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On certain transformations of poly-basic bilateral hypergeometric series

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Abstract

In this paper, we have established certain transformations of basic hypergeometric series with more than one base. Some of these lead to the relationship between product of two q-series. These results, in turn, lead to very interesting transformations of bi-basic and poly-basic q-series. A few of the results which are representative of the many results obtained are presented in this article.
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1. Introduction

A systematic theory of bi-basic hypergeometric series was established by Agarwal and Verma [2,3], yet not much break through could be achieved though a good number of results involving more than one base do exist in the literature. It has been a challenging job to develop a systematic theory of transformations of basic hypergeometric series with several bases.

In a series of communications, Denis et al. [4,5], Denis and Singh [6], Singh [9] making use of several series identities and sums of partial series, succeeded in establishing a number of transformations of poly-basic series.

Recently Gasper [7], made use of the following identity:

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} a_j \tag{1}$$

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and using a known indefinite summation established a transformation of a ${}_{10}\phi_9$ with four independent bases.

In this paper, we make use of the series identity:

$$\sum_{k=-m}^{n} a_{k+m} \sum_{j=0}^{n-k} A_j = \sum_{k=-m}^{n} A_{k+m} \sum_{j=0}^{n-k} a_j, \tag{2}$$

to establish transformations of poly-basic bilateral hypergeometric series in terms of another similar series, not necessarily having the same number of bases. Obviously, (2) follows from (1) by a shift of summation index.

In this article, we show through a few examples how many bi-basic bilateral transformations can be established. These transformations give rise to product transformation formulae, which for special values of the parameters lead to special transformation formulae for basic hypergeometric series.

2. Notation and definitions

A basic hypergeometric series is one where each of the parameters is a basic number, with the base being, say, $|q| < 1$. A generalization of this series is to have some parameters not all having the same base. By bi-basic hypergeometric series is meant a basic hypergeometric series in which some of the numerator and denominator parameters have the base q and the other numerator/denominator parameters have a different base, say, $|q_1| < 1$. A generalized bi-basic hypergeometric function in one variable is defined as

$$\Phi\left[\begin{matrix}(a)\colon(b);q,q_1;z\\(c)\colon(d);q^i,q_1^j\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[(a);q]_n[(b);q_1]_n z^n q^{i\ n(n-1)/2} q_1^{j\ n(n-1)/2}}{[q;q]_n[(c);q]_n[(d);q_1]_n}, \tag{3}$$

where (a) represents the sequence of A-parameters: $a_1 a_2 \cdots a_A$, and

$$[(a);q]_n = [a_1,a_2,\ldots,a_A;q]_n = [a_1;q]\cdots[a_A;q]_n \qquad \text{with}$$
$$[a;q]_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \qquad [a;q]_0 = 1.$$

The series (3) converges for $(|q|,|q_1|) < 1, |z| < \infty$, when $i,j > 0$ and $(|q|,|q_1|,|z|) < 1$, when $i = j = 0$. We also define a poly-basic hypergeometric series of one variable as

$$\Phi\left[\begin{matrix}a_1,a_2,\ldots,a_r\colon c_{1,1},\ldots,c_{1,r_1};\ldots;c_{m,1},\ldots,c_{m,r_m};q,q_1,\ldots q_m;z\\ b_1,b_2,\ldots,b_s\colon d_{1,1},\ldots,d_{1,s_1};\ldots;\ldots;d_{m,1},\ldots,d_{m,s_m}\end{matrix}\right]$$
$$=\sum_{n=0}^{\infty}\frac{[a_1,a_2,\ldots,a_r;q]_n z^n}{[q,b_1,b_2,\ldots,b_s;q]_n}\prod_{j=1}^{m}\frac{[c_{j,1},\ldots,c_{j,r_j};q_j]_n}{[d_{j,1},\ldots,d_{j,s_j};q_j]_n}. \tag{4}$$

A sum of terms u_r, where the index r is in the interval $[-\infty, \infty]$ is called a bilateral series, convergent under appropriate conditions, and the series may terminate on either or both sides. Most of the other notations are standard as in [8].

