

CARDINALITY AND INVARIANT SUBSPACES

LOUIS DE BRANGES DE BOURCIA*

ABSTRACT. A question which was posed by Felix Hausdorff [12] concerns the accessibility of the cardinality of an uncountable set from smaller cardinalities. If the set is not a union of a class of smaller cardinality of sets of smaller cardinality, the question is whether a greatest cardinality exists which is less than the cardinality of the given set. Although the question remains unanswered, and may indeed be unanswerable, satisfactory foundations of analysis are now obtained when the issue is left undecided. A generalization of the Hahn–Banach theorem is obtained in locally hyperconvex spaces which are modules over a Weierstrass algebra. A construction of invariant subspaces is an application. If an algebra of continuous transformations of a Hilbert space into itself is closed in the weak topology induced by the trace class and does not contain the identity transformation, then a nontrivial proper closed subspace exists which is a common invariant subspace for the elements of the algebra.

A Weierstrass algebra is an associative algebra with unit over the rational numbers which admits a conjugation with positivity properties and a related Hausdorff topology. The conjugation is an anti-automorphism ξ into ξ^- of order two. A self-conjugate element of the algebra is said to be nonnegative if it is a sum

$$\xi_0^- \xi_0 + \dots + \xi_r^- \xi_r$$

with ξ_0, \dots, ξ_r elements of the algebra for some nonnegative integer r . It is assumed that ξ_0, \dots, ξ_r all vanish if the sum vanishes.

Locally hyperconvex topologies are defined for two-sided modules over a Weierstrass algebra. A class of nonnegative transformations of the module into itself is assumed given. A nonnegative transformation is linear when the module is treated as an algebra over the rational numbers. The sum of two nonnegative transformations is nonnegative. The inverse of a nonnegative transformation is nonnegative when it exists. An example of a nonnegative transformation is defined by taking c into $\xi^- c \xi$ for every element ξ of the Weierstrass algebra. If inverses exist for nonnegative transformations P and Q and if T is a nonnegative transformation such that $1 - T$ is nonnegative, then an inverse exists for the nonnegative transformation

$$(1 - T)P + TQ.$$

A hyperconvex combination

$$(1 - T)a + Tb$$

*Research supported by the National Science Foundation.

of elements a and b of the module is defined using a nonnegative transformation such that $1 - T$ is nonnegative. A subset of the module is said to be hyperconvex if it contains the hyperconvex combinations of every pair of its elements.

A Hausdorff topology for a module over a Weierstrass algebra is said to be locally hyperconvex if addition is continuous as a transformation of the Cartesian product of the module with itself into the module, if every open set is a union of hyperconvex open sets, and if every hyperconvex open set has an absorption property: Whenever a is an element of the set and b is an element of the module, an invertible nonnegative transformation T exists such that $1 - T$ is nonnegative and such that the hyperconvex combination

$$(1 - T)a + tb$$

belongs to the set.

A Weierstrass algebra is treated as a module over itself for the definition of a locally hyperconvex topology. A Weierstrass algebra is required to have a locally hyperconvex topology. A locally hyperconvex topology of a two-sided module over a Weierstrass algebra is required to be compatible with the locally hyperconvex topology of the algebra: A continuous transformation of the Cartesian product of the Weierstrass algebra with itself into the module is defined by taking a pair of elements ξ and η of the algebra into the element $\xi^{-c}\eta$ of the module for every element c of the module.

If a locally hyperconvex topology of a module over a Weierstrass algebra is given, then a related locally hyperconvex topology is constructed using the concept of a hyperdisk. A nonempty hyperconvex subset of the module is said to be a hyperdisk if it is disjoint from the closure of every disjoint hyperconvex set.

If A is a hyperdisk and if B is a hyperconvex set, then the intersection of A with the closure of B is contained in the closure of the intersection of A with B . For an element of A which does not belong to the closure of the intersection of A with B belongs to a hyperconvex open set C whose intersection with A is disjoint from C . Since the intersection of B with C is a hyperconvex set which is disjoint from A , the hyperdisk A is disjoint from the closure of the intersection of B with C . Since C is an open set, the intersection of C with the closure of B is contained in the closure of the intersection of B with C . It follows that the intersection of A with C is disjoint from the closure of B .

The intersection of hyperdisks A and B is a hyperdisk if it is nonempty. For the intersection of A and B is a hyperconvex set. If a hyperconvex set C is disjoint from the intersection of A and B , then the intersection of B and C is a hyperconvex set which is disjoint from A . Since A is a hyperdisk, A is disjoint from the closure of the intersection of B and C . Since B is a hyperdisk, the intersection of B with the closure of C is contained in the closure of the intersection of B with C . It follows that the intersection of A and B is disjoint from the closure of C .

The hyperdisk topology of a locally hyperconvex space is the locally hyperconvex topology whose open sets are the unions of hyperdisk. The hyperdisk topology has the same closed hyperconvex sets as the given topology. Since every nonempty hyperconvex set which is open for the given topology is a hyperdisk, every hyperconvex set which is closed

for the given topology is closed for the hyperdisk topology. If a hyperconvex set B is closed for the hyperdisk topology, then an element of the space which does not belong to B belongs to a hyperdisk A which is disjoint from B . Since A is disjoint from the closure of B , an element of the space which does not belong to B does not belong to the closure of B .

