

Deductive systems of fuzzy logic (a tutorial)

Petr Hájek¹, Lluís Godó²

¹Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic

and

²Institut d'Investigació en Intel·ligència Artificial, CSIC
08193 Bellaterra, Catalonia, Spain

0.1 Where are truth-functions of connectives from?

Obviously, the truth-functions should behave classically on extremal truth values 0,1 and should satisfy some natural monotonicities (the truth function of conjunction (disjunction) should be non-decreasing in both arguments; the truth function of implication should be non-decreasing in the second argument but non-increasing in the first, i.g. the less true is the antecedent φ and the more is true the succedent ψ the more is true the implication $\varphi \rightarrow \psi$. ($-$) should be non-increasing.) This leads to the notion of a t-norm: (cf. [43]) this is an operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ which is commutative and associative, non-decreasing in both arguments and having 1 as unit element and 0 as zero element, i.c.

$$x * y = y * x$$

$$(x * y) * z = x * (y * z)$$

$$x \leq x' \text{ and } y \leq y' \text{ implies } x * y \leq x' * y'$$

$$1 * x = x, 0 * x = 0.$$

We shall only work with *continuous* t-norms as good candidates for truth functions of a conjunction. Each t-norm t determines uniquely its corresponding implication \Rightarrow (not necessarily continuous) satisfying, for all $x, y, z \in [0, 1]$

$$z \leq x \Rightarrow y \text{ iff } x * z \leq y.$$

For each such system we define an *evaluation* to be a mapping e assigning to each atom p its truth degree $e(p)$, $0 \leq e(p) \leq 1$; a *1-tautology* is a formula whose value is 1 for each evaluation.

We present three outstanding examples:

(1) *Lukasiewicz logic* [25] with the conjunction

$x * y = \max(x + y - 1, 0)$ and the corresponding implication

$x \Rightarrow y = 1$ for $x \leq y$ and $x \Rightarrow y = 1 - x + y$ otherwise;

(2) *Gödel logic* [9] will the conjunction

$x * y = \min(x, y)$ and the corresponding implication

$x \Rightarrow y = 1$ iff $x \leq y$ and $x \Rightarrow y = y$ otherwise;

(3) *Product logic* will the conjunction $x * y = x \cdot y$ (product) and $x \Rightarrow y = 1$ iff $x \leq y$, $x \Rightarrow y = y/x$ otherwise.

Negation ($-$) is defined as follows: $(-)x = x \Rightarrow 0$

One can show (see e.g. [33]) that each t-norm is composed in a certain way from these three examples. Thus our question reads: what is the logic of these examples?

We show that min and max are definable from $*$ and \Rightarrow .

For each continuous t-norm $*$, the following identities are true in $L(*)$:

(i) $\min(x, y) = x * (x \Rightarrow y)$,

(ii) $\max(x, y) = \min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x)$.

In the next subsection we shall present a *basic fuzzy logic* BL. Formulas provable in BL are 1-tautologies for each continuous t-norm. We shall formulate a completeness theorem formulated with the help of *residuated lattices*. Then in three following sections we shall develop logics of the three main t-norms defined above.

0.2 The basic many-valued logic

Fix a continuous t-norm $*$: you fix a propositional calculus (whose set of truth values is $[0, 1]$): This means is the truth function of the (strong) conjunc-

tion $\&$, the residuum \Rightarrow of $*$ becomes the truth function of the implication. Further connectives are defined as follows:

$$\begin{aligned}\varphi \wedge \psi & \text{ is } \varphi \& (\varphi \rightarrow \psi), \\ \varphi \vee \psi & \text{ is } ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi), \\ \neg \varphi & \text{ is } \varphi \rightarrow \bar{0}, \\ \varphi \equiv \psi & \text{ is } (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi).\end{aligned}$$

An *evaluation of propositional variables* is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0, 1]$.

This extends to each formula via truth-functions as follows:

$$\begin{aligned}e(\bar{0}) &= 0, \\ e(\varphi \rightarrow \psi) &= (e(\varphi) \Rightarrow e(\psi)), \\ e(\varphi \& \psi) &= (e(\varphi) * e(\psi)).\end{aligned}$$

A formula φ is a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e .

The following formulas are axioms of the basic logic:

$$(A1) (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

$$(A2) (\varphi \& \psi) \rightarrow \varphi$$

$$(A3) (\varphi \& \psi) \rightarrow (\psi \& \varphi)$$

$$(A4) (\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi))$$

$$(A5a) (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$$

$$(A5b) ((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$$

$$(A6) ((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$$

$$(A7) \bar{0} \rightarrow \varphi$$

The *deduction rule* of BL is modus ponens. Given this, the notions of a *proof* and of a *provable formula* in BL are defined in the obvious way

All axioms of BL are 1-tautologies in each $PC(*)$. If φ and $\varphi \rightarrow \psi$ are 1-tautologies of $PC(*)$ then ψ is also a 1-tautology of $PC(*)$. Consequently, each formula provable in BL is a 1-tautology of each $PC(*)$. Let us present a list of some formulas provable in *BL*.

BL proves the following properties of implication:

- (1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$
- (3) $\varphi \rightarrow \varphi$

BL proves the following properties of strong conjunction:

- (4) $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow \psi$
- (5) $\varphi \rightarrow (\psi \rightarrow (\varphi \& \psi))$
- (6) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \& \chi) \rightarrow (\psi \& \chi))$
- (7) $((\varphi_1 \rightarrow \psi_1) \& (\varphi_2 \rightarrow \psi_2)) \rightarrow ((\varphi_1 \& \varphi_2) \rightarrow (\psi_1 \& \psi_2))$
- (8) $(\varphi \& \psi) \& \chi \equiv \varphi \& (\psi \& \chi)$.

BL proves the following properties of min-conjunction:

- (9) $(\varphi \wedge \psi) \rightarrow \varphi, (\varphi \wedge \psi) \rightarrow \psi, (\varphi \& \psi) \rightarrow (\varphi \wedge \psi)$
- (10) $(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\varphi \wedge \psi))$
- (11) $((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))$
- (12) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

BL proves the following properties of max-disjunction:

- (13) $\varphi \rightarrow (\varphi \vee \psi), \psi \rightarrow (\varphi \vee \psi), (\varphi \vee \psi) \rightarrow (\psi \vee \varphi),$
- (14) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \vee \psi) \rightarrow \psi)$
- (15) $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$
- (16) $((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \vee \psi) \rightarrow \chi$.

BL proves the following properties of negation.

- (17) $\varphi \rightarrow (\neg \varphi \rightarrow \psi)$, in particular, $\varphi \rightarrow \neg \neg \varphi$
- (18) $(\varphi \rightarrow (\psi \& \neg \psi)) \rightarrow \neg \varphi$

$\bar{1}$ stands for $\bar{0} \rightarrow \bar{0}$.

BL proves the following:

- (19) $\bar{1}$,
- (20) $\varphi \rightarrow (\bar{1} \& \varphi)$.

BL proves the following additional properties of \wedge, \vee :

- (21) $(\varphi \wedge (\psi \wedge \chi)) \rightarrow ((\varphi \wedge \psi) \wedge \chi)$
 $((\varphi \wedge \psi) \wedge \chi) \rightarrow (\varphi \wedge (\psi \wedge \chi))$
 (associativity of \wedge),
- (22) analogous associativity for \vee ,
- (23) $\varphi \rightarrow \varphi \wedge (\varphi \vee \psi)$
 $(\varphi \vee (\varphi \wedge \psi)) \rightarrow \varphi$

BL proves

- (24) $\varphi \equiv \varphi, (\varphi \equiv \psi) \rightarrow (\psi \equiv \varphi),$
 $((\varphi \equiv \psi) \& (\psi \equiv \chi)) \rightarrow (\varphi \equiv \chi),$

- (25) $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi),$
 $(\varphi \equiv \psi) \rightarrow (\psi \rightarrow \varphi)$
(26) $(\varphi \equiv \psi) \rightarrow ((\varphi \& \chi) \equiv (\psi \& \chi)),$
(27) $(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi)),$
(28) $(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi)),$
(29) $(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)).$

BL proves the following distributive laws:

- (30) $\varphi \& (\psi \vee \chi) \equiv (\varphi \& \psi) \vee (\varphi \& \chi)$
 $\varphi \& (\psi \wedge \chi) \equiv (\varphi \& \psi) \wedge (\varphi \& \chi)$

(31) $(\varphi \wedge (\psi \vee \chi)) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
 $(\varphi \vee (\psi \wedge \chi)) \equiv ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

BL proves:

- (32) $(\varphi \vee \psi) \& (\varphi \vee \psi) \rightarrow ((\varphi \& \varphi) \vee (\psi \& \psi))$
 $(\varphi \wedge \psi) \& (\varphi \wedge \psi) \rightarrow ((\varphi \& \varphi) \wedge (\psi \& \psi))$

(33) $(\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n,$ for each $n,$
where α^n is $\alpha \& \dots \alpha, n$ times.

A *theory* over BL is a set of formulas. A *proof* in a theory T is a sequence $\varphi_1, \dots, \varphi_n$ of formulas whose each member is either an axiom of BL or a member of T (special axiom) or follows from some preceding members of the sequence using the deduction rule modus ponens.

$T \vdash \varphi$ means that φ is *provable* in T , i.e. is the last member of of a proof in T . The deduction theorem for *BL* reads as follows:

Deduction theorem: Let T be a theory and let φ, ψ be formulas.

$T \cup \{\varphi\} \vdash \psi$ iff there is an n such that $T \vdash \varphi^n \rightarrow \psi$ (where φ^n is $\varphi \& \dots \& \varphi, n$ factors).

Now we shall introduce some algebras corresponding to *BL* similarly as Boolean algebras correspond to classical logic.

A *regular residuated lattice* (or a *BL-algebra*) is an algebra

$$(L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

- (i) $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering \leq),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative, $1 * x = x$ for all x .

(iii) the following conditions hold:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z .

(2) $x \cap y = x * (x \Rightarrow y)$

(3) $x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$

(4) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

An \mathbf{L} -evaluation of propositional variables is any mapping e assigning to each propositional variable p an element $e(p)$ of \mathbf{L} . This extends in the obvious way to an evaluation of all formulas using the operations on \mathbf{L} as truth functions.

The logic BL is sound with respect to \mathbf{L} -tautologies: if φ is provable in BL then φ is an \mathbf{L} -tautology for each regular linearly ordered residuated lattice. More generally, if T is a theory over BL and T proves φ then, for each regular linearly ordered residuated lattice \mathbf{L} and each \mathbf{L} -evaluation e of propositional variables assigning the value 1 to all the axioms of T , $e(\varphi) = 1$.

Classes of provably equivalent formulas (w.r.t. a theory T) form a regular residuated lattice.

Completeness theorem. For each formula φ the following three things are equivalent:

(i) φ is provable in BL,

(ii) for each linearly ordered regular residuated lattice \mathbf{L} , φ is an \mathbf{L} -tautology;

(iii) for each regular residuated lattice \mathbf{L} , φ is an \mathbf{L} -tautology.

We shall generalize this completeness theorem as follows:

(1) An *axiom schema* given by a formula

$\Phi(p_1, \dots, p_n)$ is the set of all formulas

$\Phi(\varphi_1, \dots, \varphi_n)$ resulting by the substitution of φ_i for p_i ($i = 1, \dots, n$) in $\Phi(p_1, \dots, p_n)$.

(2) A logical calculus \mathcal{C} is a *schematic extension* of BL if it results from BL by adding some (finitely or infinitely many) axiom schemata to its axioms.

(The deduction rule remains to be modus ponens.)

(3) Let \mathcal{C} be a schematic extension of BL and let \mathbf{L} be a lattice. \mathbf{L} is a \mathcal{C} -lattice if all axioms of \mathcal{C} are \mathbf{L} -tautologies.

Completeness. Let \mathcal{C} be a schematic extension of BL and let φ be a formula. The following are equivalent:

- (i) \mathcal{C} proves φ ,
- (ii) φ is an \mathbf{L} -tautology for each linearly ordered \mathcal{C} -lattice \mathbf{L} ,
- (iii) φ is an \mathbf{L} -tautology for each \mathcal{C} -lattice \mathbf{L} .