3. Main transformations

We show how we can establish our main transformations of bilateral basic hypergeometric series through a few selected examples. First we choose to exhibit a simple example: Let us take

$$a_k = \frac{[a, y; q_1]_k q_1^k}{[q_1, ayq_1; q_1]_k} \quad \text{and} \quad A_k = \frac{[\alpha, \beta; q]_k q^k}{[q, \alpha\beta q; q]_k}, \tag{5}$$

in (2) and use the known [1, App.II (8)] partial sum result:

$${}_2\Phi_1 \left[\begin{matrix} a, y; q; q \\ ayq \end{matrix} \right]_N = \frac{[aq, yq; q]_N}{[q, ayq; q]_N}, \tag{6}$$

to get

$$\begin{aligned} &{}_4\Psi_4 \left[\begin{matrix} aq_1^m, yq_1^m; q^{-n}, q^{-n}/\alpha\beta; q_1, q; q_1 \\ q_1^{l+m}, ayq_1^{l+m}; q^{-n}/\alpha, \ q^{-n}/\beta \end{matrix} \right] \\ &= \frac{[\alpha, \beta; q]_m [q_1, ayq_1; q_1]_m [aq_1, yq_1, q_1]_n [q, \alpha\beta q; q]_n q^m}{[q, \alpha\beta q; q]_m [a, y; q_1]_m [q_1, ayq_1; q_1]_n [\alpha q, \beta q; q]_n q_1^m} \\ &\quad \times {}_4\Psi_4 \left[\begin{matrix} \alpha q^m, \beta q^m; q_1^{-n}, q_1^{-n}/ay; q, q_1; q_1 \\ q^{l+m}, \alpha\beta q^{l+m}; q_1^{-n}/a, \ q_1^{-n}/y \end{matrix} \right], \end{aligned} \tag{7}$$

where the Ψ function represents the bi-basic bilateral series.

To illustrate the power of this method, we give an advanced example: let

$$a_k = \frac{(1 - ap^k q^k)(1 - bp^k q^{-k})[a, b; p]_k [c, a/bc; q]_k q^k}{(1 - a)(1 - b)[q, aq/b; q]_k [ap/c, bcp; p]_k}$$

and

$$A_k = \frac{[d, q_1\sqrt{d}, -q_1\sqrt{d}, e, f, g; q_1]_k q_1^k}{[q_1, \sqrt{d}, -\sqrt{d}, dq_1/e, dq_1/f, dq_1/g; q_1]_k}, \quad (d = efg). \tag{8}$$

Use these in (1) and make use of the following partial sum [1, App.II (25)]:

$${}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix} \right]_N = \frac{[aq, bq, cq, dq; q]_N}{[q, aq/b, aq/c, aq/d; q]_N}, \quad (a = bcd) \tag{9}$$

and the partial sum [9, App. II (35)]:

$$\sum_{k=0}^{n}\frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;\,p]_k[c,a/bc;q]_kq^k}{(1-a)(1-b)[q,aq/b;q]_k[ap/c,bcp;\,p]_k}$$

$$=\frac{[ap,bp;\,p]_n[cq,aq/bc;q]_n}{[q,aq/b;q]_n[ap/c,bcp;\,p]_n}, \tag{10}$$

we get:

$$ {}_{10}\Psi_{10}\left[\begin{matrix} ap^{m+1}q^{m+1};bp^{m+1}/q^{m+1};ap^m,bp^m;cq^m,aq^m/bc;\\ ap^mq^m;bp^m/q^m;ap^{m+1}/c;bcp^{m+1};q^{1+m},aq^{m+1}/b;\\ q_1^{-n},eq_1^{-n}/d,fq_1^{-n}/d,gq_1^{-n}/d;\,pq,\,p/q,\,p,q,q_1;q\\ ;q_1^{-n}/d,q_1^{-n}/e,q_1^{-n}/f,q_1^{-n}/g \end{matrix}\right]$$

$$=\frac{[a;\,pq]_m[b;\,p/q]_m[q,cq/b;q]_m[ap/c,bcp;\,p]_m}{[apq;\,pq]_m[bp/q;\,p/q]_m[a,b;\,p]_m[c,a/bc;q]_m}$$

$$\times\frac{[q_1,dq_1/e,dq_1/f,dq_1/g;q_1]_n[d,q_1\sqrt{d},-q_1\sqrt{d},e,f,g;q_1]_m}{[dq_1,eq_1,fq_1,gq_1;q_1]_n[q_1,\sqrt{d},-\sqrt{d},dq_1/e,dq_1/f,dq_1/g;q_1]_m}$$