The closure of a hyperconvex set B with respect to a locally hyperconvex topology is hyperconvex. For if u and v are elements of the closure of B and if A is a hyperconvex open set containing the origin, then elements a and b of B exist such that $u - a$ and $v - b$ belong to A . An element of the hyperconvex span of u and v is a hyperconvex combination

$$(1 - T)u + Tv$$

with T a nonnegative transformation such that $1 - T$ is nonnegative. Since B is hyperconvex, the hyperconvex combination

$$(1 - T)a + Tb$$

belongs to B . Since A is hyperconvex, the difference

$$[(1 - T)u + Tv] - [(1 - T)a + Tb] = (1 - T)(u - a) + T(v - b)$$

belongs to A .

If B is a nonempty hyperconvex set and if s is an element of the locally hyperconvex space which does not belong to B , then a hyperconvex set $B(s)$ is constructed so that B is contained in $B(s)$ and so that s belongs to the closure of $B(s)$. The set $B(s)$ is the set of hyperconvex combinations

$$(1 - T)s + Tc$$

with c an element of B and with T an invertible nonnegative transformation such that $1 - T$ is nonnegative. Every hyperconvex open set which contains s contains an element of $B(s)$ by the definition of a locally hyperconvex topology. It is sufficient by a translation to verify hyperconvexity of $B(s)$ when s is the origin. A hyperconvex combination

$$(1 - T)Pa + TQb$$

of elements Pa and Qb of $B(s)$ is constructed from elements a and b of B with Ta nonnegative transformation such that $1 - T$ is nonnegative and with P and Q invertible nonnegative transformations such that $1 - P$ and $1 - Q$ are nonnegative. Then

$$R = (1 - T)P + TQ$$

is a nonnegative transformation such that

$$1 - P = (1 - T)(1 - P) + T(1 - Q)$$

is nonnegative. Since P and Q are invertible, R is invertible. A nonnegative transformation S such that $1 - S$ is nonnegative is obtained as a solution of the equations

$$RS = TQ$$

and

$$R(1 - S) = (1 - T)P.$$

Since the set B is hyperconvex,

$$c = (1 - S)a + Sb$$

is an element of B . The hyperconvex combination

$$(1 - T)Pa + TQb = Rc$$

of elements Pa and Qb of the set $B(s)$ is then an element Rc of the set $B(s)$.

The conjugate dual space of a locally hyperconvex space \mathcal{H} over a Weierstrass algebra is the essentially unique vector space \mathcal{H}^* over the complex numbers which is in duality with \mathcal{H} and which represents the functionals linear over the rational numbers and continuous for the hyperdisk topology. The pairing

$$\langle a, b \rangle$$

of an element a of \mathcal{H} with an element b of \mathcal{H}^* is linear over the rational numbers as a function of a for fixed b and conjugate linear over the complex numbers as a function of b for fixed a . If b is an element of \mathcal{H}^* , b^- is the linear functional on \mathcal{H} defined by the scalar product

$$b^- a = \langle a, b \rangle$$

for every element a of \mathcal{H} . The functional is continuous from the hyperdisk topology of \mathcal{H} to the Euclidean topology of the complex numbers. Every linear functional on \mathcal{H} which is continuous from the hyperdisk topology of \mathcal{H} to the Euclidean topology of the complex numbers is represented by a unique element of \mathcal{H}^* . An element b of \mathcal{H}^* is said to be hyperlinear if the inverse image under b^- of every convex subset of the complex plane is a hyperconvex subset of \mathcal{H} .

The hyperconvex Hahn–Banach theorem is an existence theorem for hyperlinear elements of the conjugate dual space of a locally hyperconvex space.

Theorem 1. *If a hyperdisk A of a locally hyperconvex space \mathcal{H} is disjoint from a hyperconvex subset B of \mathcal{H} , then a hyperlinear element b of the conjugate dual space \mathcal{H}^* exists such that b^- maps A and B into disjoint subsets of the complex plane.*

Proof of Theorem 1. It can be assumed that the set B is nonempty. A maximal hyperconvex set which contains B and is disjoint from A exists by the Zorn lemma. It is sufficient to give a proof of the theorem with B is a maximal hyperconvex set which is disjoint from

A . Since the closure of B is hyperconvex and is disjoint from A , B is a closed hyperconvex set. It will be shown that the complement of B is hyperconvex.

If u belongs to the complement of B , a hyperconvex set $B(u)$ is constructed as the set of hyperconvex combinations

$$(1 - P)u + Pa$$

of u and elements a of B with Pa nonnegative transformation such that $1 - P$ is nonnegative and such that P and $1 - P$ are invertible. Since the closure of $B(u)$ contains u and every element of B , the element of $B(u)$ can be chosen in A .

If v belongs to the complement of B , a hyperconvex set $B(v)$ is obtained as the set of hyperconvex combinations

$$(1 - Q)v + Qb$$

of v and elements b of B with Q a nonnegative transformation such that $1 - Q$ is nonnegative and such that Q and $1 - Q$ are invertible. Since the closure of $B(v)$ contains v and every element of B , the element of $B(v)$ can be chosen in A .

A hyperconvex combination

$$(1 - V)u + Vv$$

of u and v is defined using a nonnegative transformation V such that $1 - V$ is nonnegative. Since the nonnegative transformation

$$(1 - P)V + (1 - Q)(1 - V)$$

is invertible, a nonnegative transformation T such that $1 - T$ is nonnegative exists which satisfies the equation

$$T(1 - Q)(1 - V) = (1 - T)(1 - P)V.$$

The nonnegative transformations

$$R = (1 - T)P + TQ$$

and

$$1 - R = (1 - T)(1 - P) + T(1 - Q)$$

are invertible. A nonnegative transformation U such that $1 - U$ is nonnegative exists which satisfies the identities

$$R(1 - U) = (1 - T)P$$

and

$$RU = TQ.$$

The identities

$$(1 - R)(1 - V) = (1 - T)(1 - P)$$

and

$$(1 - R)V = T(1 - Q)$$

are satisfied. Since the identity

$$\begin{aligned} & (1 - T)[(1 - P)u + Pa] + T[(1 - Q)v + Qb] \\ &= (1 - R)[(1 - V)u + Vv] + R[(1 - U)a + Ub] \end{aligned}$$

is satisfied, the hyperconvex combination of elements

$$(1 - P)u + Pa$$

and

$$(1 - Q)v + Qb$$

of elements of A is an element of A which is a hyperconvex combination of

$$(1 - V)u + Vv$$

and the element

$$(1 - U)a + Ub$$

of B .

