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On the logarithmic coefficients of Bazilevič functions

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ABSTRACT

The objective of the present paper is to study the logarithmic coefficients of Bazilevič functions. We obtain the inequality $|\gamma_n| \leqslant An^{-1}\log n$ (A is an absolute constant) which holds for Bazilevič functions.

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1. Introduction

Throughout the paper, $\mathscr{A}$ denotes the class of analytic functions $f(z)$ in the unit disk $U = \{z \in C : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$.

Let α and β be real numbers with $\alpha > 0$. A function $f(z) \in \mathscr{A}$ is called Bazilevič functions of type (α, β) if

$$f(z) = \left[(\alpha + i\beta)\int_0^z p(t)(g(t))^{\alpha} t^{i\beta-1} dt\right]^{\frac{1}{\alpha+i\beta}}, \tag{1.1}$$

for a starlike (univalent) function $g(z)$ in U and an analytic function $p(z)$ with $p(0) = 1$ satisfying $Re\{p(z)\} > 0$ in U. We denote by $B(\alpha, \beta)$ the class of Bazilevič functions of type (α, β). For the sake of brevity we shall simply denote by $B(\alpha)$ instead of $B(\alpha, 0)$ and we shall call a function in $B(\alpha)$ a Bazilevič functions of type α.

Let $S, \mathscr{K}, S^*$ and S_c denote the subclasses of $\mathscr{A}$ of functions univalent, convex, starlike and close-to-convex, respectively. We also denote by $\mathscr{P}$ the class of analytic functions $p(z)$ with $p(0) = 1$ and $Re\{p(z)\} > 0$ in U. Note that $\mathscr{P}$ is known as the Carathéodory class.

Let $\alpha > 0$ and $\beta \in R$. In view of (1.1), for $f(z) \in \mathscr{A}$, we readily see that $f(z) \in B(\alpha, \beta)$ if and only if

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} \in \mathscr{P}, \tag{1.2}$$

for some $g(z) \in S^*$ (see [1]).

Bazilevič [2] shows that $B(\alpha, \beta) \subset S$ for $\alpha > 0$, $\beta \in R$. Later, Sheil-Small [3] extends it to the case $\alpha \geqslant 0$ and gives a geometric characterization for $B(\alpha, \beta)$. So far, Bazilevič functions form the largest known subclass of S which has concrete expressions.

It is well known that the inclusion relations

$$\mathscr{K} \subset S^* \subset S_c \subset B(\alpha) \subset B(\alpha, \beta) \subset S$$

are valid. See [2,4,5] for further information.

Associated with each $f(z)$ in S is a well defined logarithmic function

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n, \quad z \in U. \tag{1.3}$$

The numbers γ_n are called the logarithmic coefficients of $f(z)$. Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$. It is clear that $|\gamma_1| \leqslant 1$ for each $f(z) \in S$. The problem of the best upper bounds for $|\gamma_n|$ is still open. In fact even

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the proper order of magnitude is still not known. It is known, however, for the starlike functions that the best bound is $|\gamma_n| \leqslant \frac{1}{n}$ and that this is not true in general [4, P.151]; [6, P.898]; [7, P.140] and [8].

In the paper [9] it is pointed out that the inequality $|\gamma_n| \leqslant An^{-1}\log n$ (A is an absolute constant) which holds for circularly symmetric functions.

In a recent paper [10], it is presented that the inequality $|\gamma_n| \leqslant \frac{1}{n}$ holds also for close-to-convex functions. However, it is pointed out in [11] that there are some errors in the proof and, hence, the result is not substantiated. It is proved in [12] that there exists a function $f(z) \in S_c$ such that $|\gamma_n| > \frac{1}{n}$. Furthermore, it is proved in [13] that the inequality $|\gamma_n| \leqslant An^{-1}\log n$ holds for close-to-convex functions, where A is an absolute constant.

In the present paper, we study the logarithmic coefficients of Bazilevič functions $B(\alpha,\beta)$. Also, we obtain the inequality $|\gamma_n| \leqslant An^{-1}\log n$ (A is an absolute constant) which holds for Bazilevič functions $B(\alpha,\beta)$.