$$\times\left(\frac{q_1}{q}\right)^m\frac{[ap,bp;\,p]_n[cq,aq/bc;q]_n}{[q,aq/b;q]_n[ap/c,bcp;\,p]_n}$$

$$\times{}_{10}\Psi_{10}\left[\begin{matrix} dq_1^m,q_1^{1+m}\sqrt{d},-q_1^{1+m}\sqrt{d},eq_1^m,fq_1^m,gq_1^m;\\ q_1^{1+m},q_1^m\sqrt{d},-q_1^m\sqrt{d},dq_1^{1+m}/e,dq_1^{1+m}/f;dq_1^{1+m}/g;\\ ;q^{-n},bq^{-n}/a;cp^{-n}/a,\,p^{-n}/bc;q_1,q,\,p;q_1\\ ;q^{-n}/c,bcq^{-n}/a,\,p^{-n}/a,\,p^{-n}/b \end{matrix}\right],\quad (d=efg). \tag{11}$$

We have established bilateral basic hypergeometric series for the choices of the following:

$$a_k=\frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;\,p]_k[c,a/bc;q]_kq^k}{(1-a)(1-b)[q,aq/b;q]_k[ap/c,bcp;\,p]_k}$$

and

$$A_k=\frac{(1-AP^kQ^k)(1-BP^kQ^{-k})[A,B;P]_k[C,A/BC;Q]_kQ^k}{(1-A)(1-B)[Q,AQ/B;Q]_k[AP/C,BCP;P]_k}, \tag{12}$$

which in conjunction with the partial sum [10] yields a transformation for a ${}_{10}\Psi_{10}$.

The following choice:

$$a_k=\frac{(1-ap^kq^k)[a;\,p]_k[b;q]_kb^{-k}}{(1-a)[q;q]_k[ap/b;\,p]_k}\quad\text{and}\quad A_k=\frac{(c,q_1\sqrt{c},-q_1\sqrt{c},d;q_1)_k}{(q_1,\sqrt{c},-\sqrt{c},cq_1/d;q_1)_kd^k}, \tag{13}$$

along with the partial sum results [1, App. II (23)]

$$ {}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e \end{matrix}\right]_N = \frac{[aq, eq; q]_N}{[q, aq/e; q]_N e^N} \tag{14} $$

and [9, App. II (34)]

$$ \sum_{k=0}^{n} \frac{(1-ap^k q^k)[a;p]_k [c;q]_k c^{-k}}{(1-a)[q;q]_k [ap/c;p]_k} = \frac{[ap;p]_n [cq;q]_n c^{-n}}{[q;q]_n [ap/c;p]_n}, \tag{15} $$

results in a transformation between a ${}_5\Psi_5$ and a ${}_6\Psi_6$. We have established a number of similar transformations and these will be reported elsewhere and can also be obtained from the authors.

The transformations we have obtained suggest product theorems. To illustrate, the transformation [7] leads to:

$$ {}_2\Psi_2\left[\begin{matrix} aq_1^m, yq_1^m; q_1; q_1 x \\ q_1^{1+m}, ayq_1^{1+m} \end{matrix}\right] {}_2\Phi_1\left[\begin{matrix} \alpha q, \beta q; q; x \\ \alpha\beta q \end{matrix}\right] = \frac{[\alpha, \beta; q]_m [q, ayq_1; q_1]_m}{[q, \alpha\beta q; q]_m [a, y; q_1]_m} \left(\frac{q}{q_1}\right)^m $$
$$ \times {}_2\Psi_2\left[\begin{matrix} \alpha q^m, \beta q^{1+m}; q; qx \\ q^{1+m}, \alpha\beta q^{1+m} \end{matrix}\right] {}_2\Phi_1\left[\begin{matrix} aq_1, yq_1; q_1; x \\ ayq_1 \end{matrix}\right], \quad (|q_1 x|, |x|, |qx| < 1). \tag{16} $$

It is to be noted that if we take all the bases equal in a particular transformation, then we get the corresponding transformation for a basic hypergeometric function having only one base. Further, specializing the parameters can also lead to interesting results. The final example we present here is to illustrate these aspects.