This completes the proof that the complement of B is hyperconvex. Since the hyperconvex set B is closed, the hyperconvex complement of B is open. A continuous hyperlinear functional exists which maps B and its complement into disjoint convex subsets of the real line. The hyperlinear functional is represented by an element of \mathcal{H}^* .

This completes the proof of the theorem.

A locally hyperconvex space admits a strongest locally hyperconvex topology. A hyperconvex set is open for the strongest locally hyperconvex topology if for every element a of the set and for every element b of the space, a hyperconvex combination

$$(1 - T)a + Tb$$

belongs to the set with T an invertible nonnegative transformation such that $1 - T$ is nonnegative.

A characterization of hyperdisks is an application of the proof of the hyperconvex Hahn–Banach theorem. A nonempty hyperconvex set, which is open for the strongest locally hyperconvex topology, is a hyperdisk if, and only if, every hyperlinear functional which maps the set into a proper subset of the complex plane is continuous.

The hyperweak topology of a locally hyperconvex space \mathcal{H} is the weakest topology with respect to which b^- is continuous for every hyperlinear element b of the conjugate dual space \mathcal{H}^* . The space \mathcal{H} is a Hausdorff space in the hyperweak topology by the Hahn–Banach theorem. Addition is a continuous transformation of the Cartesian product of \mathcal{H} with itself into \mathcal{H} when \mathcal{H} is considered in the hyperweak topology. Every hyperweakly open set is

a union of hyperweakly open hyperconvex sets. If a is an element of a hyperweakly open hyperconvex set and if b is an element of \mathcal{H} , then

$$(1 - T)a + Tb$$

belongs to the set for an invertible nonnegative transformation T such that $1 - T$ is nonnegative. The transformation can be defined to take c into $\xi^-c\xi$ for a nonzero rational multiple of the identity transformation. If c is an element of \mathcal{H} , the transformation of the Cartesian product of the Weierstrass algebra with itself into \mathcal{H} which takes a pair of elements ξ and η of the algebra into $\xi^-c\eta$ is continuous when \mathcal{H} is considered in the hyperweak topology since it is continuous when \mathcal{H} is considered in the given locally hyperconvex topology and since the inclusion of \mathcal{H} in itself is continuous from the given locally hyperconvex topology into the hyperweak topology. The hyperweak topology is then a locally hyperconvex topology. A hyperconvex set is closed for the hyperweak topology if, and only if, it is closed for the given locally hyperconvex topology. A hyperlinear functional is continuous for the hyperweak topology if, and only if, it is continuous for the given locally hyperconvex topology.

A center for a hyperconvex subset of a locally hyperconvex space is an element a of the set such that $2a - b$ belongs to the set whenever b belongs to the set. A hyperconvex subset of a locally hyperconvex space is said to be centered at a if a is a center of the set.

The conjugate dual space \mathcal{H}^* of a locally hyperconvex space is treated as a two-sided module over the Weierstrass algebra. If ξ and η are elements of the Weierstrass algebra and if b is an element of \mathcal{H}^* , then $\xi^-b\eta$ is the element of \mathcal{H}^* defined by the identity

$$(\xi^-b\eta)^-a = b^-(\xi a\eta^-)$$

for every element a of \mathcal{H} . A nonnegative transformation T , which by definition map \mathcal{H} into \mathcal{H} , has an adjoint T^* which maps \mathcal{H}^* into \mathcal{H}^* and which is defined by the identity

$$(T^*b)^-a = b^-(Ta)$$

for every element a of \mathcal{H} and every element b of \mathcal{H}^* . A hyperconvex combination

$$(1 - T^*)a + T^*b$$

of elements a and b of \mathcal{H}^* is defined using the adjoint T^* of a nonnegative transformation T . A subset of \mathcal{H}^* is said to be hyperconvex if it contains the hyperconvex combinations of every pair of elements. A center of a hyperconvex subset of \mathcal{H}^* is an element a of the set such that $2a - b$ belongs to the set whenever b belongs to the set. A hyperconvex subset of \mathcal{H}^* is said to be centered at a if a is a center of the set.

If \mathcal{H}^* is the conjugate dual space of a locally hyperconvex space \mathcal{H} , the pairing

$$\langle b, a \rangle = \langle a, b \rangle^-$$

between an element b of \mathcal{H}^* and an element a of \mathcal{H} is defined as the complex conjugate of the pairing between an element a of \mathcal{H} and an element b of \mathcal{H}^* . If a is an element of \mathcal{H} , a^- is the linear functional on \mathcal{H}^* defined by the identity

$$a^-b = \langle b, a \rangle$$

for every element b of \mathcal{H}^* . The weak topology of \mathcal{H}^* is the weakest topology with respect to which a^- is continuous for every element a of \mathcal{H} .

A construction of weakly compact hyperconvex sets of hyperlinear elements of the conjugate dual space of a locally hyperconvex space is an application of the compactness of Cartesian products of compact Hausdorff spaces.

