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Tetrahedral Symmetry and the Graviton

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Abstract

We attempt to give a purely geometric interpretation of the Standard Model and trace of the graviton. The concept of a "point" particle is replaced by fundamental symmetries assumed to characterize physical space. In particular the construction rests on Equation (1) which is an irreducible representation (or IR) of the Dirac ring, in line with some of Einstein's last ideas that view "particles" not as objects sitting in space-time but rather as parts of space-time itself.Essentially (1), governing the nucleon,imposes tetrahedral symmetry and consequently the Clebsch cubic of Fig.4 that carries the famous 27 lines that will be identified with E_6 and the Standard Model.A section is the Cayley cubic of Fig.6 that shows a circle belonging to a photon γ. To find the other half of the Standard Model we study E_7 , homomorphic to the binary octahedral group,, that introduces the product $\gamma.\gamma$ with spin 2 thought to represent the graviton.

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1 Introduction

In his book [14] Penrose, Section 22.1,refers to a point particle as a mysterious spread out wavy thing in the quantum formalism. Such a "spread out wave" may account for the Uncertainty Principle and for quantum entanglement where entangled "particles" ,described by the wave function,may interact at a distance. 'Thus instead of each particle individually having its own state vector we find that the entire quantum system requires a thoroughly self-entangled single state vector'. Such an entangled state vector has been constructed recently by de Wet[7]

Again Biedenharn and Louck [3] in Chapter 4 point out that the concept of an elementary (point) particle in physics becomes synonymous with the fundamental symmetries assumed to characterize space. Einstein [9] in his last published work suggested that a discreetly based algebraic theory might be the way forward in future physics. In the preface to his book on the Foundations of Quantum Theory Kahan [11] states that " Particles may be said to be representations of the Lorentz group, just as the electron may be said to be the equation of Dirac".

In fact this was precisely the approach adopted by de Wet[5] long ago that is expressed by the fundamental equation (1) below. Comparatively recently Barth and Nieto [2] showed how this relation leads to tetrahedral symmetry which is certainly a discreetly based algebraic theory and among other things describes the nuclear quadrupole [7].

However in this contribution we will be looking more closely at the Clebsch cubic, which is invariant under the binary tetrahedral group T_d and carries the famous 27 lines that correspond to the 27 fundamental weights of the exceptional Lie group E_6 (cf. eg. Hunt [10] Section 4.1) Slansky [15] has related these weights in Table 21 (reproduced in Fig.8) to elements of half the Standard Model and we shall see how the remainder are incorporated by turning to E_7 homomorphic to the binary octahedral group O by the MacKay correspondence [13].

So in this way we pass from an IR of the Lorentz group to "particles" found at high energies. Specifically Fig.6 illustrates the Cayley cubic, which is a subspace of the Clebsch cubic given by equation (9),and shows 3 quarks at the apices of a tetrahedron as well as the gluon paths between them. These are asymptotic to the 9 lines on the cubic surface.

In Ref.[8] de Wet shows that Fig.6 will also be valid for the many nucleon case, so there is no need to introduce any binding forces and mechanics is replaced by a geometry that makes no attempt to fix the exact location of nucleons, only the quadrupoles in which they move! Then from a purely geometric viewpoint a "particle" is a singularity in space-time that is a function of energy, much as the sun distorts space to form the planetary orbits without any need to postulate a gravitational force. In other words the masses of the ultra short-lived particles found in giant accelerators are a function of the energies put in.

Immediately above the line ef of Fig.6 are the 2 arcs of a circle shown in more detail in Fig.7. This is the U(1) symmetry of Table 8 and describes a photon γ represented by $e^{i\theta}$ But the fundamental representation of E_7 in equation (11) suggests that the two U(1) representations may describe gravitons $e^{i\theta}.e^{i\theta}$.

We will begin with tetrahedral symmetry and introduce the graviton in Section 3.

2 Tetrahedral Symmetry

We will need the fundamental IR or one-form

$$\frac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \tag{1}$$

of a minimal left ideal of the center D of the Dirac ring [6].Here we use Eddington's transparent notation (related to the Dirac matrices by (2) below)and associate the commuting operators E_{23}, E_{14} and E_{05} respectively with independent rotations in 3-space,4-space and isospace which correspond to the spin σ, parity π and charge T_3 carried by a single nucleon. E_4 is the unit 4×4 unit matrix while ψ_2, ψ_3 and ψ_4 are half angles of rotation or 'turns', e is a primitive idempotent. To understand how E_{14} is a parity operator we observe that a rotation through π about t will send x to -x without inverting the time axis but instead change to a left-handed coordinate system.Eddington's E-numbers are mapped into the Dirac matrices by

$$\gamma_\nu = iE_{0\nu}, E_{\mu\nu} = E_{\rho\mu}E_{\rho\nu} = -E_{\nu\mu},$$
$$E^2_{\mu\nu} = -1, E_{\mu\nu}E_{\sigma\tau} = E_{\sigma\tau}E_{\{mu\nu} = iE_{\lambda\rho}, \mu < \nu = 1, \ldots, 5 \tag{2}$$

