SECOND HANKEL DETERMINANT FOR BI-STARLIKE AND BI-CONVEX FUNCTIONS OF ORDER β

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Abstract. In the present investigation the authors obtain upper bounds for the second Hankel determinant $H_2(2)$ of the classes bi-starlike and bi-convex functions of order β, represented by $S_\sigma^*(\beta)$ and $K_\sigma(\beta)$, respectively. In particular, the estimates for the second Hankel determinant $H_2(2)$ of bi-starlike and bi-convex functions which are important subclasses of bi-univalent functions are pointed out.

1. Introduction and definitions

Let $\mathcal{A}$ denote the family of functions f analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. The Koebe one-quarter theorem (see [7]) ensures that the image of $\mathcal{U}$ under every $f \in \mathcal{S}$ contain a disk of radius $1/4$. So, every $f \in \mathcal{S}$ has an inverse function f^{-1} satisfying $f^{-1}(f(z)) = z$ $(z \in \mathcal{U})$ and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f);\ r_0(f) \geq 1/4)$$

where $f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1).

Two of the most famous subclasses of univalent functions are the class $\mathcal{S}^*(\beta)$ of starlike functions of order β and the class $\mathcal{K}(\beta)$ of convex functions of order β. By definition, we have

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta;\ z \in \mathcal{U};\ 0 \leq \beta < 1 \right\}$$

and

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta;\ z \in \mathcal{U};\ 0 \leq \beta < 1 \right\}.$$

The classes consisting of starlike and convex functions are usually denoted by $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{K} = \mathcal{K}(0)$, respectively.

For $0 \leq \beta < 1$, a function $f \in \sigma$ is in the class $\mathcal{S}_\sigma^*(\beta)$ of *bi-starlike functions of order* β, or $\mathcal{K}_\sigma(\beta)$ of *bi-convex functions of order* β if both f and its inverse map f^{-1} are, respectively, starlike or convex of order β. These classes were introduced by Brannan and Taha [2] in 1985. Especially the classes $\mathcal{S}_\sigma^*(0) = \mathcal{S}_\sigma^*$ and $\mathcal{K}_\sigma(0) = \mathcal{K}_\sigma$ are *bi-starlike* and *bi-convex functions*, respectively. In 1967, Lewin [17] showed that for every functions $f \in \sigma$ of the form (1.1), the second coefficient of f satisfy the inequality $|a_2| < 1.51$. In 1967, Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. Later, Netanyahu [18] proved that $\max_{f\in\sigma} |a_2| = 4/3$. In 1985, Kedzierawski [13] proved Brannan and Clunie's conjecture for $f \in \mathcal{S}_\sigma^*$. In 1985, Tan [25] obtained the bound for a_2 namely $|a_2| < 1.485$ which is the best known estimate for functions in the class σ. Brannan and Taha [2] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes $\mathcal{S}_\sigma^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$. Recently, Deniz [6] and Kumar et al. [15] both extended and improved the results of Brannan and Taha [2] by generalizing their classes using subordination. The problem of estimating coefficients $|a_n|$, $n \geq 2$ is still open. However, a lot of results for $|a_2|$, $|a_3|$ and $|a_4|$ were proved for

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some subclasses of σ (see [3], [5], [9], [11], [21], [23], [24], [26], [27]). Unfortunatelly, none of them are not sharp.

One of the important tools in the theory of univalent functions is Hankel Determinants which are utility, for example, in showing that a function of bounded characteristic in $\mathcal{U}$, i.e., a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [4]. The Hankel determinants [19] $H_q(n)$ $(n = 1, 2, ..., \; q = 1, 2, ...)$ of the function f are defined by

$$H_q(n) = \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{bmatrix} \quad (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$. For example, we can know that the functional $H_2(1) = a_3 - a_2^2$ is known as the Fekete-Szegö functional and they consider the further generalized functional $a_3 - \mu a_2^2$ where μ is some real number (see, [8]). In 1969, Keogh and Merkes [14] proved the Fekete-Szegö problem for the classes $\mathcal{S}^*$ and $\mathcal{K}$. Someone can see the Fekete-Szegö problem for the classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ at special cases in the paper of Orhan *et.al.* [20]. On the other hand, very recently Zaprawa [28], [29] have studied on Fekete-Szegö problem for some classes of bi-univalent functions. In special cases, he gave Fekete-Szegö problem for the classes $\mathcal{S}_\sigma^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$. In 2014, Zaprawa [28] proved the following resuts for $\mu \in \mathbb{R}$,