Theorem 2. *If a hyperdisk A of a locally hyperconvex space \mathcal{H} is centered at the origin, then the set B of elements b of the conjugate dual space \mathcal{H}^* such that the real part of b^- maps A into the interval $(-1, 1)$ is a weakly compact hyperconvex set which is centered at the origin. The set A is the set of elements a of \mathcal{H} such that the real part of a^- maps the set of hyperlinear elements of B into the interval $(-1, 1)$.*

Proof of Theorem 2. The set B is clearly hyperconvex and centered at the origin. Since every element of \mathcal{H} is mapped into A by an invertible nonnegative transformation T such that $1 - TY$ is nonnegative, the action of b^- on \mathcal{H} for an element of \mathcal{H}^* is determined by the restriction to A of c^- when c is the element of \mathcal{H}^* such that $b = T^*c$. Since the interval $[-1, 1]$ is a compact Hausdorff space I , the set I^A of all functions defined on A with values in I is a compact Hausdorff space in the Cartesian product topology. A hyperlinear element b of \mathcal{H}^* belongs to B if, and only if, the restriction of b^- to A belongs to I^A . The closure in I^A of the functions represented by elements of B consists of functions which have hyperlinear extensions to \mathcal{H} . Since A is a hyperdisk, these hyperlinear functionals are continuous. Since the hyperlinear functionals are represented by elements of \mathcal{H}^* , they are represented by elements of B . Since B determines a compact subset of I^A , the set B is weakly compact.

If an element a of \mathcal{H} belongs to A , then the real part of a^- maps B into the interval $(-1, 1)$. It will be shown that the real part of a^- does not map the set of hyperlinear elements of B into the interval $(-1, 1)$ when an element a of \mathcal{H} does not belong to A . A hyperlinear element b of \mathcal{H}^* then exists by the hyperconvex Hahn–Banach theorem such that b^-a does not belong to the image of A under b^- . The choice of b is made so that the real part of b^-a does not belong to the image of A under the real part of b^- . Since the hyperdisk A is centered at the origin, the image of A is a convex open subset of the real line which is centered at the origin. Since the real part of b^-a does not belong to the set, the choice of b can be made so that the real part of b^- maps A into the interval $(-1, 1)$ and does not map a into the interval. Then b is a hyperlinear element of B such that the real part of a^- does not map b into the interval $(-1, 1)$.

This completes the proof of the theorem.

A converse construction of hyperdisks of a locally hyperconvex space is made from weakly compact hyperconvex subsets of the conjugate dual space.

Theorem 3. *If a weakly compact hyperconvex set B of the conjugate dual space \mathcal{H}^* of a locally hyperconvex space \mathcal{H} is centered at the origin, then the set A of elements a of \mathcal{H} such that the real part of a^- maps B into the interval $(-1, 1)$ is a hyperdisk which is centered at the origin. The set B is the set of elements b of \mathcal{H}^* such that the real part of b^- maps A into the interval $(-1, 1)$.*

Proof of Theorem 3. It will first be shown that B is the set of elements b of \mathcal{H}^* such that the real part of b^- maps A into the interval $(-1, 1)$. The proof is an application of the hyperconvex Hahn–Banach theorem in which \mathcal{H}^* is treated as a locally hypervonvex space over the complex numbers as Weierstrass algebra. A nonnegative transformation for \mathcal{H}^* is multiplication by a nonnegative number. The given locally hyperconvex space \mathcal{H} is then a subspace of the conjugate dual space \mathcal{H}^\wedge of \mathcal{H}^* . The space \mathcal{H}^\wedge is a vector space over the complex numbers whose elements are the linear combinations with complex coefficients of elements of \mathcal{H} . The set A is a subset of the set A^\wedge of elements a of \mathcal{H}^\wedge such that the real part of a^- maps B into the interval $(-1, 1)$. The set A is dense in A^\wedge considered in the weak topology induced by \mathcal{H}^* . If an element b of \mathcal{H}^* does not belong to the weakly compact convex set B , then it does not belong to some weakly open convex set which contains B . Since B is centered at the origin, the weakly open set can be chosen to be centered at the origin. An element a of \mathcal{H}^\wedge exists such that the real part of a^- maps the weakly open set into the interval $(-1, 1)$ but does not map b into the interval. Since A is weakly dense in A^\wedge , an element a of A exists such that the real part of a^- maps B into the interval $(-1, 1)$ but does not map b into the interval. The real part of b^- does not map A into the interval $(-1, 1)$.

The set A^\wedge is hyperconvex and centered at the origin since the set B is hyperconvex and centered at the origin. The space \mathcal{H}^\wedge is a locally hyperconvex space over the complex numbers as Weierstrass algebra when considered in the weak topology induced by \mathcal{H} . It will be shown that A^\wedge is an open subset of \mathcal{H}^\wedge considered in its strongest locally hyperconvex topology. If a is an element of A^\wedge and if G is an element of \mathcal{H}^\wedge , a number t in the interval $(0, 1]$ exists such that

$$(1 - t)a + tc$$

belongs to A^\wedge . Since B is weakly compact, a positive number κ exists such that the real part of c^- maps B into the interval $(-\kappa, \kappa)$. Since the real part of a^- maps B into the interval $(-1, 1)$, a positive number ϵ exists such that the real part of a^- maps B into the interval $(\epsilon - 1, 1 - \epsilon)$. When the positive number t is chosen so small that

$$t\kappa < \epsilon$$

and

$$t \leq 1,$$

then

$$(1 - t)a + tc$$

belongs to A^\wedge .

