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A Cauchy problem in nonlinear heat conduction

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Abstract
A Cauchy problem on the semiline for a nonlinear diffusion equation is considered, with a boundary condition corresponding to a prescribed thermal conductivity at the origin. The problem is mapped into a moving boundary problem for the linear heat equation with a Robin-type boundary condition. Such a problem is then reduced to a linear integral Volterra equation of II type which admits a unique solution.

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The nonlinear diffusion equation

$$u_t = \left(\frac{u_x}{u^2}\right)_x \qquad u = u(x, t) \tag{1}$$

is a well-known mathematical model for heat conduction in high polymer systems [1] and in simple monoatomic metals of Storm type [2]. Fixed and moving boundary problems for equation (1) have been solved in the past through a linearizing transformation which allows us to reduce equation (1) to the linear heat equation [3, 4]. In the following we limit our consideration to materials of Storm type and consider for equation (1) an initial/boundary value problem on the semiline with a prescribed thermal conductivity at the origin. We show that the corresponding problem for the linear heat equation is a semiline problem with a moving boundary and a Robin-type boundary condition. Such problem is then solved, i.e. reduced to a linear integral equation of Volterra II type which admits a unique solution. An explicit example is also discussed.

We start our analysis by observing that the thermal variable u in equation (1) is related to the temperature distribution of the system through the relation [3]

$$u = \int_{T_0}^{T} \rho c_p(T')\,\mathrm{d}T', \tag{2}$$

where ρ and $c_p(T)$ represent in turn the density (assumed to be constant) and the specific heat of the system. In our model the thermal variable u represents therefore the heat energy (quantity of heat for unitary length) propagating through a semi-infinite one-dimensional metallic rod. Moreover we observe that for materials of Storm type we can write

$$\left(\frac{u_x}{u^2}\right)_x = k(T), \tag{3}$$

$k(T)$ being the thermal conductivity of the material.

Let us now analyse for equation (1) the initial/boundary value problem on the semiline $0 \leqslant x < \infty$, characterized by the following initial and boundary data:

$$u(x,0) = u_0(x), \qquad 0 \leqslant x < \infty \tag{4a}$$

$$u(\infty,t) = \gamma > 0, \qquad u_x(\infty,t) = 0, \quad t \geqslant 0 \tag{4b}$$

$$\frac{u_x(0,t)}{u^2(0,t)} = \alpha > 0, \tag{4c}$$

where α and γ are positive constants. Due to (3), the boundary condition (4c) corresponds, from the physical point of view, to a prescribed constant thermal conductivity at the origin.

Next we introduce the hodograph transform

$$u(x,t) = [v(z,t)]^{-1} \tag{5a}$$

with

$$\frac{\partial z}{\partial x} = u(x,t) \tag{5b}$$

$$\frac{\partial z}{\partial t} = -\left(\frac{1}{u(x,t)}\right)_x \tag{5c}$$

whose compatibility, $\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial^2 z}{\partial t \partial x}$, is guaranteed by (1).

Under the above transformation equation (1) is mapped into

$$v_t = v_{zz} \tag{6}$$

over the domain $\alpha t \leqslant z < \infty$, with the initial datum

$$v(z,0) \equiv v_0(z_0) = [u_0(x)]^{-1}, \tag{7a}$$

where

$$z_0 \equiv z_0(x) = \int_0^x \mathrm{d}x' u_0(x'). \tag{7b}$$

Moreover, the boundary conditions (4b), (4c) become

$$v(\infty,\, t) = \frac{1}{\gamma}, \qquad v_z(\infty,\, t) = 0 \tag{7c}$$

$$\alpha\, v(\alpha\, t,\, t) + v_z(\alpha\, t,\, t) = 0. \tag{7d}$$

The initial/boundary value problem for the nonlinear diffusion equation (1), with the initial datum (4a) and the boundary conditions (4b), (4c) is then mapped into an initial/boundary value problem for the linear heat equation (6) over a domain with a linearly moving boundary, characterized by the initial datum (7a) and boundary conditions (7c), (7d). We observe that (7d) is a Robin-type boundary condition at the moving boundary. In order to solve this problem we introduce the fundamental kernel of the heat equation

$$K(z-\xi, t-t') = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t-t'}} \exp\left[-\frac{1}{4}\frac{(z-\xi)^2}{(t-t')}\right] \tag{8}$$