Remark. Results of the present section are new; but rely very heavily on related results of Höhle [21].

0.3 Łukasiewicz logic

This logic results by extending BL by the following axiom (L4):

$$((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi) \quad (\text{L4})$$

; Similarly as classical logic, Łukasiewicz logic \mathbf{L} may be alternatively developed from implication \rightarrow and negation \neg (or just from \rightarrow and 0); the truth function of negation is $(-)x = x \Rightarrow 0 = 1 - x$. We can define two different conjunctions and disjunctions:

$$\varphi \& \psi \text{ is } \neg(\varphi \rightarrow \neg\psi), \quad x * y = \max(x + y - 1, 0)$$

$$\varphi \underline{\vee} \psi \text{ is } \neg(\neg\varphi \& \neg\psi), \quad x \underline{\cup} y = \min(x + y, 1)$$

$$\varphi \vee \psi \text{ is } (\varphi \rightarrow \psi) \rightarrow \psi, \quad x \cup y = \max(x, y)$$

$$\varphi \wedge \psi \text{ is } \neg(\neg\varphi \vee \neg\psi), \quad x \cap y = \min(x, y)$$

The following are the original axioms of Łukasiewicz logic:

$$\varphi \rightarrow (\psi \rightarrow \varphi) \quad (\text{L1})$$

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \quad (\text{L2})$$

$$(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \quad (\text{L3})$$

$$((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi) \quad (\text{L4})$$

The only deduction rule is modus ponens; the definition of a proof is as in classical logic (relative to our set of axioms).

As mentioned above, this set of axioms is equivalent to BL + (L4)

Completeness of this set of axioms was conjectured by Lukasiewicz in Thirties, but first proved by Rose and Rosser [39]; a good proof can be found in [10]. The relevant algebras are particular regular residuated lattices called *MV*-algebras.

Needless to say, details are non-trivial and laborious but the structure is the same in all our three logics.

Completeness. A formula φ is provable in Lukasiewicz logic *L* iff it is a 1-tautology of Lukasiewicz logic.

Remark. Observe the difference from the completeness theorem for *BL*: here we do and work with all linearly ordered regular residuated lattices but with just one: the real interval $[0,1]$ with the truth functions of Lukasiewicz logic.

0.4 Gödel logic

Kurt Gödel (born 1906 in Brno, now Czech Republic), probably the most important mathematical logician, published in 1932 an extremely short paper [9] concerning intuitionistic logic (a subsystem of classical logic with a different meaning of connectives; e.g. $\varphi \vee \neg\varphi$ is not provable). Gödel's aim was to show that there is no finitely valued logic for which axioms of intuitionistic logic would be complete. For this purpose he created a semantics of (possibly infinite-valued) propositional calculus which is now called Gödel logic *G*. (Needless to say, this was more than three decades before fuzzy sets have been defined).

Gödel logic has the following connectives: $\rightarrow, \wedge, \vee, \neg$ (implication, conjunction, disjunction, negation; negation may be replaced by 0). The semantics is as follows (cf. Sect. 2.3):

$$\begin{aligned} x \Rightarrow y &= 1 \text{ if } x \leq y, \quad x \Rightarrow y = y \text{ otherwise,} \\ x \cap y &= \min(x, y), \\ x \cup y &= \max(x, y), \\ (-)x &= 1 \text{ for } x = 0, \quad (-)x = 0 \text{ for } x > 0. \end{aligned}$$

The axioms are as follows (G1 - G11 are axioms of intuitionistic logic, G12 is an axiom of "linearity"):

$$(G1) (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

- (G2) $\varphi \rightarrow (\varphi \vee \psi)$
- (G3) $\psi \rightarrow (\varphi \vee \psi)$
- (G4) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$
- (G5) $(\varphi \wedge \psi) \rightarrow \varphi$
- (G6) $(\varphi \wedge \psi) \rightarrow \psi$
- (G7) $(\chi \rightarrow \varphi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow (\varphi \wedge \psi)))$
- (G8) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \wedge \psi) \rightarrow \chi)$
- (G9) $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
- (G10) $(\varphi \wedge \neg\varphi) \rightarrow \psi$
- (G11) $(\varphi \rightarrow (\varphi \wedge \neg\varphi)) \rightarrow \neg\varphi$
- (G12) $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$

It is an easy checking to show that all these are 1-tautologies. The deduction rule is modus ponens; this defines the notion of a proof.

One can show that G is equivalently axiomatized by BL plus $\varphi \rightarrow (\varphi \& \varphi)$ – idempotence of $\&$. It follows easily that $\varphi \& \psi$ is equivalent to $\varphi \wedge \psi$ so that $\&$ is redundant.

Completeness theorem: Each 1-tautology is provable. Again here the proof is rather non-trivial with a different class of algebras, called Heyting algebras or pseudo-boolean algebras. We have no room for details; [10] is recommended for a readable elaborated proof originally given by Dummett [7].

Deduction theorem is valid for G: $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash (\varphi \rightarrow \psi)$. Note that G is *the only* many-valued logic having the deduction theorem, more precisely: if a logic contains a conjunction given by a t-norm and the corresponding implication \rightarrow , is completely axiomatized and satisfies the deduction theorem then the t-norm is minimum and hence \rightarrow is Gödel implication.

Gödels logic satisfies the following form of *strong completeness*: Say that a theory *semantically entails* φ if for each evaluation e there is a conjunction α of finitely many axioms of T such that $e(\alpha) \leq e(\varphi)$. (Observe that in classical logic this is equivalent to saying that φ is true in each model of T .)

Strong completeness: For each theory T and formula φ , $T \vdash \varphi$ iff T semantically entails φ .

Note that the easy part of this equivalent (soundness) implies that if $T \vdash \varphi$ and $e(\alpha) \geq r$ for each axiom α of T then $e(\varphi) \geq r$. The difficult part can be obtained by combining the (normal) completeness of G with the techniques of Takeuti and Titani [45].

0.5 Product logic.

The logic based on the product t-norm has been considerably less investigated than the two preceding ones (see [1]). The paper [18] investigates product logic and proves completeness theorem using a class of algebras called *product algebras*. There are several open problems related to this (rather interesting and unjustly overlooked) logic.

We write \odot instead of $\&$.

The *axioms* of Π are those of BL plus

$$(II1) \quad \neg\neg\chi \rightarrow ((\varphi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\varphi \rightarrow \psi)),$$

$$(II2) \quad \varphi \wedge \neg\varphi \rightarrow \bar{0}.$$

The axioms are 1-tautologies over the algebra $[0, 1]_P$ of the truth functions.

Π proves the following formulas:

$$(1) \quad \neg(\varphi \odot \psi) \rightarrow \neg(\varphi \wedge \psi)$$

$$(2) \quad (\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$$

$$(3) \quad \neg\varphi \vee \neg\neg\varphi$$

The axiom (II2) can be equivalently replaced by each of the following formulas:

$$\neg(\varphi \odot \varphi) \rightarrow \neg\varphi, \quad (\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi, \quad \neg\varphi \vee \neg\neg\varphi.$$

Following the general approach we define a Π -*algebra* (or *product algebra*) to be a regular residuated lattice satisfying

$$\begin{aligned} \neg\neg z &\leq ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y)), \\ x \cap \neg x &= \mathbf{0}. \end{aligned}$$

Using this notion one proves the following

Completeness theorem.

- (1) A formula φ is provable in the product logic Π iff it is a 1-tautology of the product logic.
- (2) Let T be a finite theory over Π , φ a formula. T proves φ over the product logic iff it is true in each model of T (in the sense of Π).

0.6 Rational Pavelka logic

Till now we have been interested almost exclusively only in axiomatizing 1-tautologies, i.e. proving formulas that are absolutely true. But in fuzzy logic we are interested in deriving consequences from assumptions that are only partially true, true in some degree. (We met a result of this type at the end of 2.5 - for Gödel logic.) Logics of partial truth were studied, in a very general manner, as early as in the seventies by the Czech mathematician Jan Pavelka [34] and since then have been substantially simplified; We refer to [14] but here we describe a still simpler version. It is very different from the original Pavelka's version and looks as an "innocent" extension of Łukasiewicz's \mathbb{L} ; but the main completeness result of Pavelka still holds.

The idea is as follows: assume that $e(\varphi) = r$; then for each φ , $e(\psi) \geq r$ iff $e(\varphi \rightarrow \psi) = 1$. Thus if φ is a formula whose value is r in *all* evaluations then the axiom $\varphi \rightarrow \psi$ would just postulate that ψ is at least r -true.

Thus we extend the language of \mathbb{L} by adding truth constant \bar{r} for some $r \in [0, 1]$ as new atomic formulas, postulating that $e(\bar{r}) = r$ for each evaluation (we already have had $\bar{0}$ and $\bar{1}$). Our choice will be to add truth constants \bar{r} for each *rational* $r \in [0, 1]$ (thus we have truth constants for a countable dense recursively representable set of reals from $[0, 1]$, this is all we need).

Thus for example if φ, ψ are formulas then $(\varphi \rightarrow \bar{0.7}) \& (\bar{0.4} \rightarrow \neg\psi)$ is a formula. We have some obvious tautologies like $\neg\bar{0.7} \equiv \bar{0.3}$ and $\bar{0.7} \rightarrow \bar{0.5} \equiv \bar{0.8}$; in general, for each rational $r, s \in [0, 1]$ we have

$$\begin{aligned} \text{(P1)} \quad & \neg\bar{r} \equiv \overline{(-)r}, \\ \text{(P2)} \quad & (\bar{r} \rightarrow \bar{s}) \equiv \overline{r \Rightarrow s} \end{aligned}$$

We add these schemas as new logical axioms; the resulting logic (with the language extended by truth constants and axioms extended by (P1), (P2)) will be called RPL (rational propositional logic or rational Pavelka logic). The only deduction rule is modus ponens.

If φ is a formula and $r \in [0, 1]$ is rational then (φ, r) denotes just the formula $(\bar{r} \rightarrow \varphi)$ (saying that φ is at least r -true). We have some derived deduction rules.

Lemma. Let T be a theory in RPL (a set of special axioms); for each formula α , $T \vdash \alpha$ means that α is provable in T .

- (1) If $T \vdash (\varphi, r)$ and $T \vdash (\varphi \rightarrow \psi, s)$, then $T \vdash (\psi, r * s)$.
- (2) If $T \vdash (\varphi, r)$ then $T \vdash (\bar{s} \rightarrow \varphi, s \Rightarrow r)$.

Definition. Let T be a theory in RPL. (1) The *truth degree* of φ in T is

$\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a model of } T\}$.
 (2) *The provability degree* of φ in T is

$$|\varphi|_T = \sup\{r \mid T \vdash (\varphi, r)\}.$$

Thus $\|\varphi\|_T$ is the infimum of values of φ in models of T ; $|\varphi|_T$ is the supremum of rationals r such that $T \vdash \bar{r} \rightarrow \varphi$.

Completeness theorem for RPL: Let T be a theory in RPL; then, for each formula φ , $\|\varphi\|_T = |\varphi|_T$.

This is a very pleasing and elegant result (invented originally by Pavelka); the proof is moderately difficult (much easier than the proof of completeness of L , but using the fact that we have the Rose-Rosser's complete axiom system for L).

Remarks. (1) A *fuzzy theory* is a fuzzy set of formulas, i.e. a mapping T associating to each formula φ the degree $T(\varphi)$ of being an axiom. An evaluation e is a *model* of T if for each φ , $e(\varphi) \geq T(\varphi)$, i.e. each formula is at least as much true, as the theory demands. It is natural to assume that each $T(\varphi)$ is a rational number. The notion of a fuzzy theory is central in Pavelka's approach but we see that it is superfluous; if you define $T' = \{(\varphi, T(\varphi)) \mid \varphi \text{ formula}\}$ (thus for each φ , if $T(\varphi) = r$ we put $(\bar{r} \rightarrow \varphi)$ into T') then T' is a (crisp) theory having the same models as T .