The set A is shown to be a hyperdisk by showing that a hyperlinear element b of the conjugate dual space of \mathcal{H} for the strongest locally hyperconvex topology belongs to \mathcal{H}^* if b^- maps A into a proper subset of the complex plane. Since A is hyperconvex and centered at the origin, it can be assumed that the real part of b^- maps A into the interval $(-1, 1)$. The desired conclusion holds since b belongs to B by an argument at the start of the proof.

This completes the proof of the theorem.

The completion of a locally hyperconvex space \mathcal{H} in its hyperdisk topology is a locally hyperconvex space \mathcal{H}^\wedge over the same Weierstrass algebra. The conjugate dual space of the completion coincides as a set with the conjugate dual space \mathcal{H}^* of \mathcal{H} . The inclusion of \mathcal{H}^* in itself is continuous from the weak topology induced by \mathcal{H}^\wedge into the weak topology induced by \mathcal{H} . The hyperconvex subsets of \mathcal{H}^* , which are compact in the weak topology induced by \mathcal{H} , are the hyperconvex subsets of \mathcal{H}^* , which are compact in the weak topology induced by \mathcal{H}^\wedge . The weak topology induced by \mathcal{H} and the weak topology induced by \mathcal{H}^\wedge coincide on these sets. A nonempty convex subset of \mathcal{H}^* is said to be a subdisk if it is disjoint from the weak closure of every disjoint hyperconvex set whose weak closure is weakly compact. If A is a subdisk and if B is a hyperconvex set whose weak closure is weakly compact, then the intersection of A with the weak closure of B is contained in the weak closure of the intersection of A with B . The intersection of two subdisks is a subdisk if it is nonempty.

The space \mathcal{H}^* is a Hausdorff space in a topology whose open sets are the unions of subdisks. Addition is continuous as a transformation of the Cartesian product of \mathcal{H}^* with itself into \mathcal{H}^* when \mathcal{H}^* is considered with the subdisk topology. Multiplication is continuous as a transformation of the Cartesian product of \mathcal{H}^* with the complex numbers into \mathcal{H}^* when \mathcal{H}^* is considered with the subdisk topology.

A linear functional on \mathcal{H}^* is continuous for the subdisk topology if, and only if, its restriction to every hyperconvex set whose weak closure is weakly compact is weakly continuous. Every linear functional on \mathcal{H}^* which is continuous for the weak topology is continuous for the subdisk topology. A linear functional is continuous for the weak topology if, and only if, it is represented by an element of \mathcal{H} . The completion of \mathcal{H} is the essentially unique vector space \mathcal{H}^\wedge , which contains \mathcal{H} and which is in duality with \mathcal{H}^* , such that the linear functionals on \mathcal{H}^* which are continuous for the subdisk topology are the linear functionals which are represented by elements of \mathcal{H}^\wedge . The pairing of an element a of \mathcal{H} with an element b of \mathcal{H}^* is assumed to be equal to the pairing of a as an element of \mathcal{H}^\wedge with b as an element of \mathcal{H}^* . The compact hyperconvex subsets of \mathcal{H}^* for the weak topology induced by \mathcal{H}^\wedge are compact hyperconvex subset of \mathcal{H}^* for the weak topology induced by \mathcal{H} .

If ξ and η are elements of the Weierstrass algebra, the transformation b into $\xi^- b \eta$ maps weakly compact hyperconvex subsets of \mathcal{H}^* into weakly compact hyperconvex subsets of \mathcal{H}^* . The transformation of \mathcal{H} into itself which takes a into $\xi a \eta^-$ has an extension a into $\xi a \eta^-$ of \mathcal{H}^\wedge into itself which is defined by the identity

$$b^-(\xi a \eta^-) = (\xi^- b \eta)^- a$$

for every element b of \mathcal{H}^* .

A locally hyperconvex topology of \mathcal{H}^\wedge is defined using the linear functionals on \mathcal{H}^* represented by elements of \mathcal{H}^\wedge . The locally hyperconvex topology of \mathcal{H}^\wedge is the topology of uniform convergence of linear functionals on hyperconvex subsets of \mathcal{H}^* which are compact in the weak topology induced by \mathcal{H} . Every hyperdisk of \mathcal{H}^\wedge is an open set when \mathcal{H}^\wedge is considered with this topology. The hyperdisk topology of \mathcal{H} is then the subspace topology which \mathcal{H} inherits from \mathcal{H}^\wedge . The space \mathcal{H} is dense in the space \mathcal{H}^\wedge by the hyperconvex Hahn–Banach theorem.

A subset of the conjugate dual space \mathcal{H}^* of a locally hyperconvex space \mathcal{H} is said to be bounded if its image under a^- is a bounded subset of the complex plane for every element a of \mathcal{H} . A weakly compact subset of \mathcal{H}^* is bounded since its image under a^- is a compact subset of the complex plane for every element a of \mathcal{H} . A weakly compact subset of \mathcal{H}^* is also weakly closed. The weak compactness of weakly closed and bounded hyperconvex subsets of \mathcal{H}^* is a hypothesis in the open mapping theorem and in the closed graph theorem. An equivalent hypothesis is made on hyperconvex subsets of \mathcal{H} .

Theorem 4. *Every weakly closed and bounded hyperconvex subset of the conjugate dual space \mathcal{H}^* of a locally hyperconvex space \mathcal{H} is weakly compact if, and only if, every nonempty hyperconvex subset of \mathcal{H} which is open for the strongest locally hyperconvex topology and whose closure is equal to its closure for the strongest locally hyperconvex topology is a hyperdisk.*

Proof of Theorem 4. A hyperconvex subset of \mathcal{H}^* which is weakly closed and bounded is contained in a hyperconvex subset of \mathcal{H}^* which is centered at the origin as well as being weakly closed and bounded. Compactness of the weakly closed and bounded subset follows from its inclusion in a weakly compact subset which is centered at the origin.