2. Main results

First, we give the following lemmas:

Lemma 1 [13]. *Let $f(z) \in S$. Then, for $z = re^{i\theta}$, $\frac{1}{2} \leqslant r < 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta \leqslant 1 + \frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}, \tag{2.1}$$

and

$$\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr \leqslant 1 + 2log\frac{1}{1-r}. \tag{2.2}$$

Lemma 2. *Let $f(z) \in S$, $\tau \in C$. Then, $z = re^{i\theta}$, $0 < r < 1$,*

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\tau}\right) = \tau\frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right). \tag{2.3}$$

Proof. It is clear that

$$\frac{zf'(z)}{f(z)} = \frac{1}{i}\frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right) + 1. \tag{2.4}$$

It follows that

$$Re\frac{zf'(z)}{f(z)} = Im\left\{\frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right)\right\} + 1 = \frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right) + 1. \tag{2.5}$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{1}{i\tau}\frac{\partial}{\partial\theta}\left(\log\left(\frac{f(z)}{z}\right)^{\tau}\right) + 1, \tag{2.6}$$

then

$$Re\frac{zf'(z)}{f(z)} = \frac{1}{\tau}Im\left\{\frac{\partial}{\partial\theta}\left(\log\left(\frac{f(z)}{z}\right)^{\tau}\right)\right\} + 1 = \frac{1}{\tau}\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\tau}\right) + 1. \tag{2.7}$$

From (2.5) and (2.7) we obtain

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\tau}\right) = \tau\frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right). \qquad \square$$

Theorem 1. *Let $f(z) \in B(\alpha,\beta)$. Then, for $n \geqslant 2$,*

$$|\gamma_n| \leqslant An^{-1}\log n, \tag{2.8}$$

where A is an absolute constant, and the exponent -1 is the best possible.

Proof. If $f(z) \in B(\alpha,\beta)$, then there exist $g(z) \in S^*$ such that $Re\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} > 0, \alpha > 0, \beta \in R$. Write $p(z) = \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}$, then $Rep(z) > 0$. It is clear that

$$p(z) = 2Rep(z) - \overline{p(z)}.$$

From (1.3), we obtain

$$\frac{zf'(z)}{f(z)} = 1 + z\left(\log\frac{f(z)}{z}\right)' = 1 + \sum_{k=1}^{\infty} 2k\gamma_k z^k. \tag{2.9}$$

Then, for $z = re^{i\theta}(0 < r < 1)$ and $n = 2,3,\ldots$

$$2n\gamma_n = \frac{1}{2\pi i}\oint_{|z|=r} \frac{zf'(z)}{f(z)} z^{-n-1} dz.$$

Hence, we obtain

$$\begin{aligned}|2n\gamma_n r^n| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} e^{-in\theta} d\theta\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} p(z)\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta} e^{-in\theta} d\theta\right| \\ &\leqslant \frac{1}{2\pi}\left|\int_0^{2\pi} 2Rep(z)\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta} e^{-in\theta} d\theta\right| + \frac{1}{2\pi}\left|\int_0^{2\pi} \overline{p(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{i\beta} e^{-in\theta} d\theta\right| = I_1 + I_2. \end{aligned} \tag{2.10}$$

Now we estimate two terms I_1 and I_2. Write

$$\frac{zf'(z)}{f(z)} = u(re^{i\theta}) + iv(re^{i\theta}). \tag{2.11}$$

(a)

$$\begin{aligned} I_1 &\leqslant \frac{1}{\pi}\int_0^{2\pi} Rep(z)\left|\frac{g(z)}{f(z)}\right|^{\alpha}\left|\frac{z}{f(z)}\right|^{i\beta} d\theta \leqslant \frac{1}{\pi}\left|\int_0^{2\pi} p(z)\left|\frac{g(z)}{f(z)}\right|^{\alpha}\left|\frac{z}{f(z)}\right|^{i\beta} d\theta\right| = \frac{1}{\pi}\left|\int_0^{2\pi} \frac{zf'(z)}{f(z)} e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}} d\theta\right| \\ &\leqslant \frac{1}{\pi}\left|\int_0^{2\pi} u(re^{i\theta}) e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}} d\theta\right| + \frac{1}{\pi}\left|\int_0^{2\pi} v(re^{i\theta}) e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}} d\theta\right| = 2(J_1 + J_2). \end{aligned} \tag{2.12}$$