$$f \in \mathcal{S}_\sigma^*(\beta) \Rightarrow \left|a_3 - \mu a_2^2\right| \leq \begin{cases} 1-\beta; & \frac{1}{2} \leq \mu \leq \frac{3}{2} \\ 2(1-\beta)\left|\mu - 1\right|; & \mu \geq \frac{3}{2} \text{ and } \mu \leq \frac{1}{2} \end{cases}$$

and

$$f \in \mathcal{K}_\sigma(\beta) \Rightarrow \left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1-\beta}{3}; & \frac{2}{3} \leq \mu \leq \frac{4}{3} \\ (1-\beta)\left|\mu - 1\right|; & \mu \geq \frac{4}{3} \text{ and } \mu \leq \frac{2}{3} \end{cases}.$$

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2 a_4 - a_3^2$. The bounds for the second Hankel determinant $H_2(2)$ obtained for the classes $\mathcal{S}^*$ and $\mathcal{K}$ in [12]. Recently, Lee et al. [16] established the sharp bound to $|H_2(2)|$ by generalizing their classes using subordination. In their paper, one can find the sharp bound to $|H_2(2)|$ for the functions in the classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$.

In this paper, we seek upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for functions f belonging to the classes $\mathcal{S}_\sigma^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$.

Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $\mathcal{P}$: $\mathcal{U} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\Re p(z) > 0$.

To establish our main results, we shall require the following lemmas.

Lemma 1.1. [22] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \tag{1.2}$$

then the sharp estimate $|c_k| \leq 2$ $(k = 1, 2, ...)$ holds.

Lemma 1.2. [10] *If the function $p \in \mathcal{P}$ is given by the series (1.2), then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.3}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)\left(1 - |x|^2\right)z, \tag{1.4}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2. Main results

Our first main result for the class $\mathcal{S}_\sigma^*(\beta)$ as follows:

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_\sigma^*(\beta)$, $0 \leq \beta < 1$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{4}{3}(1-\beta)^2\left(4\beta^2 - 8\beta + 5\right), & \beta \in \left[0, \frac{29-\sqrt{137}}{32}\right] \\ (1-\beta)^2\left(\frac{13\beta^2 - 14\beta - 7}{16\beta^2 - 26\beta + 5}\right), & \beta \in \left(\frac{29-\sqrt{137}}{32}, 1\right). \end{cases} \tag{2.1}$$

Proof. Let $f \in \mathcal{S}_\sigma^*(\beta)$ and $g = f^{-1}$. Then

$$\frac{zf'(z)}{f(z)} = \beta + (1-\beta)p(z) \text{ and } \frac{wg'(w)}{g(w)} = \beta + (1-\beta)q(w) \tag{2.2}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + ...$ and $q(w) = 1 + d_1 w + d_2 w^2 + ...$ in $\mathcal{P}$.

Comparing coefficients in (2.2), we have

$$a_2 = (1-\beta)c_1, \tag{2.3}$$
$$2a_3 - a_2^2 = (1-\beta)c_2, \tag{2.4}$$
$$3a_4 - 3a_3 a_2 + a_2^3 = (1-\beta)c_3 \tag{2.5}$$

and

$$-a_2 = (1-\beta)d_1, \tag{2.6}$$
$$3a_2^2 - 2a_3 = (1-\beta)d_2, \tag{2.7}$$
$$-10a_2^3 + 12a_3 a_2 - 3a_4 = (1-\beta)d_3. \tag{2.8}$$

From (2.3) and (2.6), we arrive at

$$c_1 = -d_1 \tag{2.9}$$

and

$$a_2 = (1-\beta)c_1. \tag{2.10}$$

Now, from (2.4), (2.7) and (2.10), we get that

$$a_3 = (1-\beta)^2 c_1^2 + \frac{(1-\beta)}{4}(c_2 - d_2). \tag{2.11}$$

Also, from (2.5) and (2.8), we find that

$$a_4 = \frac{2}{3}(1-\beta)^3 c_1^3 + \frac{5}{8}(1-\beta)^2 c_1 (c_2 - d_2) + \frac{1}{6}(1-\beta)(c_3 - d_3). \tag{2.12}$$

Thus, we can easily establish that

$$\begin{aligned} \left|a_2 a_4 - a_3^2\right| = \Big| &-\frac{1}{3}(1-\beta)^4 c_1^4 + \frac{1}{8}(1-\beta)^3 c_1^2 (c_2 - d_2) \\ &+\frac{1}{6}(1-\beta)^2 c_1 (c_3 - d_3) - \frac{1}{16}(1-\beta)^2 (c_2 - d_2)^2 \Big|. \end{aligned} \tag{2.13}$$