A hyperconvex subset of \mathcal{H} containing the origin which is open for the strongest locally hyperconvex topology and whose closure is equal to its closure for the strongest locally hyperconvex topology contains a nonempty hyperconvex set which is centered at the origin as well as being open for the strongest locally hyperconvex topology and having its closure equal to its closure for the strongest locally hyperconvex topology. The given hyperconvex set is a hyperdisk if it contains a hyperdisk which is centered at the origin.

If a hyperconvex subset B of \mathcal{H}^* is centered at the origin as well as being weakly closed and bounded, then the set A of elements a of \mathcal{H} such that the real part of a^- maps B into a closed subset of the interval $(-1, 1)$ is a hyperconvex subset of \mathcal{H} which is centered at the origin as well as being open for the strongest locally hyperconvex topology and having its closure equal to its closure for the strongest locally hyperconvex topology. If A is a hyperdisk, then the set of elements b of \mathcal{H}^* such that the real part of b^- maps A into the interval $(-1, 1)$ is weakly compact. Since B is a weakly closed subset of a weakly compact set, it is weakly compact.

If a hyperconvex set A is centered at the origin as well as being open for the strongest locally hyperconvex topology and having its closure equal to its closure for the strongest locally hyperconvex topology, then the set of elements b of \mathcal{H}^* such that the real part of b^- maps the closure of A into the interval $(-1, 1)$ is a bounded hyperconvex set which is

centered at the origin. If the closure of B is weakly compact, then the set of elements a of \mathcal{H} such that the real part of a^- maps the closure of B into the interval $(-1, 1)$ is a hyperdisk which is centered at the origin. Since the hyperconvex set A is open for the strongest locally hyperconvex topology and since its closure for the strongest locally hyperconvex topology contains a hyperdisk, A is a hyperdisk.

This completes the proof of the theorem.

A hyperdisk is an example of a nonempty hyperconvex subset of a locally hyperconvex space which is open for the strongest locally hyperconvex topology and whose closure is equal to its closure for the strongest locally hyperconvex topology. The intersection of two hyperconvex sets which are open for the strongest locally hyperconvex topology and whose closure is equal to its closure for the strongest locally hyperconvex topology is a hyperconvex set which is open for the strongest locally hyperconvex topology and whose closure is equal to its closure for the strongest locally hyperconvex topology. A locally hyperconvex space is said to be primitive if every nonempty hyperconvex set which is open for the strongest locally hyperconvex topology and whose closure is equal to its closure for the strongest locally hyperconvex topology is an open set. If a locally hyperconvex space is primitive, then every hyperdisk is an open set. If a locally hyperconvex space is not primitive, then it admits a primitive locally hyperconvex topology whose open sets are the unions of the nonempty hyperconvex sets which are open for the strongest locally hyperconvex topology and whose closure for the given topology is equal to its closure for the strongest locally hyperconvex topology. The inclusion of the space in itself is continuous from the primitive locally hyperconvex topology into the given locally hyperconvex topology.

A computation of weakly compact hyperconvex subsets of the conjugate dual space of a locally hyperconvex space is an underlying concept in a proof [1] of the Stone–Weierstrass theorem.

Theorem 5. *A weakly compact hyperconvex subset of the conjugate dual space of a locally hyperconvex space is the weakly closed convex span of its hyperlinear elements if it is centered at the origin.*

Proof of Theorem 5. Assume that a hyperconvex subset B of the conjugate dual space \mathcal{H}^* of a locally hyperconvex space \mathcal{H} is weakly compact and centered at the origin. The set of hyperlinear elements of B is weakly compact since it is weakly closed. The set A of elements a of \mathcal{H} such that the real part of a^- maps B into the interval $(-1, 1)$ is a hyperdisk which is centered at the origin. The set B is the set of elements b of \mathcal{H}^* such that the real part of b^- maps A into the interval $(-1, 1)$. It has been seen that A is the set of element a of \mathcal{H} such that the real part of a^- maps the set of hyperlinear elements of B into the interval $(-1, 1)$. It follows that A is the set of element a of \mathcal{H} such that the real part of a^- maps the weakly closed convex span of the hyperlinear elements of B into the interval $(-1, 1)$. The closed convex span of the hyperlinear elements of B is a weakly compact convex subset of B . The desired conclusion is an application of the Hahn–Banach theorem for the weak topology of \mathcal{H}^* . If an element b of \mathcal{H}^* does not belong to the weakly

closed convex span of the hyperlinear element of B , then an element a of \mathcal{H} exists such that a^- maps the weakly closed convex span of the hyperlinear element of B into a set which does not contain the image of b . The choice of a can be made so that the real part of a^- maps the weakly closed convex span of the hyperlinear elements of B into the interval $(-1, 1)$ and does not map b into the interval. Since a belongs to A , b does not belong to B .

This completes the proof of the theorem.

A locally hyperconvex space is said to have the hyperconvex Krein–Šmulyan property if a convex subset of the conjugate dual space is weakly closed whenever it has a weakly compact intersection with every weakly compact hyperconvex set.