(2) The set of all formulas is a recursive set and the syntax is recursive; thus we may call a theory T *recursive* if T is a recursive set of formulas. Note that $|\varphi|_T$ may be irrational; on the other hand, if $r > 0$ is rational then we can construct a recursive theory T such that the set of all φ such that $|\varphi|_T \geq r$ is "badly" non-recursive (for experts: it may be Π_2 -complete; see [14] for details).

(3) We can similarly extend other logics, e.g. Gödel logic or product logic but unfortunately we cannot hope for Pavelka style completeness (as Pavelka himself tells us) since the truth function of implication is not continuous in these logics. To see this take the theory $T = \{p \rightarrow (\frac{\bar{1}}{n}) \mid n \text{ natural}\}$; then

$$\|p \rightarrow 0\|_T = 1 \text{ for each of } L, G, P;$$

$$|p \rightarrow 0|_T = 1 \text{ for } L \text{ but } |p \rightarrow 0|_T = 0 \text{ for both } G \text{ and } P \text{ (verify).}$$

Note that RPL satisfies the same generalized deduction theorem as L (and of course does not satisfy the classical deduction theorem).

1 Predicate calculi

1.1 The classical predicate calculus

In the present section we assume the reader to have some basic knowledge of the classical predicate calculus. In this subsection we survey the basic notions and facts, for comparison with their many-valued generalizations. We shall restrict ourselves to calculi without function symbols. Details may be found e.g. in [27].

A *language* consists of *predicates* P, Q, \dots , *object constants* c, d, \dots , *object variables* x, y, \dots . Each predicate is assigned a positive natural number as its *arity*. If P is an n -ary predicate and t_1, \dots, t_n are variables and/or constants then $P(t_1, \dots, t_n)$ is an atomic *formula*. Non-atomic formulas are from atomic ones using *connectives* \rightarrow, \neg and the *universal quantifier* \forall : if φ, ψ are formulas and x is an object variable then $\varphi \rightarrow \psi$, $\neg\varphi$, $(\forall x)\varphi$ are *formulas*. The variable x is bound in $(\forall x)\varphi$; other variables are free/bound in φ iff they are free/bound in $(\forall x)\varphi$. A variable is free/bound in $\neg\varphi$ iff it is such in φ ; it is free/bound in $\varphi \rightarrow \psi$ iff it is such in φ or in ψ . A formula is *closed* if it has no free variable.

Other connectives are introduced as abbreviations as in propositional quantifier; the *existential quantifier* \exists is defined thus: $(\exists x)\varphi$ abbreviates $\neg(\forall x)\neg\varphi$.

An *interpretation* of a language L is given by the following:

- a non-empty *domain* M ,
- for each n -ary predicate P , an n -ary *relation* $r_P \subseteq M^n$ (set of n -tuples of elements of M)
- for each constant c , an element $m_c \in M$.

The interpretation is *witnessed* if each element $m \in M$ is the meaning of a constant c , $m = m_c$. (This can be achieved by extending the language by some additional constants.) For each closed formula φ and each interpretation

$$\mathbf{M} = \langle M, (r_P)_{P \text{ predicate}}, (m_c)_{c \text{ constant}} \rangle,$$

The *truth value* of φ in \mathbf{M} is defined as follows:

- If $P(c, \dots, d)$ is a closed atomic formula then $\|P(c, \dots, d)\|_{\mathbf{M}} = 1$ iff $\langle m_c, \dots, m_d \rangle \in r_P$ (the tuple of meanings of c, \dots, d is in the relation r_P which is the meaning of P); otherwise $\|P(c, \dots, d)\|_{\mathbf{M}} = 0$;
- $\|\varphi \rightarrow \psi\|_{\mathbf{M}} \Rightarrow \|\varphi\|_{\mathbf{M}} \Rightarrow \|\psi\|_{\mathbf{M}}$, $\|\neg\varphi\|_{\mathbf{M}} = (-) \|\varphi\|_{\mathbf{M}}$;
- $\|(\forall x)\varphi\|_{\mathbf{M}} = \min_c \|\varphi(c)\|_{\mathbf{M}}$, where $\varphi(c)$ results from φ by substituting the constant c for (free occurrences of) x .

We write $M \models \varphi$ for $\|\varphi\|_{\mathbf{M}} = 1$ and read: φ is true in \mathbf{M} . If φ is not closed then $\mathbf{M} \models \varphi$ means that $M \models (\forall x_1) \dots (\forall x_n)\varphi$, where x_1, \dots, x_n are the variables free in φ .

A *theory* is a set of formulas (special axioms). \mathbf{M} is a *model* of a T if each $\varphi \in T$ is true in \mathbf{M} .

Logical axioms: axioms of classical propositional calculus plus

$$(A1) \quad (\forall x)\varphi \rightarrow \varphi(t)$$

where t is either a constant or an object variable free for x in φ (this is a simple condition preventing “clash of free and bound variables”) - the *substitution* axiom,

$$(A2) \quad (\forall x)(\nu \rightarrow \varphi) \rightarrow (\nu \rightarrow (\forall x)\varphi)$$

where ν is a formula in which x is not free.

Deduction rules: Modus ponens and *generalization:* from φ derive $(\forall x)\varphi$.

A *proof* in a theory T is a sequence $\varphi_1, \dots, \varphi_n$ of formulas (not necessarily closed) such that each φ_i either is a logical axiom or belongs to T (is a special axiom) or results from some previous formulas(s) using one of the deduction rules. A formula φ is *provable* in T (notation: $T \vdash \varphi$) if φ is the last member of a proof in T .

Gödel's completeness theorem: $T \vdash \varphi$ iff φ is true in each model of T . In particular, φ is a tautology (true in all interpretations) iff $\vdash \varphi$ (φ is provable using only logical axioms).

1.2 The basic fuzzy predicate logic

A *predicate language* consists of *predicates* P, Q, \dots , each together with its *arity* and *object constants*. c, d, \dots . *Logical symbols* are *object variables* x, y, \dots , *connectives* $\&, \rightarrow$, *truth constants* $\bar{0}, \bar{1}$ and *quantifiers* \forall, \exists . Other connectives ($\wedge, \vee, \neg, \equiv$) are defined as in propositional calculus. *Terms* are object variables and object constants.

Atomic formulas have the form $P(t_1, \dots, t_n)$ where P is a predicate of arity n and t_1, \dots, t_n are terms. If φ, ψ are formulas and x is an object variable then $\varphi \rightarrow \psi$, $\varphi \& \psi$, $(\forall x)\psi$, $(\exists x)\varphi$, $\bar{0}$, $\bar{1}$ are formulas; each formula results from atomic formulas by iterated use of this rule.

Let \mathcal{J} be a predicate language and let \mathbf{L} be a regular residuated lattice. An \mathbf{L} -structure $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ for \mathcal{J} , $M \neq \emptyset$, for each n -ary predicate P a \mathbf{L} -fuzzy n -ary relation $r_P : M^n \rightarrow \mathbf{L}$ on M and for each object constant c , m_c is an element of M .

An \mathbf{M} -evaluation of *object variables* is a mapping v assigning to each object variable x an element $v(x) \in M$. *Values* of terms and formulas are defined as follows: $\|\!|x\|\!|_{M,v} = v(x)$; $\|\!|c\|\!|_{M,v} = m_c$.

$$\begin{aligned} & \|P(t_1, \dots, t_n)\|_{M,v}^L = r_P(\|t_1\|_{M,v}, \dots, \|t_n\|_{M,v}); \\ & \|\varphi \rightarrow \psi\|_{M,v}^L = \|\varphi\|_{M,v}^L \Rightarrow \|\psi\|_{M,v}^L; \\ & \|\varphi \& \psi\|_{M,v}^L = \|\varphi\|_{M,v}^L * \|\psi\|_{M,v}^L; \\ & \|\bar{0}\|_{M,v} = \mathbf{0}; \quad \|\bar{1}\|_{M,v} = \mathbf{1}; \\ & \|(\forall x)\varphi\|_{M,v}^L = \inf\{\|\varphi\|_{M,v'}^L \mid v \equiv_x v'\}; \\ & \|(\exists x)\varphi\|_{M,v}^L = \sup\{\|\varphi\|_{M,v'}^L \mid v \equiv_x v'\}; \end{aligned}$$

provided the infimum/supremum exists in the sense of \mathbf{L} ;

The structure \mathbf{M} is \mathbf{L} -safe if all the needed infima and suprema exist, i.e. $\|\varphi\|_{M,v}^L$ is defined for all φ, v .

$$\|\varphi\|_{\mathbf{M}} = \inf\{\|\varphi\|_{M,v} \mid v \text{ } \mathbf{M} \text{ - evaluation}\}.$$

A formula φ of a language \mathcal{J} is an \mathbf{L} -tautology if $\|\varphi\|_{\mathbf{M}} = \mathbf{1}_{\mathbf{L}}$ for each safe \mathbf{L} -structure \mathbf{M} .

The following are *logical axioms on quantifiers*:

- ($\forall 1$) $(\forall x)\varphi(x) \rightarrow \varphi(t)$ (t substitutable for x in $\varphi(x)$)
- ($\exists 1$) $\varphi(t) \rightarrow (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
- ($\forall 2$) $(\forall x)(\nu \rightarrow \varphi) \rightarrow (\nu \rightarrow (\forall x)\varphi)$ (x not free in ν)
- ($\exists 2$) $(\forall x)(\varphi \rightarrow \nu) \rightarrow ((\exists x)\varphi \rightarrow \nu)$ (x not free in ν)
- ($\forall 3$) $(\forall x)(\nu \vee \varphi) \rightarrow (\nu \vee (\forall x)\varphi)$ (x not free in ν)

The predicate calculus \mathcal{CV} (over a given predicate language \mathcal{J}) has the following axioms:

- all formulas resulting from the axioms of \mathcal{C} by substituting arbitrary formulas of \mathcal{F} for propositional variables, and
- the axioms $(\forall 1), (\forall 2), (\exists 1), (\exists 2), (\forall 3)$ for quantifiers and deduction rules
- modus ponens (from $\varphi, \varphi \rightarrow \psi$ infer ψ) and
- generalization (from φ infer $(\forall x)\varphi$).

In particular, we are interested in $\text{BL}\forall$ and three stronger logics: $\text{L}\forall$ (Łukasiewicz), $\text{G}\forall$ (Gödel), $\text{P}\forall$ (product). Also note in passing that if \mathcal{C} is the classical propositional calculus (as described above) then in $\mathcal{C}\forall$ the axioms $(\forall 3), (\exists 1), (\exists 2)$ are redundant (provable from the rest); $(\forall 1), (\forall 2)$ are the usual axioms of the classical predicate logic.

The axioms $(\forall 1)$ – $(\forall 3), (\exists 1)$ – $(\exists 2)$ are \mathbf{L} -tautologies for each regular residuated lattice \mathbf{L} .

(Soundness of provability.) Let \mathcal{C} be a schematic extension of BL , let T be a theory in the language of T over $\mathcal{C}\forall$, let φ be a formula of T . If $T \vdash \varphi$ (φ is provable in T) then $\|\varphi\|_M^L = 1$ for each \mathcal{C} -lattice \mathbf{L} and each \mathbf{L} -model \mathbf{M} of T .

Let φ be an arbitrary formula, ν a formula not containing x freely. Then $\text{BL}\forall$ proves the following:

- (1) $(\forall x)(\nu \rightarrow \varphi) \equiv (\nu \rightarrow (\forall x)\varphi)$
- (2) $(\forall x)(\varphi \rightarrow \nu) \equiv ((\exists x)\varphi \rightarrow \nu)$
- (3) $(\exists x)(\nu \rightarrow \varphi) \rightarrow (\nu \rightarrow (\exists x)\varphi)$
- (4) $(\exists x)(\varphi \rightarrow \nu) \rightarrow ((\forall x)\varphi \rightarrow \nu)$

The converse implications in (3), (4) are *not* provable in BL . We shall see later that neither of them is a tautology of $\text{G}\forall$; the converse of (3) is but the converse of (4) is not a tautology of $\text{P}\forall$; and both converses are tautologies of $\text{L}\forall$.