According to Lemma 1.2 and (2.9), we write

$$\left.\begin{array}{c} 2c_2 = c_1^2 + x(4 - c_1^2) \\ 2d_2 = d_1^2 + x(4 - d_1^2) \end{array}\right\} \Longrightarrow c_2 - d_2 = \frac{4 - c_1^2}{2}(x - y) \tag{2.14}$$

and

$$\begin{aligned} 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)\left(1 - |x|^2\right) z, \\ 4d_3 &= d_1^3 + 2(4 - d_1^2)d_1 y - d_1(4 - d_1^2)y^2 + 2(4 - d_1^2)\left(1 - |y|^2\right) w, \end{aligned}$$

(2.15)

$$c_3 - d_3 = \frac{c_1^3}{2} + \frac{c_1\left(4 - c_1^2\right)}{2}(x + y) - \frac{c_1\left(4 - c_1^2\right)}{2}(x^2 + y^2) + \frac{\left(4 - c_1^2\right)}{2}\left(\left(1 - |x|^2\right) z - \left(1 - |y|^2\right) w\right).$$

for some x, y, z, w with $|x| \leq 1, |y| \leq 1, |z| \leq 1$ and $|w| \leq 1$.Using (2.14) and (2.15) in (2.13), and applying the triangle inequality we have

$$\begin{aligned}
\left|a_2 a_4 - a_3^2\right| &= \Bigg| -\frac{1}{3}(1-\beta)^4 c_1^4 + \frac{1}{16}(1-\beta)^3 c_1^2(4-c_1^2)(x-y) \\
&+ \frac{1}{6}(1-\beta)^2 c_1 \left[\frac{c_1^3}{2} + \frac{(4-c_1^2)c_1}{2}(x+y) - \frac{(4-c_1^2)c_1}{4}(x^2+y^2) + \frac{(4-c_1^2)}{2}\Big((1-|x|^2)z - (1-|y|^2)w\Big)\right] \\
&- \frac{1}{64}(1-\beta)^2 (4-c_1^2)^2 (x-y)^2 \Bigg| \\
\leq\ & \frac{1}{3}(1-\beta)^4 c_1^4 + \frac{1}{12}(1-\beta)^2 c_1^4 + \frac{1}{6}(1-\beta)^2 c_1(4-c_1^2) \\
&+ \left[\frac{1}{16}(1-\beta)^3 c_1^2(4-c_1^2) + \frac{1}{12}(1-\beta)^2 c_1^2(4-c_1^2)\right](|x|+|y|) \\
&+ \left[\frac{1}{24}(1-\beta)^2 c_1^2(4-c_1^2) - \frac{1}{12}(1-\beta)^2 c_1(4-c_1^2)\right](|x|^2+|y|^2) + \frac{1}{64}(1-\beta)^2 (4-c_1^2)^2(|x|+|y|)^2.
\end{aligned}$$

Since $p \in \mathcal{P}$, so $|c_1| \leq 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Thus, for $\lambda = |x| \leq 1$ and $\mu = |y| \leq 1$ we obtain

$$\left|a_2 a_4 - a_3^2\right| \leq T_1 + T_2(\lambda + \mu) + T_3(\lambda^2 + \mu^2) + T_4(\lambda + \mu)^2 = F(\lambda, \mu)$$

where

$$\begin{aligned}
T_1 &= T_1(c) = \frac{(1-\beta)^2}{12}\left[\left(1 + 4(1-\beta)^2\right)c^4 - 2c^3 + 8c\right] \geq 0, \\
T_2 &= T_2(c) = \frac{1}{48}(1-\beta)^2 c^2(4-c^2)(7-3\beta) \geq 0, \\
T_3 &= T_3(c) = \frac{1}{24}(1-\beta)^2 c(4-c^2)(c-2) \leq 0, \\
T_4 &= T_4(c) = \frac{1}{64}(1-\beta)^2 (4-c_1^2)^2 \geq 0.
\end{aligned}$$