Applying the part of integration, (2.5) and (2.3), we have

$$\begin{aligned} J_1 &\leqslant \frac{1}{2\pi}\left|\int_0^{2\pi} e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}} d\theta\right| + \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right) e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}} d\theta\right| \\ &\leqslant 1 + \frac{1}{2\sqrt{\alpha^2+\beta^2}\pi}\left|\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(e^{iarg\left(\frac{f(z)}{z}\right)^{\alpha+i\beta}}\right) e^{-iarg\left(\frac{g(z)}{z}\right)^{\alpha}} d\theta\right| \\ &= 1 + \frac{1}{2\sqrt{\alpha^2+\beta^2}\pi}\left|\int_0^{2\pi} e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}} \frac{\partial}{\partial\theta}\left(arg\left(\frac{g(z)}{z}\right)^{\alpha}\right) d\theta\right| \\ &\leqslant 1 + \frac{\alpha}{2\sqrt{\alpha^2+\beta^2}\pi}\int_0^{2\pi}\left(\left|\frac{\partial}{\partial\theta}(argg(z))\right| + \left|\frac{\partial z}{\partial\theta}\right|\right) d\theta. \end{aligned} \tag{2.13}$$

Since $g(z) \in S^*$, we have (see [5])

$$\frac{\partial}{\partial\theta}(argg(z)) > 0 \quad \text{and} \quad \int_0^{2\pi} \frac{\partial}{\partial\theta}(argg(z)) = 2\pi. \tag{2.14}$$

By applying (2.14), from (2.13) we obtain

$$J_1 = 1 + \frac{\alpha}{2\sqrt{\alpha^2+\beta^2}\pi}\left(\int_0^{2\pi} \frac{\partial}{\partial\theta}(argg(z)) d\theta + \int_0^{2\pi} r d\theta\right) = 1 + \frac{\alpha}{2\sqrt{\alpha^2+\beta^2}\pi}(2\pi + 2\pi r) < 1 + \frac{2\alpha}{\sqrt{\alpha^2+\beta^2}}. \tag{2.15}$$

By the Cauchy–Riemann condition, we obtain, for $0 < r_0 < r < 1$

$$v(re^{i\theta}) - v(r_0 e^{i\theta}) = \int_{r_0}^{r} \frac{\partial v(re^{i\theta})}{\partial r} dr = -\int_{r_0}^{r} \frac{1}{r}\frac{\partial u(re^{i\theta})}{\partial\theta} dr. \tag{2.16}$$

By (2.16), we obtain

$$J_2 \leqslant \frac{1}{2\pi}\left|\int_0^{2\pi} v(r_0e^{i\theta})e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}}d\theta\right| + \frac{1}{2\pi}\left|\int_0^{2\pi}\int_{r_0}^{r}\frac{1}{r}\frac{\partial u(re^{i\theta})}{\partial\theta}e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}}drd\theta\right| = J_{21} + J_{22}. \tag{2.17}$$

Taking $r_0 = \frac{1}{2}$, it follows that

$$J_{21} \leqslant \max_{\theta\in[0,2\pi]}\left|v(r_0e^{i\theta})\right| \leqslant \max_{\theta\in[0,2\pi]}\left|\frac{r_0f'(r_0e^{i\theta})}{f(r_0e^{i\theta})}\right| \leqslant \frac{1+r_0}{1-r_0} = 3. \tag{2.18}$$

By the part of integration, we obtain

$$J_{22} = \frac{1}{2\pi}\left|\int_{r_0}^{r}\int_0^{2\pi}\frac{1}{r}u(re^{i\theta})e^{iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}}\left(\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\alpha+i\beta}\right) - \frac{\partial}{\partial\theta}\left(\left(\frac{g(z)}{z}\right)^{\alpha}\right)\right)d\theta dr\right|. \tag{2.19}$$

By (2.3) and (2.5), we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\alpha+i\beta}\right) - \frac{\partial}{\partial\theta}\left(\left(\frac{g(z)}{z}\right)^{\alpha}\right)\right| &= \left|(\alpha+i\beta)\frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right) - \alpha\frac{\partial}{\partial\theta}\left(arg\frac{g(z)}{z}\right)\right| \\ &= \left|(\alpha+i\beta)\left(Re\frac{zf'(z)}{f(z)} - 1\right) - \alpha\left(Re\frac{zg'(z)}{g(z)} - 1\right)\right| \\ &= \left|(\alpha+i\beta)Re\frac{zf'(z)}{f(z)} - \alpha Re\frac{zg'(z)}{g(z)} - i\beta\right| \\ &\leqslant \sqrt{\alpha^2+\beta^2}\left|\frac{zf'(z)}{f(z)}\right| + \alpha\left|\frac{zg'(z)}{g(z)}\right| + |\beta|.\end{aligned} \tag{2.20}$$