For arbitrary formulas φ, ψ , $\text{BL}\forall$ proves the following:

- (5) $(\forall x)(\varphi \rightarrow \psi) \rightarrow ((\forall x)\varphi \rightarrow (\forall x)\psi)$
- (6) $(\forall x)(\varphi \rightarrow \psi) \rightarrow ((\exists x)\varphi \rightarrow (\exists x)\psi)$

$$(7) ((\forall x)\varphi \& (\exists x)\psi) \rightarrow (\exists x)(\varphi \& \psi)$$

For arbitrary φ and for ν not containing x freely, BL \forall proves

$$(9) (\exists x)(\varphi \& \nu) \equiv ((\exists x)\varphi \& \nu),$$

$$(10) (\exists x)(\varphi \& \varphi) \equiv ((\exists x)\varphi \& (\exists x)\varphi).$$

BL \forall proves the following:

$$(11) (\exists x)\varphi \rightarrow \neg(\forall x)\neg\varphi$$

$$(12) \neg(\exists x)\varphi \rightarrow (\forall x)\neg\varphi$$

Completeness. Let T be a theory over \mathcal{CV} . For each formula φ , T proves φ iff for each linearly ordered \mathcal{C} -algebra \mathbf{L} and each safe \mathbf{L} -model of T , $\|\varphi\|_{\mathbf{L}}^T = 1$.

1.3 Łukasiewicz predicate logic

L \forall proves

$$(\exists x)\varphi \equiv \neg(\forall x)\neg\varphi.$$

$$(\forall x)(\varphi \& \nu) \rightarrow ((\forall x)\varphi \& \nu).$$

Axioms ($\exists 1$)($\exists 2$)($\forall 3$) are redundant (provable from the others).

Lemma. L \forall proves the following:

$$(\nu \rightarrow (\exists x)\varphi) \rightarrow (\exists x)(\nu \rightarrow \varphi),$$

$$((\forall x)\varphi \rightarrow \nu) \rightarrow (\exists x)(\varphi \rightarrow \nu).$$

Theorem. There is no recursive axiomatic system complete with respect to L \forall -tautologies (over $[0, 1]_{\mathbf{L}}$). Moreover, the last set is Π_2 -complete.[36, 37, 42]

1.4 Rational Pavelka quantification logic

We extend Łukasiewicz predicate logic by propositional constants \bar{r} for each rational $r \in [0, 1]$; for each \mathbf{M} , $\|\bar{r}\|_{\mathbf{M}} = r$. The *axioms* of RPL \forall are those of RPL plus (A1), (A2) from 3.1 plus

We introduce (φ, r) as abbreviation of $(\bar{r} \rightarrow \varphi)$ as above; given a theory T , we define the provability degree and truth degree as above:

$$|\varphi|_T = \sup\{r \mid T \vdash (\varphi, r)\},$$

$$\|\varphi\|_T = \inf\{\|\varphi\|_{\mathbf{M}} \mid M \text{ a model of } T\}.$$

(We should say that for a non-closed φ , $\|\varphi\|_{\mathbf{M}}$ is defined as $\|(\forall x_n)\dots(\forall x_n)\varphi\|_{\mathbf{M}}$ analogously as above; \mathbf{M} is a *model* of T if $\|\varphi\|_{\mathbf{M}} = 1$ for each $\varphi \in T$.)

We have the following Pavelka-style

Completeness theorem (see [16]). For each theory T and formula φ ,

$$\|\varphi\|_T = |\varphi|_T,$$

i.e. the truth degree equals the provability degree. Let T be a recursive theory. For each positive $r \in [0, 1]$, the set $Pr(T, r)$ of all φ such that $|\varphi|_T \geq r$ is Π_2 ; there is a recursive theory T such that $Pr(T, 1)$ is Π_2 -complete. (See again [16].)

Thus RPL \forall is an elegant fuzzy predicate calculus with truth degree equal to provability degree; on the other hand, it badly undecidable. For details see [16] and its predecessors, in particular, [30].

1.5 Gödel predicate logic

This logic is, in contradistinction to Łukasiewicz logic, recursively axiomatizable.

Logical axioms are those of Gödel propositional logic (see 2.5) plus the axioms $(\forall 1)$, $(\forall 2)$, $(\forall 3)$, $(\exists 1)$, $(\exists 2)$ of BL \forall (see above) *Deduction rules* are modus ponens and generalization. The logic is *sound* in the following sense: if $T \vdash \varphi$ then for each M there is a conjunction α of finitely many elements of T such that $\|\alpha\|_T \leq \|\varphi\|_T$. It follows that if all axioms of T are 1-true in M ($\|\alpha\|_{\mathbf{M}} = 1$) and $T \vdash \varphi$ then $\|\varphi\|_{\mathbf{M}} = 1$ too. Moreover, if M is such that $\|\alpha\|_{\mathbf{M}} \geq r$ for some r and all $\alpha \in T$ and if $T \vdash \varphi$ then $\|\varphi\|_{\mathbf{M}} \geq r$.

Completeness $T \vdash \varphi$ iff for each M there is a conjunction α of finitely many elements of T such that $\|\alpha\|_{\mathbf{M}} \leq \|\varphi\|_{\mathbf{M}}$. In particular, φ is a 1-tautology ($\|\varphi\|_{\mathbf{M}} = 1$ for all \mathbf{M}) iff $\vdash \varphi$.

Hence, in contradistinction to Łukasiewicz predicate logic (and Rational Quantification Logic), the set of all 1-tautologies of Gödel predicate logic is recursively enumerable.

Historical remark. Recursive axiomatizability of Gödel predicate logic was first shown by Takeuti and Titani [45] using an auxiliary deduction rule. M. Baaz showed that the rule is superfluous (still unpublished).

We have surveyed two main systems of fuzzy predicate calculus: Łukasiewicz's calculus (with its extension RPL \forall à la Pavelka-Novák) and Gödel's calculus (à la Takeuti-Titani). The investigation of a predicate calculus based on the product conjunction remains to be a future task. We know that the set of 1-tautologies of $\Pi\forall$ is not recursively enumerable; moreover, it is Π_2 -hard.

2 General fuzzy logics

In this section we describe a very general approach to the syntax and semantics of fuzzy logics, developed by Pavelka [34]. This approach does not assume any truth functionality.

2.1 Formulas and models

We have a set *Form* of *formulas*. These may be formulas of some propositional logic, predicate logic, or quite abstract entities. *Semantics* is given by a set \mathcal{S} whose elements are called *models*. Each model is a mapping $M : Form \rightarrow [0, 1]$; thus M assigns to each formula the degree in which it is true (in the model).

For example, *Form* consists of formulas of Łukasiewicz propositional calculus and \mathcal{S} consists of all $e : Form \rightarrow [0, 1]$ obeying the truth functions of connectives, i.e. $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$, $e(\neg\varphi) = (-)e(\varphi)$.

Any $T : Form \rightarrow [0, 1]$ may be understood as a *fuzzy theory*; $T(\varphi)$ is the degree in which φ is an axiom. An $M \in \mathcal{S}$ is a *model of T* if, for each φ , $M(\varphi) \geq T(\varphi)$ (each formula is at least as true as the theory T demands).

For each fuzzy theory T and formula φ , let $\|\varphi\|_T = \inf\{M(\varphi) \mid M \text{ is a model of } T\}$ (the truth degree of φ for T).

2.2 Provability

We shall work with *graded formulas*, i.e. pairs (φ, x) where φ is a formula and $x \in [0, 1]$. An n -ary *deduction rule* assigns to

some n -tuples $(\varphi_1, x_1, \dots, (\varphi_n, x_n))$ of graded formulas a graded formula $(r'(\varphi_1, \dots, \varphi_n), r''(x_1, \dots, x_n))$ (r', r'' are appropriate functions).

The function r'' is assumed to preserve all (infinite) suprema, i.e. if $\sup_{n \in I} (x_n) = y$ then $\sup_{n \in I} (r''(\dots, x_n, \dots)) = r''(\dots, \sup_{n \in I} x_n, \dots)$.

For example, recall the *fuzzy modus ponens* in Łukasiewicz logic:

$$\frac{(\varphi, x), (\varphi \rightarrow \psi, y)}{(\psi, x * y)}.$$

A theory T is *closed under* the rule (r', r'') if for each tuple $\varphi_1, \dots, \varphi_n$ of formulas, $T(r'(\varphi_1, \dots, \varphi_n)) \geq r''(T(\varphi_1), \dots, T(\varphi_n))$, i.e. if $T(\varphi_i) = x_i$ and $T(r'(\varphi_1, \dots, \varphi_n)) = y$ then from $(\varphi_1, x_1), \dots, (\varphi_n, x_n)$ the rule derives $(r'(\varphi_1, \dots, \varphi_n), r''(x_1, \dots, x_n))$ and T demands $r'(\varphi_1, \dots, \varphi_n)$ to be at least y -true, $y \geq r''(x_1, \dots, x_n)$.

A *deductive structure* is given by a fuzzy theory A (of logical axioms) and a set \mathcal{R} of deduction rules. For each fuzzy theory T , there is a unique theory $T' \supseteq T$ such that $T' \supseteq A$ and T' is closed under each rule from \mathcal{R} . T' is denoted $Cn_{A, \mathcal{R}}(T)$.

A *graded proof* in T (given A, \mathcal{R}) is a set of graded formulas $(\varphi_1, x_1), \dots, (\varphi_n, x_n)$ such that each (φ_i, x_i) either is a logical axiom ($A(\varphi_i) = x_i$) or is an axiom of T ($T(\varphi_i) = x_i$) or (φ_i, x_i) results by a rule $R \in \mathcal{R}$ from some previous graded formulas. The *provability degree* $|\varphi|_T$ is $\sup\{r \mid T \vdash (\varphi, r)\}$ (where $T \vdash (\varphi, x)$ obviously means that (φ, x) is the last member of a proof).

The condition of *sup* preservation guarantees that for each φ $|\varphi|_{T'} = Cn_{A, \mathcal{R}}(T)(\varphi)$.

The deductive structure (A, \mathcal{R}) is *sound* for the semantics \mathcal{S} if for each theory T and each formula φ , $|\varphi|_T \leq \|\varphi\|_T$ ($|\varphi|_T$ being defined using (A, \mathcal{R}) , $\|\varphi\|_T$ using \mathcal{S}). It is *complete* if $|\varphi|_T = \|\varphi\|_T$.

3 Equality and similarity

3.1 Classical predicate calculus with equality

Classical predicate calculus is often extended with to deal with the relation of equality. This is achieved by introducing in the language a new predicate E of arity 2 and adding to the axioms of classical predicate calculus (i.e.

the three axioms of classical propositional calculus plus the two axioms on quantifiers ($\forall 1$) and ($\forall 2$)) the following two axioms for equality:

$$(E1) \quad E(x, x)$$

$$(E2) \quad E(x, y) \rightarrow (P(\dots, x, \dots) \rightarrow P(\dots, y, \dots))$$

where P is any predicate of the (extended) language. Then any extension of classical predicate calculus including these axioms is called a PC system with equality. Let T be such an extension. Then it is easy to show that T proves the following formulas about the equality predicate:

$$(\forall x)E(x, x)$$

$$(\forall x)(\forall y)(E(x, y) \rightarrow E(y, x))$$

$$(\forall x)(\forall y)(\forall z)(E(x, y) \rightarrow (E(y, z) \rightarrow E(x, z)))$$

Thus since each of these must be true in any model of T , the predicate E has to be interpreted by an equivalence relation (reflexive, symmetric and transitive), but not necessarily as an equality $=$. However it can be also proved that any consistent PC system with equality has a model where E is interpreted by $=$.

3.2 Many-valued predicate calculi with fuzzy equality

The fuzzy counterpart of classical equivalence relations is the following notion of fuzzy similarity relation, also known as fuzzy equality relations.