Theorem 6. *A locally hyperconvex space \mathcal{H} has the hyperconvex Krein–Šmulyan property if a countable basis exists for the neighborhoods of the origin in the hyperdisk topology.*

Proof of Theorem 6. A sequence of hyperdisks A_n , which are centered at the origin, exists such that A_{n+1} is contained in A_n for every positive integer n and such that every neighborhood of the origin for the hyperdisk topology contains some set A_n . Define B_n for every positive integer n as the set of elements b of the conjugate dual space \mathcal{H}^* such that the real part of b^- maps A_n into the interval $(-1, 1)$. Then B_n is a weakly compact hyperconvex subset of \mathcal{H}^* which is centered at the origin. The set B_n is contained in the set B_{n+1} for every positive integer n . Every weakly compact hyperconvex subset of \mathcal{H}^* is contained in some set B_n . A convex subset C of \mathcal{H}^* will be shown weakly closed if it has a weakly compact intersection with every weakly compact hyperconvex set. It needs to be shown that an element of \mathcal{H}^* which does not belong to C does not belong to the closure of C . It can by a translation be assumed that the element is the origin. Since the intersection of C with B_n is a weakly compact convex set which does not contain the origin, a weakly open convex set U_n exists which contains the origin but whose closure is disjoint from the intersection of C with B_n . The sets U_n can be constructed inductively by the Hahn–Banach theorem so that U_{n+1} always contains the closure of the intersection of U_n with B_n . A weakly open convex subset U exists which is disjoint from C but which contains the intersection of the closure of U_n with B_n for every positive integer n .

This completes the proof of the theorem.

A transformation T of a locally hyperconvex space \mathcal{P} into a locally hyperconvex space \mathcal{Q} is said to be hyperlinear if \mathcal{P} and \mathcal{Q} are modules over the same Weierstrass algebra, if the transformations are linear when \mathcal{P} and \mathcal{Q} are treated as vector spaces over the rational numbers, if the identity

$$T(\xi^- c\eta) = \xi^-(Tc)\eta$$

holds for every element c of \mathcal{P} when ξ and η are elements of the Weierstrass algebra, and if the transformation maps hyperconvex subsets of \mathcal{P} into hyperconvex subsets of \mathcal{Q} . If the transformation is continuous from a locally hyperconvex topology of \mathcal{P} to a locally hyperconvex topology of \mathcal{Q} , then it is continuous from the hyperweak topology of \mathcal{P} to the hyperweak topology of \mathcal{Q} since the closed hyperconvex sets for a locally hyperconvex

topology are identical with the closed hyperconvex sets for the hyperweak topology and since continuity of a hyperlinear transformation is decided by closures of hyperconvex sets. The transformation is then continuous from the hyperdisk topology of \mathcal{P} to the hyperdisk topology of \mathcal{Q} . For if A is a hyperdisk of \mathcal{Q} , the inverse image of A in \mathcal{P} is a hyperconvex set. If a hyperconvex set C of \mathcal{P} is disjoint from the inverse image of A , then the image of C in \mathcal{Q} is a hyperconvex set which is disjoint from A . Since A is a hyperdisk, the closure of the image of C in \mathcal{Q} is a hyperconvex set B which is disjoint from A . Since T is continuous from the hyperweak topology of \mathcal{P} to the hyperweak topology of \mathcal{Q} , the inverse image of B in \mathcal{P} is a closed hyperconvex set. Since C is contained in the inverse image of B , the closure of C is contained in the inverse image of B . Since the inverse image of B is disjoint from the inverse image of A , the closure of C is disjoint from the inverse image of A . This completes the verification that the inverse image of A is a hyperdisk.

The hyperconvex closed graph theorem applies to hyperlinear transformations in primitive locally hyperconvex spaces which have the hyperconvex Krein–Šmulyan property.

Theorem 7. *A hyperlinear transformation T of a primitive locally hyperconvex space \mathcal{P} into a locally hyperconvex space \mathcal{Q} which has the hyperconvex Krein–Šmulyan property is continuous if it has a closed graph in the Cartesian product of \mathcal{P} and \mathcal{Q} .*

Proof of Theorem 7. Continuity of T is proved by showing that the domain of the adjoint T^* of T contains every element of the conjugate dual space \mathcal{Q}^* of \mathcal{Q} . The adjoint takes an element a of \mathcal{Q}^* into an element b of the conjugate dual space \mathcal{P}^* of \mathcal{P} when the identity

$$a^- Tc = b^- c$$

holds for every element c of the domain of T , which is \mathcal{P} by hypothesis. If every element of \mathcal{Q}^* belongs to the domain of T^* , then T is continuous from the hyperweak topology of \mathcal{P} to the hyperweak topology of \mathcal{Q} . Since \mathcal{P} is then continuous from the hyperdisk topology of \mathcal{P} to the hyperdisk topology of \mathcal{Q} and since \mathcal{P} is a primitive locally hyperconvex space, T is continuous from the given locally hyperconvex topology of \mathcal{P} to the given locally hyperconvex topology of \mathcal{Q} .

The adjoint T^* is a relation which has a closed graph in the Cartesian product of \mathcal{Q}^* , which is considered in the weak topology induced by \mathcal{Q} , and \mathcal{P}^* , which is considered in the weak topology induced by \mathcal{P} . The adjoint T^* is a transformation since the domain of T is dense in \mathcal{P} . Since T has a closed graph, T is the adjoint of T^* . Since T is a transformation, the domain of T^* is in \mathcal{Q}^* .