Let us choose:

$$ a_k = \frac{[\alpha, \beta; q_1]_k q_1^k}{[q_1, \alpha\beta q_1; q_1]_k} \quad \text{and} \quad A_k = \frac{(1-ap^k q^k)[a;p]_k [b;q]_k b^{-k}}{(1-a)[q;q]_k [ap/b;p]_k} \tag{17} $$

in (1) and make use of the partial sum results (6) and (15). We get:

$$ {}_4\Psi_4\left[\begin{matrix} \alpha q_1^m, \beta q_1^m; q^{-n}; bp^{-n}/a; q_1, q, p; q_1/b \\ q_1^{1+m}, \alpha\beta q_1^{1+m}; q^{-n}/b; p^{-n}/a \end{matrix}\right] $$
$$ = \frac{[q_1, \alpha\beta q_1; q_1]_m [q;q]_n [ap/b;p]_n [apq;pq]_m [a;p]_m [b;q]_m}{[\alpha, \beta; q_1]_m [ap;p]_n [bq;q]_n [a;pq]_m [q;q]_m [ap/b;p]_m} $$
$$ \times \frac{[\alpha q_1, \beta q_1; q_1]_n b^n}{[q_1, \alpha\beta q_1; q_1]_n (bq_1)^m [q;q]_n [ap/b;p]_n} $$
$$ \times {}_5\Psi_5\left[\begin{matrix} ap^{1+m}q^{1+m}; ap^m; bq^m; q_1^{-n}, q_1^{-n}/\alpha\beta; pq, p, q, q_1; 1/b \\ ap^m q^m; ap^{1+m}/b; q_1^{1+m}; q_1^{-n}/\alpha, q_1^{-n}/\beta \end{matrix}\right]. \tag{18} $$

The above transformation (18) suggests the following product theorem:

$$
{}_2\Psi_2\left[\begin{matrix}\alpha q_1^m,\beta q_1^m;q_1;q_1bx\\ q_1^{1+m},\alpha\beta q_1^{1+m}\end{matrix}\right]\quad {}_2\Phi_1\left[\begin{matrix}bq:ap;q,p;x\\ -:ap/b\end{matrix}\right]
$$

$$
=\frac{[q_1,\alpha\beta;q_1]_m[apq;pq]_m[a;p]_m[b;q]_m}{[\alpha,\beta;q_1]_m[a;pq]_m[q;q]_m[ap/b;p]_m(bq_1)^m}
$$

$$
\times{}_3\Psi_3\left[\begin{matrix}a(pq)^{1+m}:ap^m:bp^m;\ pq,p,q;x\\ a(pq)^m:ap^{1+m}/b:q^{1+m}\end{matrix}\right]\quad {}_2\Phi_1\left[\begin{matrix}\alpha q_1,\beta q_1;q_1;bx\\ \alpha\beta q_1\end{matrix}\right]. \tag{19}
$$

Taking $m=0$ in (19), we get:

$$
{}_2\Phi_1\left[\begin{matrix}\alpha,\beta;q_1;q_1bx\\ \alpha\beta q_1\end{matrix}\right]\quad {}_2\Phi_1\left[\begin{matrix}bq:ap;q,p;x\\ -:ap/b\end{matrix}\right]
$$

$$
={}_3\Phi_2\left[\begin{matrix}b:apq:a;q,pq,p;x\\ -:a:ap/b\end{matrix}\right]\quad {}_2\Phi_1\left[\begin{matrix}\alpha q_1,\beta q_1;q_1;bx\\ \alpha\beta q_1\end{matrix}\right],\quad (|bx|,|x|<1). \tag{20}
$$

Furthermore, if we now set $b\to 1$ in (20), we get:

$$
{}_2\Phi_1\left[\begin{matrix}\alpha,\beta;q_1;q_1x\\ \alpha\beta q_1\end{matrix}\right]=(1-x){}_2\Phi_1\left[\begin{matrix}\alpha q_1,\beta q_1;q_1;x\\ \alpha\beta q_1\end{matrix}\right],\quad (|x|<1). \tag{21}
$$

To conclude, in this short article we have shown that starting from a modified Gasper [7] identity, it is possible to establish transformations of a poly-basic bilateral hypergeometric series in terms of a similar series, not necessarily having the same number of bases. Only a few examples have been shown here to illustrate our methodology.

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