Let W be a set and let L be a linearly ordered residuated lattice. A binary L -fuzzy relation S on W (i.e. a mapping $S : W \times W \rightarrow L$) is a **-similarity relation* if it satisfies the following properties ([47]):

1. reflexivity: $S(w, w) = 1$
2. symmetry: $S(w, w') = S(w', w)$
3. *-transitivity: $S(w, w') * S(w', w'') \leq S(w, w'')$

When $S(W \times W) = \{0, 1\}$, S is clearly an equivalence relation on W . For simplicity we shall assume that L is the interval $[0,1]$ with the structure given by a t-norm $*$ and its residuum \Rightarrow . Let us discuss our three basic t-norms:

- $*$ = minimum: then S is a similarity relation in the sense of Zadeh [48]). Especially, each level-cut $S_\alpha = \{(w, w') \mid S(w, w') \leq \alpha\}$ is an equivalence relation, and $1 - S$ defines a pseudo-ultrametric;

- $*$ = product: this type of fuzzy relation goes back to Menger [28] and has been studied by Ovchinnikov [32];
- $*$ = Lukasiewicz conjunction, i.e. $a * b = \max(0, a + b - 1)$. This type of fuzzy relation is studied by Ruspini [40], Bezdek and Harris [2] who call it a likeness relation. Then $1 - S$ is a pseudo-metric.

A similarity is thus a notion dual to a distance. A $*$ -similarity relation will be called *fuzzy equality* if in addition it verifies the following separating property:

$$S(w, w') = 1 \text{ iff } w = w'.$$

If S is a fuzzy equality, the 1-cut of S (that is, $\{(w, w') \mid S(w, w') = 1\}$) is just the equality on W .

Therefore, in order to define what a many-valued logical system with equality is, it seems natural to add the following axioms for fuzzy equality to our basic many-valued predicate logic $BL\forall$:

$$(E1) \quad E(x, x)$$

$$(E2) \quad E(x, y) \rightarrow (P(\dots, x, \dots) \rightarrow P(\dots, y, \dots))$$

where P is any predicate.

Then, analogously to classical predicate logic, in any model of any theory of containing (E1), (E2) the predicate E must be interpreted by a fuzzy relation which must be a $*$ -similarity relation and moreover, the interpretations of the rest of predicates have to be *extensional* [22]. Indeed, the following formulas

$$(\forall x)E(x, x)$$

$$(\forall x)(\forall y)(E(x, y) \rightarrow E(y, x))$$

$$(\forall x)(\forall y)(\forall z)(E(x, y) \rightarrow (E(y, z) \rightarrow E(x, z)))$$

directly corresponding to the reflexivity, symmetry and $*$ -transitivity properties of the similarity relations, are also provable in any theory over $BL\forall$ containing (E1) and (E2). Moreover, if S is the interpretation of E and μ_P is the interpretation of the predicate P (we consider P of arity 1), since (E2) is a 1-tautology, it follows that

$$S(a, b) \leq \mu_P(a) \Rightarrow \mu_P(b)$$

that is,

$$\mu_P(a) * S(a, b) \leq \mu_P(b),$$

which is the condition for the fuzzy set μ_P to be extensional [22], which in turn is a generalization of the classical condition

$$\text{if } a \in A \text{ and } a \equiv b \text{ then } b \in A$$

for a subset A to behave well with respect to an equivalence relation \equiv , or in other words, the condition for A to be a union of equivalence classes.

Finally, let us mention, again analogously to the classical case, that the interpretation of the equality predicate in models of theories with equality need not be a fuzzy equality in the above sense. However, for any consistent theory with equality there is a model where the equality predicate is interpreted as a fuzzy equality relation. The proof is as follows.

Let T be a consistent theory with equality over \mathcal{CV} , and let $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ a model for T . Let $S = r_E$ be the interpretation of E in the model M . It is clear that S must be a $*$ -similarity relation. Define the equivalence relation on M as follows: $a \sim b$ iff $S(a, b) = 1$, and denote the equivalence class containing a by $[a]$. Now define a new structure $\mathbf{M}' = \langle M', (r'_P)_P, (m'_c)_c \rangle$, where $M' = M / \sim$, $r'_P([a]) = [r_P(a)]$ and $m'_c = [m_c]$. It can be checked that \mathbf{M}' is a model of T and $S' = r'_E$ is a fuzzy equality.

3.3 Similarity-based logical systems

One of the possible semantics of fuzzy sets is in terms of similarity, namely a grade of membership of an item in a fuzzy set can be viewed as the degree of resemblance between this item and prototypes of the fuzzy set. In such a framework, an interesting question is how to devise a logic of similarity able to account for the proximity between interpretations.

A variety of uncertain reasoning models has been captured in the modal framework by equipping the set of boolean interpretations or possible worlds with a suitable uncertainty measure (see e.g. [15]). It is thus tempting to model similarity-based reasoning by equipping a set of possible worlds with a proximity or generalized metric structure.

Similarity relations and fuzzy sets can be closely related. Namely let $A \subseteq \Omega$ be a non-empty subset of Ω . Then a similarity relation S allows us to define the non-empty normalized fuzzy set A^* of elements *close* to A as follows:

$$\mu_{A^*}(w) = \max_{w' \in A} S(w, w')$$

. Conversely, any non-empty fuzzy set F on Ω can be viewed as deriving from a $*$ -similarity relation S and a subset A such that

$$A = \{w \mid \mu_F(w) = 1\} (\neq \emptyset)$$

$$S(w, w') = \min(\mu_F(w) \Rightarrow \mu_F(w'), \mu_F(w') \Rightarrow \mu_F(w))$$

This is due to Valverde's theorem of representation of similarity relations by fuzzy sets [47], based on residuation. This result gives a formal justification to the fact that a degree of membership $\mu_F(w)$ in a fuzzy set can be interpreted as a degree of similarity of w to *prototypes* of F , which form the set A .

Moreover it points out that if q is a proposition in a formal propositional language L , of which Ω is the (finite) set of interpretations, then the similarity induces a fuzzy proposition q^* whose (fuzzy) set of models is $[q^*] = [q]^*$, defined by means of the fuzzy relation S , where $[q]$ denotes the (classical) set of models of q (the set of interpretations where q is true). Intuitively q^* means *approximately q, not far from q*, where “approximately”, “not far from” is mathematically expressed by the similarity relation S .

Analogously to what we have said in the introduction on fuzziness and probability, the similarity-based approach in the frame of truth-functional fuzzy logic has to distinguish between a crisp proposition q and its fuzzy counterpart *approximately q*, keeping strictly in mind that *approximately p&q*, i.e. $(p \& q)^*$ is not equivalent to *approximately p and approximately q*, i.e. $(p^* \& q^*)$. Then one may be safely truth-functional.

But our aim in this section is to describe another approach that consists in considering for eac q the corresponding *approximately q*, i.e. in defining a graded satisfaction relation on the formulas of the original given propositional language as follows:

$$w \models_S^\alpha q \text{ iff } \mu_{[q^*]}(w) \geq \alpha$$

That is, in the finite case, $w \models_S^\alpha q$ if there exists a model w' of q which is α -similar to w . In other words, w belongs to the α -cut of $[q^*]$ that will be denoted by $[q^*]_\alpha$. The degree of approximate satisfaction of q by w in the sense of S has been introduced by Ruspini [41], and shall be denoted

$$I_S(q \mid w) = \mu_{[q^*]}(w) = \max_{w' \models q} S(w, w').$$

Note that, identifying each interpretation w of Ω with the conjunction of literals made true by w , we have that $I_S(w' \mid w) = S(w, w')$. Thus, one may

have $w \models_S^\alpha w'$ for $w' \neq w$. Indeed it means that w and w' are close enough to each other in the sense that $S(w, w') \geq \alpha$. Note that $w \models_S^\alpha w'$ is equivalent to $w' \models_S^\alpha w$ since S is symmetric.

The graded satisfaction relation can be extended over a graded entailment between (boolean) propositions in the obvious way:

$$p \models_S^\alpha q \text{ iff } w \models_S^\alpha q \text{ for each } w \text{ model of } p$$

In other words, $p \models_s^\alpha q$ holds if each model of p is similar, at least to the degree α , to some model of q . An equivalent definition is $p \models_S^\alpha q$ iff $I_S(q | p) \geq \alpha$, where $I_S(q | p) = \inf_{w \models p} I_S(q | w)$ is the Ruspini's *implication measure* of q given p . The graded entailment has been characterized in terms of the following properties [4]:

- Nested:* If $p \models^\alpha q$ then $p \models^\beta q$, for $\beta \leq \alpha$.
- Extremals:* $p \models^1 q$ iff $p \models q$; $p \models^0 q$.
- *-Transitivity:* If $p \models^\alpha q$ and $q \models^\beta r$ then $p \models^{\alpha*\beta} r$.
- Left Or:* $p \vee q \models^\alpha r$ iff $p \models^\alpha r$ and $q \models^\alpha r$.
- Right Or:* If r has a single model then $r \models^\alpha p \vee q$ iff $r \models^\alpha p$ or $r \models^\alpha q$.
- Consistency preservation:* If $p \not\models \top$ then $p \models^\alpha \top$ only when $\alpha = 0$.
- Continuity from below:* If $p \models^\beta q$ for each $\beta < \alpha$ then $p \models^\alpha q$.

One can understand this as a general fuzzy logic in the sense of Section 4. But one has to be aware of the fact that such a logic cannot be truth-functional. Namely given S , the truth-value evaluation $I_S(q | w)$ of q associated to the interpretation w is truth-functional neither for the negation nor for the conjunction since only the following inequalities hold in the general case:

$$I_S(\neg q | w) \geq 1 - I_S(q | w)$$

$$I_S(p \wedge q | w) \leq \min(I_S(p | w), I_S(q | w))$$

However for disjunction we do have that $[p \vee q]^* = [p]^* \cup [q]^*$, hence $I_S(p \vee q | w) = \max(I_S(p | w), I_S(q | w))$. This fact stresses the difference between similarity logic and many other logics underlying fuzzy sets like the truth-functional fuzzy logics described in Section 2. This lack of truth-functionality has also been noticed in the theory of rough sets (Pawlak, 1991). Rough sets are a theory of similarity based on equivalence relations that handles upper and lower approximations of sets. The lack of truth-functionality is thus not due to the fuzziness of similarity.

A natural logical setting for similarity-based reasoning is the one of modal logics which is tailored to account for relations on the set of interpretations. The similarity relation S can be considered as a family of nested accessibility relations R_α on the set of possible worlds Ω defined as $wR_\alpha w'$ iff $S(w, w') \geq \alpha$. Therefore, enlarging the logical language, we can define, for each α , a pair of dual modal operators \Box_α and \Diamond_α with the following standard semantics:

$$w \models \Diamond_\alpha p \text{ iff there exists } w' \text{ such that } wR_\alpha w' \text{ and } w' \models p$$

$$w \models \Box_\alpha p \text{ iff for every } w' \text{ such that } wR_\alpha w' \text{ then it holds } w' \models p$$

If the similarity relation is min-transitive, i.e.

$$S(w, w') \geq \min(S(w, w''), S(w'', w')),$$

then the accessibility relations R_α are equivalence relations, and therefore, for each α , \Box_α and \Diamond_α are a pair of dual S5 modal operators. These types of modal logics generalize rough set logics (Orlowska, 1984) and have been studied by Nakamura (1992). It is easy to check that the above defined graded satisfaction \models_S^* is directly related to the possibility operator \Diamond_α in the sense that if q is a non-modal proposition, then $w \models_S^* q$ iff $w \models \Diamond_\alpha q$.

In the following we shall describe the multi-modal system axiomatizing the graded modal operators \Box_α and \Diamond_α .

To define the language we fix a range $G \subset [0, 1]$ of possible similarity values. Further assumptions on G are that $\{0, 1\} \subseteq G$ and that, for the sake of simplicity, we shall assume that G is denumerable. Then, the multi-modal propositional language is built, in the usual way, upon a denumerable set of propositional variables p, q, \dots , connectives \rightarrow (implication) and \neg (negation), and (unary) modal operators \Diamond_α^o and \Diamond_α^c , for each $\alpha \in G$. We shall use φ, ψ, \dots to denote arbitrary formulas. We shall also use the classical definitions of \wedge and \vee in terms of \rightarrow and \neg , and furthermore $\Box_\alpha^o \varphi$ and $\Box_\alpha^c \varphi$ will stand for abbreviations of $\neg \Diamond_\alpha^o \neg \varphi$ and $\neg \Diamond_\alpha^c \neg \varphi$ respectively.

