' \colon \Phi(\lambda_p(x)) = 0\}.$$

Since Θ is a right identity modulo $\Psi^{\perp}$, we have $u - u \square \Theta \in \Psi^{\perp}$ $(u \in A_p(G))$, and so

$$(u - u \square \Theta)(\lambda_p(x)) = u(x) - u(x)\Theta(\lambda_p(x)) = 0$$

for every $u \in A_p(G)$; by taking $u(x) = 1$, this shows that $\Theta(\lambda_p(x)) = 1$. Then, for every $u \in A_p(G)$, $u - u(x)\Theta \in \Psi^{\perp}$ since

$$(u - u(x)\Theta)(\lambda_p(x)) = u(x) - u(x)\Theta(\lambda_p(x)) = 0.$$

It follows that $(u - u(x)\Theta) \square \Psi = 0$, and so

$$u \cdot \Psi = u \square \Psi = u(x)\Theta \square \Psi = u(x)\Psi,$$

which shows that Ψ is topologically x-invariant. That $M = \mathbf{C}\Psi$ follows from Proposition 5.3. $\square$

Remark 5.9. The algebras $WAP_p{}'$ and $AP_p{}'$ are commutative (for $p = 2$, see [29, Theorem 5.6, p. 50], for p arbitrary the proof is similar). So Theorem 5.8 determines also *all* the minimal right ideals in these algebras. In the other cases, namely $UC_p{}'$ and $PM_p{}'$ we have been unable to give a full answer. This seems to be difficult when G is not discrete even when it is abelian. Below we give a partial answer.

Proposition 5.10. *Let $\mathscr{A}$ be UC_p or PM_p. Then minimal right ideals M with $M^2 \neq \{0\}$ exist in $\mathscr{A}'$ if and only if G is discrete.*

Proof. Let M be a minimal right ideal in $\mathscr{A}'$ with $M^2 \neq \{0\}$. Then by Bonsall and Duncan [5, Lemma IV.30.2, p. 154], $M = \Psi \square \mathscr{A}'$ for some minimal idempotent Ψ in M. Then, by Theorem 5.4, Ψ is topologically x-invariant with $\Psi(\lambda_p(x)) = 1$ for some $x \in G$. Let

$$M_x = \{\Phi \in \mathscr{A}' \colon \Phi \text{ is topologically } x\text{-invariantand } \Phi(\lambda_p(x)) = 0\}.$$

Then M_x is a right ideal in $\mathscr{A}'$. Moreover, by $(*)$ (see Proposition 5.3),

$$\Psi \square \Phi = \Psi(\lambda_p(x))\Phi = \Phi$$

for every $\Phi \in \mathscr{A}'$ that is topologically x-invariant, and in particular for every $\Phi \in M_x$. Therefore $M_x = \Psi \square M_x \subset M$, and so M_x must be trivial. This means that there exists a unique topologically x-invariant (or equivalently invariant), element (up to a multiplicative constant) in $\mathscr{A}'$. Now by Granirer [24, Theorem 6, p. 3401], G must be discrete.

The converse for the case $\mathscr{A} = PM_p$ follows from Lemma 4.12(1), and Theorem 5.5. For the case of UC_p the converse follows from Theorem 5.8 and the commutativity of $UC_p{}'$ when G is discrete (one can prove the later statement by an argument similar to [29, Theorem 5.6, p. 50]). We leave the details for the reader. □

Acknowledgments

This work started when the M.F. was visiting the University of Alberta in Spring 2002 and put in its final form when he was visiting University of Lausanne in Winter 2003. He wishes to thank Professors A. Lau, T. Miao, and A. Derighetti for their very kind hospitality. M.F. was supported in part by Finnish academy Grants 76919 and 202084. The work of M.S.M. has been supported by a generous grant from the Swiss National Foundation for Scientific Research (FNRS). The author thanks Professors A. Derighetti and A. Lau for their warm hospitality and many helpful discussions.

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