and integrate Green's identity for the heat equation

$$\frac{\partial}{\partial \xi}\left(K\frac{\partial v}{\partial \xi} - v\frac{\partial K}{\partial \xi}\right) - \frac{\partial}{\partial t'}(Kv) = 0 \tag{9}$$

over the domain $\alpha t' < \xi < \infty$, $\varepsilon < t' < t - \varepsilon$ and let $\varepsilon \to 0$. Using (7*d*) and $K(z-\xi, 0) = \delta(z-\xi)$, we obtain

$$v(z,t) = \int_0^{+\infty} \mathrm{d}\xi\, K(z-\xi, t)v_0(\xi) + \int_0^t \mathrm{d}t' K_\xi(z-\alpha t, t-t')v(\alpha t', t'). \tag{10}$$

From (10) it follows that $v(z, t)$ has to be determined in terms of the boundary value $v(\alpha t, t)$ which is unknown; it is therefore convenient to evaluate (10) at $z = \alpha t$. By putting $v(\alpha t, t) = w(t)$, we obtain

$$w(t) = G(t) + \int_0^t \mathrm{d}t' R(t-t')w(t'), \tag{11a}$$

with

$$G(t) = \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} \mathrm{d}\xi \exp\left[-\frac{(\alpha t - \xi)^2}{4t}\right] v_0(\xi) \tag{11b}$$

and

$$R(t) = \frac{\alpha}{4\sqrt{\pi}}\frac{1}{\sqrt{t}}\mathrm{e}^{-\beta t}, \qquad \beta = \frac{\alpha^2}{4}. \tag{11c}$$

Equation (11*a*) is a linear Volterra integral equation of the convolution type with a mildly singular kernel; it admits a unique solution under the assumption that $G(t)$ is an integrable, bounded function of its argument [5]. The solution of (11*a*) can be written as

$$w(t) = G(t) + \int_0^t \mathrm{d}t'\, S(t-t')G(t'), \tag{12a}$$

where $S(t)$ is the resolvent kernel of (11*a*) given by

$$S(t) = \mathrm{e}^{-\beta t}\left\{\frac{1}{\sqrt{\pi t}} + \frac{\sqrt{\beta}}{2}\mathrm{e}^{\beta t/4}\left[1 + \mathrm{Erf}\left(\frac{\sqrt{\beta t}}{2}\right)\right]\right\}, \tag{12b}$$

with $G(t)$ given by (11*b*) and

$$\mathrm{Erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \mathrm{d}\tau\, \mathrm{e}^{-\tau^2}. \tag{12c}$$

Having established existence and uniqueness of the boundary datum $w(t)$, it then follows, via (10), existence and uniqueness of the solution of the linear problem $v(z, t)$. We can therefore conclude that, due to (5*a*), the initial/boundary value problem (1), (4*a*)–(4*c*) for the nonlinear diffusive equation admits a unique solution $u(x, t)$.

As an example, let us now consider an initial datum $u_0(x)$, compatible with the asymptotic conditions (4*b*), given by

$$u_0(x) = \gamma \tanh(x). \tag{13}$$

(11) implies, via (5*a*) and (5*b*),

$$z_0(x) = \gamma \ln\cosh(x) \tag{14a}$$

and

$$v_0(z_0) = \frac{\mathrm{e}^{z_0/\gamma}}{\gamma}(\mathrm{e}^{2z_0/\gamma} - 1)^{-1/2} \tag{14b}$$

which is the initial datum for the linear problem (6), (7*a*)–(7*d*).

When (14*b*) and (11*b*) are used, the function $G(t)$ on the right-hand side of (12*a*) takes the form

$$G(t) = \frac{1}{2\gamma}\frac{1}{\sqrt{\pi t}}\,e^{-\beta t}\int_0^{+\infty} d\xi\, e^{-\xi^2/4t}\frac{e^{(\sqrt{\beta}+1/\gamma)\xi}}{\sqrt{e^{2\xi/\gamma}-1}}. \tag{15}$$