Barth and Nieto[2] pointed out that E_{23}, E_{14} and E_{05} are a set of one of the 15 sythemes in the projective space P^3. These specify three pairs of 'fix-lines' lying in the $\pm i$ eigenspaces determined by the relation $E^2_{\mu\nu} = -1$. The six fix-lines are the edges of a fundamental tetrahedron that may be inscribed in a cube as illustrated by Fig.1.

In this way equation(1)leads us to the study of tetrahedral and cubic symmetry governed by the binary and octahedral quaternion groups T_d,O.We begin with an algebraic equation

$$z_1^4 + 2\sqrt{3}z_1^2z_2^2 - z_2^4 = 0 \tag{3}$$

for the tetrahedron first published in 1884 by Felix Klein [12]where $z_1 = x + iy, z_2 = z + it$. The real part of this equation may be put in the form of a Heisenberg or Dirac quartic

$$Ag_0 + Bg_1 + Cg_2 + Dg_3 + Eg_4 = 0$$
$$g_0 = x^4 + y^4 + z^4 + t^4, g_1 = 2(x^2y^2 + z^2t^2), g_2 = 2(x^2z^2 + y^2t^2),$$
$$g_3 = 2(y^2z^2 + x^2t^2), g_4 = 4xyzt \tag{4}$$

if we choose A=B=0,E=1 Then (3) becomes

$$(x^2z^2 + y^2t^2) - (y^2z^2 + x^2t^2) - 4xyzt = 0 \tag{5}$$

which is a classic Kummer surface [2] and is a cube shown in Fig.2 if we set t=1.

On the other hand if we choose A=0,B=C=D=1/2,t=1,E=k equation (4) yields

$$(x^2y^2 + z^2) + (y^2z^2 + x^2) + (x^2z^2 + y^2) + kxyz = 0 \tag{6}$$

that is plotted in Fig.3, with $k > 6$ and is the nuclear quadrupole that characterizes the E_2 transitions of many nuclei. This follows because in [7] it is proved that the fix-lines are not affected by representations of many nuclei constructed from tensor products of (1)with itself. In this way the 'strong force' has a geometric origin. There are also covering spaces of rotating octapoles and even hexadecupoles that would be indistinguishable from a deformed nuclear shell measured in the laboratory(ibid)

The surface shown in Fig.3 is singular at the four vertices of the fundamental tetrahedron (not shown) and that of Fig.2 is singular on the diagonal $c\dot{c}$ of Fig.1 which is vertical to the plane efg where the two tetrahedra $abcd, efg\dot{c}$ are intertwined in the cube.However in the case of the binary octahedral group O the point, $\dot{c}$, lying behind c, may be reflected into c,and still preserve the cube, but it will introduce an asyzygy where the two triples of fix-lines $ac; bc; dc$ and $e\dot{c}; f\dot{c}; g\dot{c}$ cross. This reflects quarks into anti-quarks as will appear after we have introduced the Clebsch cubic mapped into Fig.4.

The basic cubic is that of Segre,namely

$$S_3 = \sum_{i=0}^{5} x_i = \sum_{i=0}^{5} {x_i}^3 = 0 \tag{7}$$

which is invariant under the binary tetrahedral group T_d. The Nieto quintic

$$N_5 = x_0x_1x_2x_3x_4 + \ldots + x_1x_2x_3x_4x_5 = 0 \tag{8}$$

is the invariant Hessian $(\det(\partial^2 S_3/\partial x_i \partial x_j))$ of S_3 that leads us back to (6) which is a modular interpretation of N_5 (cf.[10]Section 3). However for the purposes of this contribution we shall need the Clebsch cubic

$$\begin{aligned}81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y)\\ +54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2)\\ -9(x + y + z) + 1 = 0\end{aligned} \tag{9}$$

which is a section $x_0 = 0$ of (7) (ibid,Section 4) and is shown graphically in Fig.4. Here there are 3 quarks ABC at the corners of a tetrahedron where the fourth vertex lying below o meets a vertex of a second tetrahedron with the anti-quarks EFG rotated because of an asyzygy at the junction shown in Fig.5 (which is a rotation through $\pi/2$ of Fig.4). The gluon paths between the two sets of 3 quarks are determined by skew lines on the cubic surface two of which cross at point 1 on ABC and three of which cross at the point 2 on DEF (cf

Fig.5 also). To understand how DEF are anti-quarks we shall need to assign isospin to two of the fix-lines.