A *similarity Kripke model* is a structure $\mathcal{M} = \langle W, S, \|\ \|\rangle$ where:

1. W is a non empty set of possible worlds,
2. $S : W \times W \rightarrow G$ is a $*$ -similarity fuzzy relation on W , for some t-norm $*$ on G ,
3. $\|\ \|\$ is a function that given an atomic formula p return the set $\|p\| \subseteq W$ where p is considered to be true.

The notion of a formula φ being true in a world w in a similarity Kripke model $\mathcal{M} = \langle W, S, \|\ \|\rangle$, written $(\mathcal{M}, w) \models \varphi$ is defined in the usual way, except for the modal formulas, which is defined as follows:

$$(\mathcal{M}, w) \models \diamond_{\alpha}^c \varphi \text{ iff } I_S^{\mathcal{M}}(\varphi \mid w) \geq \alpha$$

$$(\mathcal{M}, w) \models \diamond_{\alpha}^o \varphi \text{ iff } I_S^{\mathcal{M}}(\varphi \mid w) > \alpha$$

where the implication measure $I_S^{\mathcal{M}}$ is defined as follows:

$$I_S^{\mathcal{M}}(\varphi \mid w) = \sup_{(\mathcal{M}, w') \models \varphi} S(w, w').$$

Notice that \diamond_{α}^o is a normal modal operator in the sense that it has an associated accessibility relation R_{α}^o which provides it with the standard Kripke semantics:

$$(\mathcal{M}, w) \models \diamond_{\alpha}^o \varphi \text{ iff } (\mathcal{M}, w') \models \varphi \text{ for some } w' \text{ such that } (w, w') \in R_{\alpha}^o,$$

where the accessibility relation R_{α}^o is defined as

$$(w, w') \in R_{\alpha}^o \text{ iff } S(w, w') > \alpha.$$

This is not the general case for the operators \diamond_{α}^c , i.e. they do not have, in general, a corresponding accessibility relation. However they do have it whenever the *sup* appearing in the expression of $I_S^{\mathcal{M}}(\varphi \mid w)$ is reached for any φ and any w , i.e. when $I_S^{\mathcal{M}}(\varphi \mid w)$ becomes $\max_{(\mathcal{M}, w') \models \varphi} S(w, w')$. In particular, this is the case when either the range G is finite or the set of possible worlds W is finite.

Given a range G and a t-norm operation $*$ on G , the class of structures \mathcal{C}_{*}^G is the set of similarity structures $\mathcal{M} = \langle W, S, \|\ \|\rangle$ where S is a $(G, *)$ -similarity on W . The notation \mathcal{FC}_{*}^G will denote the subclass of \mathcal{C}_{*}^G consisting of similarity structures with a finite set of worlds W .

The basic similarity multi-modal logic **MS5**($G, *$) is the smallest set of sentences containing every instance of the following axiom schemes and closed under the last two inference rules:

- PL : Propositional tautologies
 K^c : $\Box_\alpha^c(\varphi \rightarrow \psi) \rightarrow (\Box_\alpha^c\varphi \rightarrow \Box_\alpha^c\psi), \forall \alpha \in G$
 K^o : $\Box_\alpha^o(\varphi \rightarrow \psi) \rightarrow (\Box_\alpha^o\varphi \rightarrow \Box_\alpha^o\psi), \forall \alpha \in G$
 T^c : $\Box_\alpha^c\varphi \rightarrow \varphi, \forall \alpha \in G$
 B^o : $\varphi \rightarrow \Box_\alpha^o\Diamond_\alpha^o\varphi, \forall \alpha \in G$
 4^c : $\Box_{\alpha*\beta}^c\varphi \rightarrow \Box_\beta^c\Box_\alpha^c\varphi, \forall \alpha \in G$
 N^c : $\Box_\alpha^c\varphi \rightarrow \Box_\beta^c\varphi, \text{ for } \beta \geq \alpha,$
 EX^c : $\Diamond_0^c\varphi,$
 EX^o : $\neg\Diamond_1^o\varphi,$
 CO : $\Box_\alpha^c\varphi \rightarrow \Box_\alpha^o\varphi, \forall \alpha \in G$
 OC : $\Box_\alpha^o\varphi \rightarrow \Box_\beta^o\varphi, \text{ for } \alpha < \beta,$
 RN^o : From φ infer $\Box_\alpha^o\varphi, \forall \alpha \in G$
 MP : From φ and $\varphi \rightarrow \psi$ infer ψ

Schemes K^i, T^i, B^i and 4^i , where i is either c or o , are direct counterpart, for the graded modal operators, of the well-known axioms of the classical $S5$ modal logic. Scheme C^c corresponds to the fact that, under the assumption of finite range G or finite set of worlds W , $I_S^M(\varphi \mid w) = 1$ only if φ is true in w . Schemes N^i stand for the nested properties of the graded modal operators, while schemes EX^i set up the extremal conditions for them. Finally, schemes OC and CO establish the obvious relations between strict and non-strict inequalities.

It is very easy to check that $\mathbf{MS5}(\mathbf{G}, *)$ is sound with respect to the class of structures \mathcal{C}_*^G , for any G and $*$. The question whether, in general, $\mathbf{MS5}(\mathbf{G}, *)$ is complete, has not been addressed yet. However there is completeness in the following particular cases.

1. For any *finite* range G , the system $\mathbf{MS5}^+(\mathbf{G}, *)$ obtained from $\mathbf{MS5}(\mathbf{G}, *)$ by adding the axiom:

$$C^c : \varphi \rightarrow \Box_1^c\varphi.$$

is complete with respecte to the class of similarity models \mathcal{C}_*^G .

2. For any *dense* range G , and $*$ = *minimum*, the system $\mathbf{MS5}^{++}(\mathbf{G}, \min)$ obtained from $\mathbf{MS5}(\mathbf{G}, \min)$ by adding the axioms:

$$\begin{aligned}
B^c &: \varphi \rightarrow \Box_\alpha^c\Diamond_\alpha^c\varphi, \text{ for } \alpha > 0 \\
C^c &: \varphi \rightarrow \Box_1^c\varphi, \text{ and} \\
4^o &: \Box_{\alpha\wedge\beta}^o\varphi \rightarrow \Box_\beta^o\Box_\alpha^o\varphi, \forall \alpha \in G,
\end{aligned}$$

is complete with respect to the class of similarity models \mathcal{FC}_{min}^G .

Remark that, for the case of G being finite, one can define the set of open modal operators $\{\Box_\alpha^o\}_{\alpha \in G}$ in terms of the closed ones $\{\Box_\alpha^c\}_{\alpha \in G}$, and therefore the system $\mathbf{MS5}^+(\mathbf{G}, *)$ admits the following much simpler axiom system:

- PL : Tautologies of propositional logic,
- K : $\Box_\alpha(\varphi \rightarrow \psi) \rightarrow (\Box_\alpha\varphi \rightarrow \Box_\alpha\psi)$,
- T : $\Box_\alpha\varphi \rightarrow \varphi$,
- B : $\varphi \rightarrow \Box_\alpha\Diamond_\alpha\varphi$,
- 4 : $\Box_{\alpha*\beta}\varphi \rightarrow \Box_\beta\Box_\alpha\varphi$,
- C : $\varphi \rightarrow \Box_1\varphi$,
- N : $\Box_\alpha\varphi \rightarrow \Box_\beta\varphi$, with $\beta \geq \alpha$,
- EX : $\Diamond_0\varphi$,

where we have written \Box_α for \Box_α^c .

As a kind of final remark, notice that it is clear that the similarity-based graded entailment relation \models_S^α introduced at the beginning of this section is fully captured inside the multi-modal systems. Namely, given a $*$ -similarity S on the set of interpretations Ω of the propositional sublanguage, if φ and ψ are non-modal formulas, then we have that

$$\varphi \models_S^\alpha \psi \text{ iff } \mathcal{M}_\mathcal{L} \models \varphi \rightarrow \Diamond_\alpha^c \psi,$$

where $\mathcal{M}_\mathcal{L} = \langle \Omega, S, \|\ \|\rangle$.

4 Appendix: Similarity and approximate reasoning

In this section we use the notions of Section 5 (subsections 1, 2) to analyze some typical patterns of approximate reasoning by the means of the notion of logical deduction, in particular, the so-called compositional rule of inference, generalized modus ponens, and the “inference” in fuzzy control. These topics have been largely discussed in the literature. We may recommend monographs [11], [24], [23]. In particular, the subsequent presentation is influenced by the work of Kruse, Gebhardt and Klawonn (see also [22]). In this section we shall use the *many-sorted* variant of fuzzy predicate calculus, which is the immediate generalization of the one-sorted case: each variable

and constant has a sort, each unary predicate has a sort, each binary predicate has a sort for its first argument and one for its second argument etc. In particular, let us agree that for each sort i , the symbol \approx_i will be always use for a fuzzy equality (similarity) predicate of the sort i for both arguments. (The index i may be omitted if clear from the context.) Let \mathcal{J} be a many-sorted predicate language and let \mathbf{L} be a regular residuated lattice. An \mathbf{L} -structure $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ for \mathcal{J} consists of the following: $M \neq \emptyset$, for each n -ary predicate P with sorts (σ, τ) a \mathbf{L} -fuzzy n -ary relation $r_P : M_\sigma \times \dots \times M_\tau \rightarrow \mathbf{L}$ and for each object constant c of the sort σ , m_c is an element of M_σ .

Example: Two sorts t, p (temperature and pressure), unary predicates Ht, Hp (high temperature, high pressure), one binary predicate F of sorts (t, p) relating temperatures and corresponding pressures. Variable x of sort t , variable y of sort p . Formula:

$$(\forall x)(\forall y)((F(x, y) \& Ht(x)) \rightarrow Hp(y))$$

saying: “for all temperatures x and pressures y , if y corresponds to x and x is a high temperature then y is a high pressure”. We elaborate this to a general approach.

4.1 The compositional rule of inference

Let us define a *variate* to be given by its *name* X and its *domain* D . X is just a symbol; D is a non-empty set. Examples are: age with the domain of integers ≤ 120 (say), temperature (with some domain), etc. Fuzzy logic notoriously uses expressions of the form “ X is A ” where A is (the name of) a fuzzy subset of D , e. g. “the age is high”. These expressions typically occur in *fuzzy rules* to be analyzed later.

How to formalize this: having n variates $(X_1, D_1), \dots, (X_n, D_n)$ we understand the D ’s as domains of a many-sorted structure interpreting a predicate language; fixed fuzzy subsets of a domain interpret some unary predicates. The name of a variate is taken to be an *object constant*, interpreted in each situation as the *actual* value of the variate. The expression “ X is A ” becomes an atomic closed formula $A(X)$. A typical rule “IF X is A THEN Y is B ” may be interpreted as $A(X) \rightarrow B(Y)$

The *compositional rule of inference* in its traditional formulation can be stated as follows:

From “ X is A ” and “ (X, Y) is R ” infer “ Y is B ” if for all $v \in D_Y$,

$$r_B(v) = \sup_{u \in D_X} (r_A(u) * r_R(u, v)).$$

where $*$ is a continuous t -norm. The relation r_B is sometimes called the *composition* of r_A and r_R , on the *image* of r_A under the relation r_R .

Observe that in fact the definition of r_B in terms of r_A and r_R is expressible in BL:

$(\forall y)(B(y) \equiv (\exists x)(A(x) \& R(x, y)))$ is 1-true in \mathbf{D} . Call the last formula *Comp*.

Lemma. $BL\forall$ proves

$$Comp \rightarrow ((A(X) \& R(X, Y)) \rightarrow B(Y)).$$

Consequently, for each structure \mathbf{D} such that $\|Comp\|_{\mathbf{D}} = 1$,
 $\|A(X) \& R(X, Y)\|_{\mathbf{D}} \leq \|B(Y)\|_{\mathbf{D}}$

Consider Zadeh’s Generalized Modus Ponens as a particular case of the Compositional Inference rule. To this end let us slightly change notation: we replace A by A^* , B by B^* and then take $R(x, y)$ to be $A(x) \rightarrow B(y)$ for some predicates A, B .