Now we need to maximize $F(\lambda, \mu)$ in the closed square $\mathbb{S} = \{(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in [0, 2)$, we conclude that

$$F_{\lambda\lambda} \cdot F_{\mu\mu} - (F_{\lambda\mu})^2 < 0.$$

Thus the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

For $\lambda = 0$ and $0 \leq \mu \leq 1$ (similarly $\mu = 0$ and $0 \leq \lambda \leq 1$), we obtain

$$F(0, \mu) = G(\mu) = (T_3 + T_4)\mu^2 + T_2\mu + T_1.$$

i. *The case* $T_3 + T_4 \geq 0$: In this case for $0 < \mu < 1$ and any fixed c with $0 \leq c < 2$, it is clear that $G'(\mu) = 2(T_3 + T_4)\mu + T_2 > 0$, that is, $G(\mu)$ is an increasing function. Hence, for fixed $c \in [0, 2)$, the maximum of $G(\mu)$ occurs at $\mu = 1$, and

$$\max G(\mu) = G(1) = T_1 + T_2 + T_3 + T_4.$$

ii. *The case* $T_3 + T_4 < 0$: Since $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \mu < 1$ and any fixed c with $0 \leq c < 2$, it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\mu + T_2 < T_2$ and so $G'(\mu) > 0$. Hence for fixed $c \in [0, 2)$, the maximum of $G(\mu)$ occurs at $\mu = 1$.

Also for $c = 2$ we obtain

$$F(\lambda, \mu) = \frac{4}{3}(1-\beta)^2 (4\beta^2 - 8\beta + 5). \tag{2.16}$$

Taking into account the value (2.16), and the cases *i* and *ii*, for $0 \leq \mu \leq 1$ and any fixed c with $0 \leq c \leq 2$,

$$\max G(\mu) = G(1) = T_1 + T_2 + T_3 + T_4.$$

For $\lambda = 1$ and $0 \leq \mu \leq 1$ (similarly $\mu = 1$ and $0 \leq \lambda \leq 1$), we obtain

$$F(1, \mu) = H(\mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + T_1 + T_2 + T_3 + T_4.$$

Similarly to the above cases of $T_3 + T_4$, we get that

$$\max H(\mu) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $G(1) \leq H(1)$ for $c \in [0,2]$, $\max F(\lambda,\mu) = F(1,1)$ on the boundary of the square $\mathbb{S}$. Thus the maximum of F occurs at $\lambda = 1$ and $\mu = 1$ in the closed square $\mathbb{S}$.

Let $K : [0,2] \to \mathbb{R}$

$$K(c) = \max F(\lambda,\mu) = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{2.17}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (2.17), yield

$$K(c) = \frac{(1-\beta)^2}{48}\left[\left(16\beta^2 - 26\beta + 5\right)c^4 + 24(2-\beta)c^2 + 48\right].$$

Assume that $K(c)$ has a maximum value in an interior of $c \in [0,2]$, by elementary calculation we find

$$K'(c) = \frac{(1-\beta)^2}{12}\left[\left(16\beta^2 - 26\beta + 5\right)c^3 + 12(2-\beta)c\right]. \tag{2.18}$$

As a result of some calculations we can do the following examine:

Case 1: Let $16\beta^2 - 26\beta + 5 \geq 0$, that is, $\beta \in \left[0, \frac{13-\sqrt{89}}{16}\right]$. Therefore $K'(c) > 0$ for $c \in (0,2)$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $c \in [0,2]$, that is, $c = 2$. Thus, we have

$$\max_{0 \leq c \leq 2} K(c) = K(2) = \frac{4}{3}(1-\beta)^2\left(4\beta^2 - 8\beta + 5\right).$$

Case 2: Let $16\beta^2 - 26\beta + 5 < 0$, that is, $\beta \in \left(\frac{13-\sqrt{89}}{16}, 1\right)$. Then $K'(c) = 0$ implies the real critical point $c_{0_1} = 0$ or $c_{0_2} = \sqrt{\frac{-12(2-\beta)}{16\beta^2-26\beta+5}}$. When $\beta \in \left(\frac{13-\sqrt{89}}{16}, \frac{29-\sqrt{137}}{32}\right]$, we observe that $c_{0_2} \geq 2$, that is, c_{0_2} is out of the interval $(0,2)$. Therefore the maximum value of $K(c)$ occurs at $c_{0_1} = 0$ or $c = c_{0_2}$ which contradicts our assumption of having the maximum value at the interior point of $c \in [0,2]$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $c \in [0,2]$, that is, $c = 2$. Thus, we have

$$\max_{0 \leq c \leq 2} K(c) = K(2) = \frac{4}{3}(1-\beta)^2\left(4\beta^2 - 8\beta + 5\right).$$