Specifically we let the fix-lines ab, ad of Fig.6 (which is Fig.1 inverted) correspond to the operators E_{05}, E_{50} with isospin $\pm 1/2$ in which case the quark charge along $de, a\ddot{a}$ will be Q=T_3+1/6=2/3 that along $bf, a\dot{a}$ would be Q=-1/2+1/6=-1/3 [7]. There is another fix-line along ac of Fig.1 that is associated with a parity operator E_{14} that will meet with gc carrying $-E_{14}$, after the reflection $\dot{c} \rightarrow c$ introduces an asyzygy, due to the fact that the coordinate system of $efg\dot{c}$ is inverted. Then by the CP- theorem reflected quarks become the anti-quarks shown in Fig.4.

The Cayley cubic

$$-5(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + xyz) + 2(xy + xz + yz) = 0 \quad (10)$$

is another section ofS_3 illustrated in Fig.6. Because the surface is singular it exhibits only 9 lines [10] and Fig.6 shows that the mesh generated by solutions of (10) is asymptotic to the lines $a\dot{a}, aa, a\ddot{a}, b\dot{b}, bf, d\dot{d}, de, c\dot{c}, ef$ In Ref.[8] de Wet links this mesh to the paths of gluons between the three up and down quarks at the apices abd of the basic tetrahedron. The fourth apex c (lying below the plane abd) is the gluon propagator.

Just above the line ef of Fig.6 are traces of a circular representation $e^{i\theta}$ of U(1) describing electro-magnetic interaction γ pointing to a bigger picture that will be discussed in the next section.The circle appears more clearly in Fig.7 and to get a wider picture we turn to the binary octahedral group O that preserves the cube of Fig.1 and is homomorphic to the Lie group E_7 by the MacKay correspondence [13]. The representation that we need is

$$E_6 \times U(1) = 27 + \overline{27} + 1 + 1 \quad (11)$$

which encompasses two sets of 27 lines generated by the tetrahedron and its self-conjugate that we shall need to describe the entire Standard Model.

3 Shades of the Graviton

Slansky [15] in Table 21, reproduced in Fig.8, identifies 12 weights of the exceptional Lie algebra E_6 with the up and down colored quarks and anti-quarks, a further 6 are associated with colored strange and anti-strange quarks. There are 8 leptons and the last weight with U(1) symmetry has been identified with the photon γ shown in Fig.7 which corresponds to the unit element of T_d. This completes the identification of the 27 lines on the Clebsch cubic with the weights of E_6. But referring to equation (11)there will be another 27 lines contributed by the self-conjugate tetrahedron that will map into a second photon and the remaining fermions of the Standard Model.

In this way the entire Standard Model is covered by E_7 but (11) also has two more unit elements arising from rotations and reflections of the two tetrahedra of Fig.1 and these unit elements will generate $e^{i\theta}.e^{i\theta} = e^{2i\theta}$ and therefore carry spin 2 . Consequently we identify them with gravitons ie. spin 2 photons ([13],section 32.1)on a double covering of the rotation group $SO(2)$ [4].

4 Conclusion

It is remarkable that the Ancient Greeks believed the the Platonic solids namely the tetrahedron,cube,octahedron, dodecahedron and icosahedron, which can be rotated and reflected into themselves, were fundamental to the structure of the universe. This contribution strives to show that they were quite right particularly in view of the fact that the binary tetrahedral T_d, octahedral O and icosahedral groups are homomorphic to the Lie algebras E_6, E_7 and E_8 [13].Furthermore T_d and O are isomorphic and all three groups can be derived from quaternions [5] which are a member of the only four division algebras. The other three being the real line, complex plane and the octonians. In other words the physical world is built with as much economy as possible in line with the thinking of classical Greece.

In this contribution we have considered only the isomorphic binary tetrahedral and octahedral groups because their representations are sufficient to cover the Standard Model , but much work has been done on associating the elementary particles with octonians.For an excellent review and comprehensive set of references see Baez [1].

Finally because quarks lie on the surfaces of Figs.4 and 6 they are anyons with fractional charge.Also Fig.6 shows lepton jets emanating from the apices a,b,d of the quark cones.

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