Definition and lemma. Let $Comp_{MP}$ be the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \& (A(x) \rightarrow B(y)))).$$

Then $BL\forall$ proves

$$(Comp_{MP} \& A^*(X) \& (A(X) \rightarrow B(Y))) \rightarrow B^*(Y).$$

This may be visualized as a deduction rule:

$$\frac{Comp_{MP}, A^*(X), A(X) \rightarrow B(Y)}{B^*(Y)}$$

moreover:

$$\begin{aligned} \|Comp_{MP} \& A^*(X) \& (A(X) \rightarrow B(Y))\|_{\mathbf{D}} &\leq \\ &\leq \|B^*(Y)\|_{\mathbf{D}}, \end{aligned}$$

The use of A, A^*, B, B^* should suggest that A^* is similar to A in some sense - and then $Comp_{MP}$ should say that B^* is similar to B in some other sense.

Notorious example: If the colour is red then the tomato is ripe; the colour is very red - what follows? But be careful: If A, B, A^* are interpreted by crisp (0, 1 - valued) subsets of the respective domains then the interpretation of B^* is also crisp and

- (i) either $r_{A^*} \subseteq r_A$, $r_{A^*} \neq \emptyset$ and $r_{B^*} = r_B$,
- (ii) or $r_{A^*} \subseteq r_A$, $r_{A^*} = \emptyset$ and $r_{B^*} = \emptyset$,
- (iii) or r_{A^*} is not a subset of r_A and then $r_{B^*} = D_Y$ (the full set).

In general, if $Comp_{MP}$ is defined as above then

$$BL\forall \vdash (\forall y)[((\exists x)(A^*(x) \& \neg A(x)) \rightarrow B^*(y))].$$

Thus for each $v \in D_Y$, $r_{B^*}(v) \geq \sup_{u \in D_X}(r_{A^*}(u) * (-)r_A(u))$.

Note also

$$Comp_{MP} \vdash (\exists x)A^*(x) \rightarrow (\forall y)(B(y) \rightarrow B^*(y)).$$

4.2 Fuzzy functions and fuzzy rules

“Fuzzy IF-THEN rules” are presented as implications but then used to construct a fuzzy relation having little to do with any implication, at least at the first glance (the relation is defined by a disjunction of conjunctions). Attempts to call

e. g. the min-conjunction a “Mamdani implication” must be strictly rejected.

The crisp situation is as follows: we have two domains M_1, M_2 and a crisp, possibly partial, function f from M_1 to M_2 . Moreover, $(u_1, v_1), \dots, (u_n, v_n) \in M_1 \times M_2$, for $i = 1, \dots, n$, $f(u_i) = v_i$. Let F be a binary predicate interpreted by f , let $\mathbf{M} = \langle M_1, M_2, f, =_1, =_2 \rangle$ where $=_i$ is identity on M_i , x -variables range on M_1 , y -variables on M_2 . The fact that f is a partial mapping is expressed by

$(\forall x, y_1, y_2)((F(x, y_1) \& F(x, y_2)) \rightarrow y_1 = y_2)$. Let c_i be the constants for u_i , and d_i for v_i .

Lemma. Under the present notation,

(1) The formula

$$\bigwedge_i F(c_i, d_i)$$

just expresses the fact that $f(u_i) = v_i$; it is true in \mathbf{M} .

(2) The formula

$$(\forall x, y) \bigwedge_i ((x = c_i) \rightarrow (y = d_i))$$

defines a relation $r \subseteq M_1 \times M_2$ whose restriction to $\{u_1, \dots, u_n\}$ coincides with the restriction of f to $\{u_1, \dots, u_n\}$ and containing all pairs (u, v) where u is distinct from all u_1, \dots, u_n and $v \in M_2$; thus $f \subseteq r$.

(3) The formula

$$(\forall x, y) \bigvee_i (x = c_i \& y = d_i)$$

defines a relation $s \subseteq M_1 \times M_2$ which is the restriction of f to $\{u_1, \dots, u_n\}$; i.e. no pair (u, v) with u distinct from all u_1, \dots, u_n belongs to s . Thus $s \subseteq f$.

Definition. F defines a (partial) fuzzy function in T with respect to \approx if T proves the following:

$$(x \approx x' \& y \approx y') \rightarrow (F(x, y) \equiv F(x', y')),$$

$$(F(x, y) \& F(x, y')) \rightarrow y \approx y'.$$

The former formula is the congruence axiom; the second says that any two images of x are similar.

Lemma. Let F define a partial fuzzy function in T w.r.t. \approx . Let c, d be constants such that $T \vdash F(c, d)$.

(1) Then $T \vdash (x \approx c \& F(x, y)) \rightarrow y \approx d$.

(2) Moreover, if $A(x)$ is $x \approx c$ and $B(y)$ is the formula given by the condition *Comp* of the compositional rule of inference from F and A , i. e. $B(y)$ is $(\exists x)(x \approx c \& F(x, y))$ then $T \vdash (B(y) \equiv y \approx d)$. (Thus the compositional rule transforms $x \approx c$ and $F(x, y)$ to $y \approx d$.)

Definition. A fuzzy relation $s : (M_1 \times M_2) \rightarrow [0, 1]$ is a *fuzzy mapping* from M_1 into M_2 w.r.t. r_1, r_2 if s is extensional, i. e. for all $x, x' \in M_1, y, y' \in M_2$,

$$r_1(x, x') * r_2(y, y') * s(x, y) \leq s(x', y')$$

and functional, i. e.

$$s(x, y) * s(x, y') \leq r_2(y, y')$$

Assume now that s is a fuzzy mapping from M_1 into M_2 (w.r.t. r_1, r_2), and that we know finitely many examples u_i, v_i ($i = 1, \dots, n$) such that $s(u_i, v_i) = 1$. Thus if F names s , c_i name u_i and d_i name v_i then $F(c_i, d_i)$ is 1-true in $\mathbf{M} = \langle M_1, M_2, r_1, r_2, s, u_i, v_i \rangle$, hence

$$x \approx c_i \& F(x, y) \rightarrow y \approx d_i$$

is 1-true; and this resembles an “IF- THEN rule”

IF x is similar to c_i THEN y is similar to d_i .

Definition. F defines a \approx -function with examples (c_i, d_i) ($i = 1, \dots, n$) in T if F defines a fuzzy function w.r.t. a similarity \approx and for $i = 1, \dots, n$, T proves $F(c_i, d_i)$.

Lemma. Let $A_i(x)$ be $x \approx c_i$, let $B_i(y)$ be $y \approx d_i$. Then T proves

$$F(x, y) \rightarrow \bigwedge_i (A_i(x) \rightarrow B_i(y)),$$

$$\bigvee_i (A_i(x) \& B_i(y)) \rightarrow F(x, y).$$

Definition. Given predicates A_i, B_i , we let $RULES(x, y)$ stand for the formula

$$\bigwedge_i (A_i(x) \rightarrow B_i(y))$$

and $MAMD(x, y)$ (resembling the name Mamdani) for the formula

$$\bigvee_i (A_i(x) \& B_i(y))$$

$$T \vdash MAMD(x, y) \rightarrow F(x, y) \rightarrow RULES(x, y).$$

One easily shows that $MAMD$ defines in T a \approx -function with examples (c_i, d_i) . In fact, $MAMD(x, y)$ defines in T the least \approx -function with examples (c_i, d_i) . *Caution:* The formula $RULES(x, y)$, i. e. $\bigwedge_i (A_i(x) \rightarrow B_i(y))$ need not define a \approx -function!

Thus keeping our assumptions on T we may ask under which conditions the two formulas, $RULES(x, y)$ and $MAMD(x, y)$ are equivalent. The following lemma gives the answer:

Lemma.

$$T \vdash \left(\bigvee_i A_i^2(x) \right) \rightarrow (MAMD(x, y) \equiv RULES(x, y))$$

$$T \cup \{\bigvee_i A_i(x)\} \vdash (\forall x, y)(MAMD(x, y) \equiv RULES(x, y)).$$

What if we just have M_i , similarities r_i and (potential) examples (u_i, v_i) ? What must be assumed to be sure that there is a fuzzy mapping s (w.r.t. r_i) such that $s(u_i, v_i) = 1$?

Lemma. If $T \vdash c_i \approx c_j \rightarrow d_i \approx d_j$ for each i, j (indices at \approx deleted) and $T \vdash MAMD(x, y) \equiv \bigvee(x \approx c_i \& y \approx d_i)$ then $MAMD$ defines a \approx -function in T and $T \vdash MAMD(c_i, d_i)$ for $i = 1, \dots, n$.

Let us be still more modest: let us have M_1, M_2 and fuzzy subsets r_{A_i} of M_1, r_{B_i} of M_2 . We ask under which conditions we may assume

- similarities s_1 on M_1 and s_2 on M_2 with respect to which r_{A_i}, r_{B_i} are extensional,
- elements $u_1, \dots, u_n \in M_1, v_1, \dots, v_n \in M_2$ such that r_{A_i} are “fuzzy singletons given by u_i with respect to s_1 ” and similarly for r_{B_i}, v_i, s_2 , and
- an s_1, s_2 -fuzzy mapping r_F “sending u_i to v_i ”.

Lemma.

(1) Define a binary predicate \approx as follows:

$$(\forall x, x')(x \approx x' \equiv \bigwedge_i (A_i(x) \equiv A_i(x'))).$$

The resulting extension T' of T is conservative, \approx is a similarity in T' and all T' proves all A_i to be extensional.

(2) Add new constants c_i and axioms $(\forall x)(A_i(x) \equiv x \approx c_i)$. The resulting theory T'' is a conservative extension of T' iff T' proves all formulas

$$(\exists x)A_i(x),$$

$$(\exists x)(A_i(x) \& A_j(x)) \rightarrow (\forall x)(A_i(x) \equiv A_j(x)).$$

Now we are ready to answer our question above:

Theorem. Assume

$$\begin{aligned} T \vdash (\exists x)A_i(x), \quad T \vdash (\exists y)B_i(y), \\ T \vdash (\exists x)(A_i(x)\&A_j(x)) \rightarrow (\forall x)(A_i(x) \equiv A_j(x)), \\ T \vdash (\exists y)(B_i(y)\&B_j(y)) \rightarrow (\forall y)(B_i(y) \equiv B_j(y)). \end{aligned}$$

Add definitions $x_1 \approx x_2 \equiv \bigwedge_i(A_i(x_1) \equiv A_i(x_2))$, $y_1 \approx y_2 \equiv \bigwedge_i(B_i(y_1) \equiv B_i(y_2))$, new constants c_i, d_i and axioms

$$A_i(x) \equiv x \approx c_i, \quad B_i(y) \equiv y \approx d_i.$$

Finally add the definition

$$MAMD(x, y) \equiv \bigvee(A_i(x)\&B_i(y)).$$

Then

- The resulting theory T^M is a conservative extension of T and \approx_1, \approx_2 are similarities.
- $MAMD$ defines in T^M a fuzzy mapping w.r.t. \approx_1, \approx_2 with the examples (c_i, d_i) iff

$$T \vdash (\exists x)(A_i(x)\&A_j(x)) \rightarrow (\exists y)(B_i(y)\&B_j(y)).$$

Finally let use discuss the (logical) principles of fuzzy control in general, without relating it to to the notion of similarity. We have rules: $A_i(x) \rightarrow B_i(y)$

We define:

$$(\forall x, y)(MAMD(x, y) \equiv \bigvee_i(A_i(x)\&B_i(y))). \quad (*)$$

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x)\&MAMD(x, y))). \quad (**)$$

Given a model $\mathbf{M} = \langle D_X, D_Y, r_{A_i}, r_{B_i} \rangle$ this defines a function associating to each fuzzy subset r_{A^*} of D_X the corresponding fuzzy subset r_{B^*} of D_Y .