In the following we concentrate our attention on the case when the parameter γ is small ($0 < \gamma < 1$) and analyse for this case the asymptotic, large t behaviour of $v(z, t)$. It is easy to check that for small γ we obtain the approximate expression

$$G(t) \cong \frac{1}{2\gamma}[1 + \mathrm{Erf}(\sqrt{\beta t})]. \tag{16}$$

In the same approximation, via (10), (8) and (14*b*), we can write the solution of the linear problem $v(z, t)$ as

$$v(z,t) \cong \frac{1}{2\gamma}\frac{1}{\sqrt{\pi t}}\int_0^{+\infty} d\xi\, \exp\left[-\frac{(z-\xi)^2}{4t}\right] + \frac{1}{4\sqrt{\pi}}\int_0^t dt'\frac{(z-\alpha t')}{(t-t')^{3/2}}\exp\left[-\frac{1}{4}\frac{(z-\alpha t')^2}{(t-t')}\right]w(t'). \tag{17}$$

The two terms on the right-hand side of (17) can be evaluated in the large time limit $t \to \infty$ (see the appendix). We denote by $v_\infty(z, t)$ the asymptotic value of $v(z, t)$ and obtain from (17)

$$v_\infty(z,t) \underset{t\ \text{large}}{\approx} \frac{1}{2\gamma}\left[1 + \frac{e^{-z^2/4t}}{\sqrt{\pi t}}\left(z - \frac{2}{\sqrt{\beta}} + O(t^{-1/2})\right)\right]. \tag{18}$$

Finally, the solution of equation (1) with initial datum (13) and boundary data (4*b*), (4*c*) is obtained (for small γ) in the large time limit as

$$u_\infty(x,t) = \left(\frac{\partial z}{\partial x}\right) \tag{19}$$

where, in virtue of (3*a*), (3*b*), $z(x, t)$ solves

$$x = \int_0^z dz'\, v_\infty(z', t) \tag{20}$$

with $v_\infty(z, t)$ given by (18).

Appendix

We write (17) in the form

$$v(z,t) = I_1(z,t) + I_2(z,t), \tag{A.1}$$

where $I_1(z, t)$ and $I_2(z, t)$ denote, respectively, the first and the second term on the right-hand side of (17).

We then get

$$I_1(z,t) = \frac{1}{2\gamma}\frac{1}{\sqrt{\pi t}}\int_0^{+\infty} d\xi\, \exp\left[-\frac{(z-\xi)^2}{4t}\right] = \frac{1}{2\gamma}\left[1 + \mathrm{Erf}\left(\frac{z}{2\sqrt{t}}\right)\right]. \tag{A.2}$$

In the large time t limit we obtain from (A.2)

$$I_1(z,t) \underset{t\ \text{large}}{\approx} \frac{1}{2\gamma}\left[1 + \frac{z}{\sqrt{\pi}}\frac{e^{-z^2/4t}}{\sqrt{t}} + O\left(\frac{e^{-z^2/4t}}{t^{3/2}}\right)\right]. \tag{A.3}$$

We now turn our attention to the asymptotic, large t, evaluation of $I_2(z,t)$. From (17) we can write

$$I_2(z,t) \cong \frac{1}{4\sqrt{\pi}}\frac{1}{\sqrt{t}}\int_0^1 \mathrm{d}u(z-\alpha\, t\, u)\exp\left[-\frac{1}{4}\frac{(z-\alpha t u)^2}{t(1-u)}\right] w(t\, u)\left(1+\frac{3}{2}u+\cdots\right). \quad \text{(A.4)}$$

In the large t limit, by using the Laplace method, we obtain from (A.4)

$$I_2(z,t) \underset{t\ \text{large}}{\approx} -\frac{2}{\sqrt{\pi\beta}}\frac{\mathrm{e}^{-z^2/4t}}{\sqrt{t}}w(0) + \frac{z}{4\sqrt{\beta}}\frac{\mathrm{e}^{-z^2/4t}}{t}w(0) + O\left(\frac{\mathrm{e}^{-z^2/4t}}{t^{3/2}}\right), \quad \text{(A.5)}$$

where, via (12a) and (16), it is $w(0) \cong 1/2\gamma$.

When (A.3) and (A.5) are used in (A.1), there immediately follows the result reported in (18).

References

[1] Bluman G and Kumei S 1980 *J. Math. Phys.* **21** 1019
[2] Storm M L 1951 *J. Appl. Phys.* **22** 940
[3] Rogers C 1985 *J. Phys. A: Math. Gen.* **18** L105
Rogers C 1986 *J. Nonlinear Mech.* **21** 249
[4] Natale M. F and Tarzia D A 2000 *J. Phys. A: Math. Gen.* **33** 395
[5] Tricomi F G 1985 *Integral Equations* (New York: Dover)