When $\beta \in \left(\frac{29-\sqrt{137}}{32}, 1\right)$ we observe that $c_{0_2} < 2$, that is, c_{0_2} is interior of the interval $[0,2]$. Since $K''(c_{0_2}) < 0$, the maximum value of $K(c)$ occurs at $c = c_{0_2}$. Thus, we have

$$\max_{0 \leq c \leq 2} K(c) = K(c_{0_2}) = K\left(\sqrt{\frac{-12(2-\beta)}{16\beta^2 - 26\beta + 5}}\right) = (1-\beta)^2\left(\frac{13\beta^2 - 14\beta - 7}{16\beta^2 - 26\beta + 5}\right).$$

This completes the proof of the Theorem 2.1. □

For $\beta = 0$, Theorem 2.1 readily yields the following coefficient estimates for bi-starlike functions.

Corollary 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}^*_\sigma$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{20}{3}.$$

Our second main result for the class $\mathcal{K}_\sigma(\beta)$ is following:

Theorem 2.3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{K}_\sigma(\beta)$, $0 \leq \beta < 1$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(1-\beta)^2}{24}\left(\frac{5\beta^2 + 8\beta - 32}{3\beta^2 - 3\beta - 4}\right) \tag{2.19}$$

Proof. Let $f \in \mathcal{K}_\sigma(\beta)$ and $g = f^{-1}$. Then

$$1 + \frac{zf''(z)}{f'(z)} = \beta + (1-\beta)p(z) \text{ and } 1 + \frac{wg''(w)}{g'(w)} = \beta + (1-\beta)q(w) \tag{2.20}$$

where $p(z) = 1 + c_1z + c_2z^2 + ...$ and $q(w) = 1 + d_1w + d_2w^2 + ...$ in $\mathcal{P}$.

Now, equating the coefficients in (2.20), we have

$$2a_2 = (1-\beta)c_1, \tag{2.21}$$

$$6a_3 - 4a_2^2 = (1-\beta)c_2, \tag{2.22}$$

$$12a_4 - 18a_3a_2 + 8a_2^3 = (1-\beta)c_3 \tag{2.23}$$

and

$$-2a_2 = (1-\beta)d_1, \tag{2.24}$$
$$8a_2^2-6a_3 = (1-\beta)d_2, \tag{2.25}$$
$$-32a_2^3+42a_3a_2-12a_4 = (1-\beta)d_3. \tag{2.26}$$

From (2.21) and (2.24), we arrive at

$$c_1=-d_1 \tag{2.27}$$

and

$$a_2=\frac{1}{2}(1-\beta)c_1. \tag{2.28}$$

Now, from (2.22), (2.25) and (2.28), we get that

$$a_3=\frac{1}{4}(1-\beta)^2c_1^2+\frac{1}{12}(1-\beta)(c_2-d_2). \tag{2.29}$$

Also, from (2.23) and (2.26), we find that

$$a_4=\frac{5}{48}(1-\beta)^3c_1^3+\frac{5}{48}(1-\beta)^2c_1(c_2-d_2)+\frac{1}{24}(1-\beta)(c_3-d_3). \tag{2.30}$$

Thus, we can easily establish that

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \Big|-\frac{1}{96}(1-\beta)^4c_1^4+\frac{1}{96}(1-\beta)^3c_1^2(c_2-d_2)\\ &\quad +\frac{1}{48}(1-\beta)^2c_1(c_3-d_3)-\frac{1}{144}(1-\beta)^2(c_2-d_2)^2\Big|.\end{aligned} \tag{2.31}$$

Using (2.14) and (2.15) in (2.31), we have

$$\begin{aligned}&\left|a_2a_4-a_3^2\right| = \Big|-\frac{1}{96}(1-\beta)^4c_1^4+\frac{1}{192}(1-\beta)^3c_1^2(4-c_1^2)(x-y)\\ &+\frac{1}{48}(1-\beta)^2c_1\left[\frac{c_1^3}{2}+\frac{(4-c_1^2)c_1}{2}(x+y)-\frac{(4-c_1^2)c_1}{4}(x^2+y^2)+\frac{(4-c_1^2)}{2}\Big((1-|x|^2)z-(1-|y|^2)w\Big)\right]\\ &-\frac{1}{288}(1-\beta)^2(4-c_1^2)^2(x-y)^2\Big|\\ \le\ &\frac{1}{96}(1-\beta)^4c_1^4+\frac{1}{96}(1-\beta)^2c_1^4+\frac{1}{48}(1-\beta)^2c_1(4-c_1^2)\\ &+\left[\frac{1}{192}(1-\beta)^3c_1^2(4-c_1^2)+\frac{1}{96}(1-\beta)^2c_1^2(4-c_1^2)\right](|x|+|y|)\\ &+\left[\frac{1}{192}(1-\beta)^2c_1^2(4-c_1^2)-\frac{1}{96}(1-\beta)^2c_1(4-c_1^2)\right](|x|^2+|y|^2)+\frac{1}{576}(1-\beta)^2(4-c_1^2)^2(|x|+|y|)^2.\end{aligned}$$