Remark. In this discussion problems of *fuzzification* and *defuzzification* are fully disregarded. We ask: *Is there any logic here?*

Definition. FC is the following two-sorted theory. The axioms are the formulas (*), (**) above (defining $MAMD$ from A_i , B_i and defining B^* from A^* , R). In addition, FC has two constants: X and Y .

Theorem. FC proves the following (over $BL\forall$):
 $[\bigwedge_i (A_i(X) \rightarrow B_i(Y)) \& \bigvee (A_i)^2(X)] \rightarrow (A^*(X) \rightarrow B^*(Y)) :$

This has double meaning: (1) Read the formula assuming that its assumptions true but also (2) assuming only that the assumptions are only sufficiently true: For example, if the rules are 1-true then $\|B^*(Y)\|_M \geq \|A^*(X)\|_M * \|\bigvee A_i^2(X)\|_M$ (* being the interpretation of $\&$).

Lemma. FC proves (over $BL\forall$) the following:

$$[(\forall x)(A^*(x) \equiv A_i(x)) \& (\exists x)A_i^2(x)] \rightarrow (\forall y)(B_i(y) \rightarrow B^*(y)),$$

$$[(\forall x)(A^*(x) \equiv A_i(x)) \& (\forall x)(\bigwedge_{i \neq j} \neg(A_i(x) \& A_j(x)))] \rightarrow (\forall y)(B^*(y) \rightarrow B_i(y)).$$

Again read the formulas as true in a model – first with the antecedent 1-true and then with the antecedent *sufficiently true*. We see that

(i) if $A^*(x)$ is sufficiently true to A_i and A_i is (sufficiently) near then B_i is sufficiently included in B^* ;

(ii) if A_i is sufficiently disjoint from all the other A_j 's and A^* is sufficiently near to A_i then B^* is sufficiently included in B_i . Obviously, these are fuzzy readings; the precise meaning is given by the formulas proved and may be expressed in more details as an exercise.

Note that instead of antecedent of the form $A_i(X)$ we could investigate $A_{i1}(X_1) \& \dots \& A_{ik}(X_k)$ or $A_{i1}(X_1) \wedge \dots \wedge A_{ik}(X_k)$; this brings no problems but is more cumbersome.

5 Conclusion

We hope that we have shown the following:

- *Fuzzy logic is neither a poor man's logic nor poor man's probability.*
Fuzzy logic (in the narrow sense) is a reasonably deep theory.

- *Fuzzy logic is a logic.* It has its syntax and semantic and notion of consequence. It is a study of consequence.
- *There are various systems of fuzzy logic, not just one.* The main two most developed systems are those of Łukasiewicz and of Gödel, the first together with its extension à la Pavelka.

In addition, we claim the following:

- *Further logical investigations of fuzzy logic are possible.* In particular, one has to apply the theory of generalized quantifiers to fuzzy logic and go further in a strictly logical analysis of things pointed out by Zadeh as “particular agenda of fuzzy logic”. Cf also [5].
- *To construct combined calculi of vagueness and of uncertainty is possible.* See [19, 20] for information; one gets many-valued modal logics.
- *Fuzzy logic in the narrow sense is a beautiful logic, but also is important for applications:* it offers foundations.

References

- [1] ALSINA C., TRILLAS E. AND VALVERDE L. On some logical connectives for fuzzy set theory. *J. Math. Anal. Appl.* 93 (1983), 15-26.
- [2] BEZDEK J.C. AND HARRIS J.D. Fuzzy partitions and relations: An axiomatic basis for clustering. *Fuzzy Sets and Systems*, 1(2), 111-127, 1978.
- [3] DUBOIS D., ESTEVA F., GARCIA P., GODO L., PRADE H. Similarity-based consequente relations. In Christine Froidevaux and Jürg Kohlas, editors, *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*, volume 946 of *Lecture Note in Artificial Intelligence*, pages 168–174, Fribourg, Switzerland, July 1995. Springer.
- [4] DUBOIS D., ESTEVA F., GARCIA P., GODO L., PRADE H. A logical approach to interpolatiion based on similarity relations *Int. Journal of Approximate Reasoning* (to appear). Also available as IIIA Research Report 96-07.

- [5] DUBOIS, D., PRADE, H. Fuzzy sets in approximate reasoning II (Logical approaches). *Fuzzy sets and systems* 40 (1991), 203-244
- [6] DUBOIS D. AND PRADE H. Similarity-based approximate reasoning. In: *Computational Intelligence Imitating Life* (J.M. Zurada, R.J. Marks II, X.C.J. Robinson, eds.), Invited Symposium, Orlando, FL, IEEE Press, 69-80, 1994.
- [7] DUMMETT, M. A propositional calculus with denumerable matrix. *Journ. Symb. Logic* 243 (1959), 97 – 106.
- [8] ESTEVA F. AND GARCIA P. AND GODO L. AND RODRIGUEZ R. A modal account of similarity-based reasoning. In: *Int. Journal of Approximate Reasoning* (To appear).
- [9] GÖDEL, K. Zum intuitionistischen Aussagenkalkül. Anzeiger Akademie der Wissenschaften Wien, Math. - naturwissensch. Klasse 69, 65-66, 1932. Also in *Ergebnisse eines mathematischen Kolloquiums* 4 (1933), 40.
- [10] GOTTWALD, S. *Mehrwertige Logik*. Akademie-Verlag, Berlin, 1988.
- [11] GOTTWALD, S. *Fuzzy Sets and Fuzzy Logic* Vieweg 1993.
- [12] HÁJEK, P. Fuzzy logic from the logical point of view In *SOFSEM'95: Theory and practice of informatics; Lecture Notes in Computer Science 1012* (Milovy, Czech Republic, 1995), M. Bartošek, J. Staudek, and j. Wiedermann,Eds.,Springer-Verlag, pp. 31–49.
- [13] HÁJEK, P. *Metamathematics of fuzzy logic* (book in preparation).
- [14] HÁJEK, P. Fuzzy logic and arithmetical hierarchy. *Fuzzy Sets and Systems* Vol. 73 (1995) 359-363
- [15] HÁJEK, P. Logics of Approximate Reasoning II. In *Proc. of the ISSEK94 Workshop on Mathematical and Statistical Methods in AI* (de Ricca G. et al., ed.) Springer-Verlag Wien 1995, pp.147-156
- [16] HÁJEK, P. Fuzzy logic and arithmetical hierarchy, II. *Studia Logica* (to appear).

- [17] HÁJEK, P., GODO, L., AND ESTEVA, F. Fuzzy logic and probability. *In: Uncertainty in Artificial Intelligence, Proc. of the 11th conference (Ed. Besnard P., Hanks S.)* Morgan-Kaufmann 1995, 237-244
- [18] HÁJEK, P., GODO, L., AND ESTEVA, F. A complete many-valued logic with product-conjunction. *Archive for Math. Logic* vol.35, (1996), p. 191-208
- [19] HÁJEK, P., HARMANCOVÁ, D., VERBRUGGE, R., L. A qualitative fuzzy possibilistic logic. *Int. J. of Approximate Reasoning*, 12 (1995) , 1-19
- [20] HÁJEK, P., HARMANCOVÁ, D., ESTEVA, F., GARCÍA, P., GODO, L. On modal logics for qualitative possibility in a fuzzy environment. *Uncertainty in AI '94*, L. de Mantaras and D. Poole, ed., Morgan-Kaufmann Publ. 1994, p.278-285
- [21] HÖHLE U. Monoidal logic. In:*Fuzzy Systems in Computer Science* (R. Kruse et al., eds.), Vieweg, Germany, 233-243, 1994.
- [22] KLAWONN F., KRUSE R. Equality relations as a basis for fuzzy control. *Fuzzy Set and Systems*, 54:54–147, 1994.
- [23] KLIR G. J., YUAN BO *Fuzzy sets and fuzzy logic*. Prentice Hall PTR 1995.
- [24] KRUSE R., GEBHARDT J., KLAWONN F. *Foundations of fuzzy systems*. John Willey and sons
- [25] ŁUKASIEWICZ, J. *Selected works*. North-Holland Publ. Comp. 1970.
- [26] MARKS II, R., J. *Fuzzy Logic Technology and Applications*. IEEE Technical Activities Board, 1994
- [27] MENDELSON, E. *Introduction to Mathematical Logic*. Van Nostrand 1964
- [28] MENGER K. Ensembles flous et fonctions aleatoires. C.R. Acad. Sci., 232, 2001- 2003, 1951.

- [29] NAKAMURA A. Applications of fuzzy-rough classifications to logics. In: *Intelligent Decision Support - Handbook of Applications and Advances of the Rough Sets Theory* (R. Slowinski, ed.), Kluwer Academic Publ., Dordrecht, 233-250, 1992.
- [30] NOVÁK, V. On the syntactico-semantical completeness of first-order fuzzy logic I, II. *Kybernetika* 26 (1990), 47-26, 134-152.
- [31] ORLOWSKA E. Modal logics in the theory of information systems. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 30(30), 213-222, 1984.
- [32] OVCHINNIKOV S. Similarity relations, fuzzy partitions and fuzzy orderings. *Fuzzy Sets and Systems*, 40, 107-126.
- [33] PARIS, J.B. The uncertain reasoner's companion – a mathematical perspective. Cambridge University Press, 1994.
- [34] PAVELKA, J. On fuzzy logic I, II, III. *Zeitschr. f. Math. Logik und Grundl. der Math.* 25 (1979), 45–52, 119-134, 447-464
- [35] PAWLAK Z. Rough Sets - Theoretical Aspects of Reasoning about Data. Kluwer Academic Publ., Dordrecht, 1991.
- [36] RAGAZ, M., E. Arithmetische Klassifikation von Formelmengen der unendlichwertigen Logik. ETH Zürich, 1981. Thesis
- [37] RAGAZ, M., E. *Die unentscheidbarkeit der einstelligen unendlichwertigen Prädikatenlogik.* *Arch. Math. Logik* 23, (1983), pp. 129-139
- [38] RASIOWA, H., SIKORSKI, R. *The mathematics of metamathematics.* Warszawa 1963
- [39] ROSE, A. AND ROSSER, J. B. Fragments of many-valued statement calculi. *Trans. A.M.S.* 87 (1958), 1-53.
- [40] RUSPINI E.H. A theory of fuzzy clustering. Proc. IEEE Conf. on Decision and Control, New Orleans, 1378-1383, 1977.
- [41] Enrique RUSPINI. On the semantics of fuzzy logic. *International Journal Of Approximate Reasoning*, (5):45–88, 1991.

- [42] SCARPELINI, B. Die Nichtaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Lukasiewicz. *Journ. Symb. Log.* 27 (1962), 159-170.
- [43] SCHWEIZER, B. AND SKLAR, A. Associative functions and abstract semi-groups. *Publ. Math. Debrecen* 10 (1963), 69-81
- [44] SCHWEIZER, B. AND SKLAR, A. *Probabilistic metric spaces*. North Holland, Amsterdam, 1983.
- [45] TAKEUTI G. AND TITANI S. Fuzzy Logic and fuzzy set theory. *Anal. Math. Logic* 32, pp. 1-32, 1992.
- [46] Enric TRILLAS and Llorenç VALVERDE. An inquiry on t-indistinguishability operator. In H. Skala et al, editor, *Aspects of Vagueness*, pages 231–256. Reidel, Dordrecht, 1984.
- [47] VALVERDE L. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, 17, 313-328.
- [48] ZADEH L.A. Similarity relations and fuzzy orderings. *Information Sciences*, 3, 177-200.
- [49] ZADEH L. The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences* 8 (1975), 199-245, 301-357
- [50] ZADEH L. A theory of approximate reasoning. In J. E. Hayes, D. Michie, and L.I. Mikulich, editors, *Machine Intelligence*, volume 9, pages 149–194. Wiley, 1979.
- [51] ZADEH, L. Fuzzy logic. *IEEE Computer* 1:83 (1988).
- [52] ZIMMERMANN, H.J. *Fuzzy Set Theory and its Applications*. Kluwer, Nijhof, second edition 1991.