By Schwarz inequality, Lemma 1 and (2.20), from (2.19) we obtain

$$\begin{aligned}J_{22} &\leqslant \frac{1}{\pi}\int_{r_0}^{r}\int_0^{2\pi}\left(\sqrt{\alpha^2+\beta^2}\left|\frac{zf'(z)}{f(z)}\right|^2 + \alpha\left|\frac{zf'(z)}{f(z)}\right|\left|\frac{zg'(z)}{g(z)}\right| + |\beta|\left|\frac{zf'(z)}{f(z)}\right|\right)d\theta dr \\ &\leqslant 2\sqrt{\alpha^2+\beta^2}\left(1+2\log\frac{1}{1-r}\right) + \frac{\alpha}{\pi}\left(\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^2 d\theta dr\right)^{\frac{1}{2}} \\ &\quad + \frac{|\beta|}{\pi}\left(\int_{r_0}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr\int_{r_0}^{r}\int_0^{2\pi}d\theta dr\right)^{\frac{1}{2}} \\ &\leqslant \left(2\sqrt{\alpha^2+\beta^2} + 2\alpha + 2|\beta|\right)\left(1+2\log\frac{1}{1-r}\right).\end{aligned} \tag{2.21}$$

By (2.18) and (2.21), from (2.17) we obtain

$$J_2 \leqslant \left(3 + 2\sqrt{\alpha^2+\beta^2} + 2\alpha + 2|\beta|\right) + \left(4\sqrt{\alpha^2+\beta^2} + 4\alpha + 4|\beta|\right)\log\frac{1}{1-r}. \tag{2.22}$$

By (2.15) and (2.22), from (2.12) we obtain

$$I_1 \leqslant \left(8 + \frac{4\alpha}{\sqrt{\alpha^2+\beta^2}} + 4\sqrt{\alpha^2+\beta^2} + 4\alpha + 4|\beta|\right) + \left(8\sqrt{\alpha^2+\beta^2} + 8\alpha + 8|\beta|\right)\log\frac{1}{1-r}. \tag{2.23}$$

(b)

$$I_2 = \frac{1}{2\pi}\left|\int_0^{2\pi}\overline{\left(\frac{zf'(z)}{f(z)}\right)}e^{-2iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}}e^{-in\theta}d\theta\right| = \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{2iarg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}}e^{in\theta}d\theta\right|. \tag{2.24}$$

From (2.9) we have

$$\frac{zf'(z)}{f(z)}e^{in\theta} = e^{in\theta}\left(1+\sum_{k=1}^{\infty}2k\gamma_k z^k\right) = e^{in\theta} + \sum_{k=1}^{\infty}2k\gamma_k r^k e^{i(n+k)\theta} = \frac{1}{i}\frac{\partial}{\partial\theta}\left(\frac{e^{in\theta}}{n} + \sum_{k=1}^{\infty}\frac{2k\gamma_k r^k e^{i(n+k)\theta}}{n+k}\right) = \frac{\partial}{\partial\theta}F(z). \tag{2.25}$$

By the part of integration, we obtain

$$I_2 = \frac{1}{\pi}\left|\int_0^{2\pi} F(z) e^{i2arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}}\left[\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\alpha+i\beta}\right) - \frac{\partial}{\partial\theta}\left(arg\left(\frac{g(z)}{z}\right)^{\alpha}\right)\right] d\theta\right|. \tag{2.26}$$

By (2.5), (2.3) and Schwartz inequality, it follows from (2.26) that

$$I_2 \leqslant 2\left(\frac{1}{2\pi}\int_0^{2\pi}|F(z)|^2 d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(|\alpha+i\beta|\left|\frac{zf'(z)}{f(z)}\right| + \alpha\left|\frac{zg'(z)}{g(z)}\right| + |\beta|\right)^2 d\theta\right)^{\frac{1}{2}} = 2(L_1L_2)^{\frac{1}{2}}. \tag{2.27}$$

Lebedev proves (see [14]) that if $f(z) \in S$ then

$$\sum_{k=1}^{\infty} k|\gamma_k| r^{2k} \leqslant \log\frac{1}{1-r}. \tag{2.28}$$

By the definition of $F(z)$ in (2.25), we obtain from (2.28)

$$L_1 = \frac{1}{n^2} + 4\sum_{k=1}^{\infty}\frac{k^2|\gamma_k|^2 r^{2k}}{(n+k)^2} \leqslant \frac{1}{n^2} + \frac{4}{n}\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k} \leqslant \frac{1}{n^2} + \frac{4}{n}\log\frac{1}{1-r}. \tag{2.29}$$