Since $p\in\mathcal{P}$, so $|c_1|\le 2$. Taking $c_1=c$, we may assume without restriction that $c\in[0,2]$. Thus, for $\lambda=|x|\le 1$ and $\mu=|y|\le 1$ we obtain

$$\left|a_2a_4-a_3^2\right|\le M_1+M_2(\lambda+\mu)+M_3(\lambda^2+\mu^2)+M_4(\lambda+\mu)^2=\Psi(\lambda,\mu)$$

where

$$\begin{aligned}M_1 &= M_1(c)=\frac{(1-\beta)^2}{96}\left[\left(1+(1-\beta)^2\right)c^4-2c^3+8c\right]\ge 0,\\ M_2 &= M_2(c)=\frac{1}{192}(1-\beta)^2c^2(4-c^2)(3-\beta)\ge 0,\\ M_3 &= M_3(c)=\frac{1}{192}(1-\beta)^2c(4-c^2)(c-2)\le 0,\\ M_4 &= M_4(c)=\frac{1}{576}(1-\beta)^2(4-c_1^2)^2\ge 0.\end{aligned}$$

Therefore we need to maximize $\Psi(\lambda,\mu)$ in the closed square $\mathbb{S}=\{(\lambda,\mu):0\le\lambda\le 1, 0\le\mu\le 1\}$. To show that the maximum of Ψ we can follow the maximum of F in the Theorem 2.1. Thus the maximum of Ψ occurs at $\lambda=1$ and $\mu=1$ in the closed square $\mathbb{S}$. Let $\Phi:[0,2]\to\mathbb{R}$ defined by

$$\Phi(c)=\max\Psi(\lambda,\mu)=\Psi(1,1)=M_1+2M_2+2M_3+4M_4. \tag{2.32}$$

Substituting the values of M_1, M_2, M_3 and M_4 in the function Φ given by (2.32), yield

$$\Phi(c) = \frac{(1-\beta)^2}{288}\left[\left(3\beta^2 - 3\beta - 4\right)c^4 + 4(8-3\beta)c^2 + 32\right].$$

Assume that $\Phi(c)$ has a maximum value in an interior of $c \in [0,2]$, by elementary calculation we find

$$\Phi'(c) = \frac{(1-\beta)^2}{72}\left[\left(3\beta^2 - 3\beta - 4\right)c^3 + 2(8-3\beta)c\right].$$

Setting $\Phi'(c) = 0$, since $0 < c < 2$, and $3\beta^2 - 3\beta - 4 < 0$ and $8 - 3\beta > 0$ for every $\beta \in [0,1)$ we have the real critical poin $c_{0_3} = \sqrt{\frac{2(3\beta-8)}{3\beta^2-3\beta-4}}$. Since $c_{0_3} \leq 2$ for every $\beta \in [0,1)$ and so $\Phi''(c_{0_3}) < 0$, the maximum value of $\Phi(c)$ corresponds to $c = c_{0_3}$, that is,

$$\max_{0<c<2} \Phi(c) = \Phi(c_{0_3}) = \Phi\left(\sqrt{\frac{2(3\beta-8)}{3\beta^2-3\beta-4}}\right) = \frac{(1-\beta)^2}{24}\left(\frac{5\beta^2+8\beta-32}{3\beta^2-3\beta-4}\right).$$

On the other hand,

$$\Phi(0) = \frac{(1-\beta)^2}{9} \text{ and } \Phi(2) = \frac{(1-\beta)^2}{6}\left(\beta^2 - 2\beta + 2\right).$$

Consequently, since $\Phi(0) < \Phi(2) \leq \Phi(c_{0_3})$ we obtain $\max\limits_{0\leq c<\leq 2} \Phi(c) = \Phi(c_{0_3})$.

This completes the proof of the Theorem 2.3. □

For $\beta = 0$, Theorem 2.3 readily yields the following coefficient estimates for bi-convex functions.

Corollary 2.4. *Let $f(z)$ given by (1.1) be in the class $\mathcal{K}_\sigma$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{3}.$$

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