By Lemma 1, it follows that

$$\begin{aligned} L_2 &= \frac{1}{2\pi}\int_0^{2\pi}(\alpha^2+\beta^2)\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta + \frac{1}{2\pi}\int_0^{2\pi}\alpha^2\left|\frac{zg'(z)}{g(z)}\right|^2 d\theta + \frac{1}{2\pi}\int_0^{2\pi}\beta^2 d\theta + \frac{1}{2\pi}\int_0^{2\pi}2|\beta|\sqrt{\alpha^2+\beta^2}\left|\frac{zf'(z)}{f(z)}\right| d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}2\alpha\sqrt{\alpha^2+\beta^2}\left|\frac{zf'(z)}{f(z)}\right|\left|\frac{zg'(z)}{g(z)}\right| d\theta + \frac{1}{2\pi}\int_0^{2\pi}2\alpha|\beta|\left|\frac{zg'(z)}{g(z)}\right| d\theta \\ &\leqslant \left(2\alpha^2+\beta^2+2\alpha\sqrt{\alpha^2+\beta^2}\right)\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right) + \beta^2 + \left(2|\beta|\sqrt{\alpha^2+\beta^2}+2\alpha|\beta|\right)\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}} \\ &\leqslant \left(2\alpha^2+\beta^2+2\alpha\sqrt{\alpha^2+\beta^2}+2|\beta|\sqrt{\alpha^2+\beta^2}+2\alpha|\beta|\right)\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right) + \beta^2. \end{aligned} \tag{2.30}$$

Combining (2.29) and (2.30), from (2.27) we obtain

$$I_2 \leqslant 2\left[\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right]^{\frac{1}{2}}\left[\left(2\alpha^2+\beta^2+2\alpha\sqrt{\alpha^2+\beta^2}+2|\beta|\sqrt{\alpha^2+\beta^2}+2\alpha|\beta|\right)\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)+\beta^2\right]^{\frac{1}{2}}. \tag{2.31}$$

Let $r = 1-\frac{1}{n}, n = 2,3,\ldots$. We obtain from (2.10), (2.23) and (2.31) that

$$|\gamma_n| \leqslant An^{-1}\log n,$$

where A is an absolute constant. Since the Koebe function $k(z) = z(1-z)^{-2} \in B(\alpha,\beta)$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$, the exponent -1 is the best possible. Thus, we have proved Theorem 1. □

Corollary 1 ([13]). *Suppose $f(z) \in S_c$ and that $f(z)$ has logarithmic coefficients $\{\gamma_n\}_{n=1}^{\infty}$. Then for $n = 2,3,\ldots$*

$$|\gamma_n| \leqslant An^{-1}\log n,$$

where A is an absolute constant.

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References

[1] Y.C. Kim, A note on growth theorem of Bazilevič functions, Appl. Math. Comput. 208 (2009) 542–546.
[2] I.E. Bazilevič, On a case of integrability in quadratures of the Löwner–Kufarev equation, Matematicheskii Sbornik (N.S.) 37 (3) (1955) 471–476. in Russian.
[3] T. Sheil-Small, On Bazilevič functions, Quart. J. Math. Oxford 23 (2) (1972) 135–142.
[4] P.L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
[5] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.
[6] P. Duren, Coefficients of univalent functions, Bull. Amer. Math. Soc. 83 (5) (1977) 891–911.
[7] J. Anderson, K. Barth, D. Brannan, Research problems in complex analysis, Bull. London Math. Soc. 9 (1977) 129–162.
[8] P. Duren, Y. Leung, Logarithmic coefficients of univalent functions, J. Anal. Math. 36 (1979) 36–43.
[9] Q. Deng, On circularly symmetric functions, Appl. Math. Lett. 23 (12) (2010) 1483–1488.

[10] M.M. Elhosh, On the logarithmic coefficients of close-to-convex functions, J. Austral. Math. Soc. (Ser. A) 60 (1996) 1–6.
[11] K. Pearce, Review of [9]. Math. Reviews, info review no. 96j: 30013, (1996).
[12] D. Girela, Logarithmic coefficients of univalent functions, Ann. Acad. Sci. Fenn. Math. 25 (2000) 337–350.
[13] Z. Ye, The logarithmic coefficients of close-to-convex functions, Bull. Inst. Math. Acad. Sinica (New Series) 3 (3) (2008) 445–452.
[14] N.A. Lebedev, Applications of area principle to problems on nonoverlapping domains, Trudy Mat. Inst. Stekliov 60 (1961) 211–231. in Russian.