The proof of the theorem is completed by showing that the domain of T^* is a weakly closed subset of \mathcal{Q}^* . The domain of T^* is hyperconvex since the transformation T is hyperlinear. Since the locally hyperconvex space \mathcal{Q} has the Krein–Šmulyan property, it is sufficient to show that the domain of T^* has a weakly compact intersection with every weakly compact hyperconvex subset of \mathcal{Q}^* . If B is a weakly compact hyperconvex subset of \mathcal{Q}^* , then the set of elements of \mathcal{P}^* of the form T^*b with b in B is hyperconvex since T is hyperlinear and is bounded since the identity

$$a^- (T^*b) = (Ta)^- b$$

holds for every element a of \mathcal{P} . Since the locally hyperconvex space \mathcal{P} is primitive, the weak closure of the set of elements of \mathcal{P}^* of the form T^*b with b in B is a weakly compact hyperconvex subset C of \mathcal{P}^* . The Cartesian product of B and C is a compact subset of the Cartesian product of \mathcal{Q}^* in the weak topology induced by \mathcal{Q} and \mathcal{P}^* in the weak topology induced by \mathcal{P} . Since the graph of T^* is a closed subset of the Cartesian product space, it has a compact intersection with the Cartesian product of B and C . It follows that the intersection of B with the domain of T^* is weakly compact.

This completes the proof of the theorem.

If a hyperlinear transformation T of a locally hyperconvex space \mathcal{P} onto a locally hyperconvex space \mathcal{Q} maps open sets for the locally hyperconvex topology of \mathcal{P} into open sets for the locally hyperconvex topology of \mathcal{Q} , then it maps open sets for the hyperweak topology of \mathcal{P} into open sets for the hyperweak topology of \mathcal{Q} . The transformation then maps open sets for the hyperdisk topology of \mathcal{P} into open sets for the hyperdisk topology of \mathcal{Q} . For if A is a hyperdisk of \mathcal{P} , the image of A in \mathcal{Q} is a hyperconvex set. If a hyperconvex subset C of \mathcal{Q} is disjoint from the image of A , then the inverse image of C in \mathcal{P} is disjoint from A . Since A is a hyperdisk, the closure of the inverse image of C in \mathcal{P} is a hyperconvex set B which is disjoint from A . The image of B in \mathcal{Q} is then a closed hyperconvex set which contains C and is disjoint from the image of A . This completes the verification that the image of A is a hyperdisk.

The hyperconvex open mapping theorem applies to continuous hyperlinear transformations in primitive locally hyperconvex spaces which have the hyperconvex Krein-Šmulyan property.

Theorem 8. *A hyperlinear transformation T of a locally hyperconvex space \mathcal{P} which has the hyperconvex Krein-Šmulyan property onto a primitive locally hyperconvex space \mathcal{Q} maps open sets into open sets if it is continuous.*

Proof of Theorem 8. Since T is hyperlinear and continuous, the kernel of T is a closed vector subspace of \mathcal{P} which is invariant under multiplication by elements of the Weierstrass algebra. The quotient space of \mathcal{P} is a locally hyperconvex space whose conjugate dual space is identified with the set of elements b of the conjugate dual space of \mathcal{P} such that b^- annihilates the kernel of T . Since the space \mathcal{P} has the hyperconvex Krein-Šmulyan property by hypothesis, the quotient space of \mathcal{P} has the hyperconvex Krein-Šmulyan property.

It can without loss of generality be assumed that the kernel of T contains no nonzero element. The inverse T^{-1} of T is then a hyperlinear transformation of \mathcal{Q} into \mathcal{P} which has a closed graph. The desired conclusions follow from the closed graph theorem.

This completes the proof of the theorem.

An example of a Weierstrass algebra is the set $\mathcal{C}(\mathcal{S})$ of all functions on a discrete space \mathcal{S} . The topology of pointwise convergence on \mathcal{S} is a locally hyperconvex topology of $\mathcal{C}(\mathcal{S})$. The topology is the only locally hyperconvex topology of $\mathcal{C}(\mathcal{S})$ when \mathcal{S} is a finite set since $\mathcal{C}(\mathcal{S})$ has finite dimension as a vector space over the complex numbers. The space $\mathcal{C}(\mathcal{S})$ is

then a primitive locally hyperconvex space which has the Krein–Šmulyan property. The same conclusions hold when \mathcal{S} is an infinite set.

Theorem 9. *If \mathcal{S} is a discrete space, the Weierstrass algebra $\mathcal{C}(\mathcal{S})$ has a unique locally hyperconvex topology. The space $\mathcal{C}(\mathcal{S})$ is a primitive locally hyperconvex space which has the Krein–Šmulyan property.*

Proof of Theorem 9. The set \mathcal{S} is a discrete subset of a Hausdorff space \mathcal{S}^\wedge such that every function $f(s)$ of s in \mathcal{S} has a unique continuous extension as a function $f(s)$ of s in \mathcal{S}^\wedge and such that every homomorphism of $\mathcal{C}(\mathcal{S})$ onto the complex numbers, which always takes the conjugate function into the conjugate number, is of the form f into $f(s)$ for a unique element s of \mathcal{S}^\wedge . The topology of pointwise convergence on \mathcal{S}^\wedge is a locally hyperconvex topology of $\mathcal{C}(\mathcal{S})$. The hyperconvex closed graph theorem is applied to show that the topology is identical with the topology of pointwise convergence on \mathcal{S} .

The Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is a locally hyperconvex space \mathcal{P} in the topology of pointwise convergence on \mathcal{S} . It will be shown that \mathcal{P} is a primitive locally hyperconvex space. A computation is made of the hyperconvex subsets of the conjugate dual space \mathcal{P}^* of \mathcal{P} which are weakly compact and centered at the origin. Such a set B is the weakly closed convex span of its hyperlinear elements. A nonzero hyperlinear element b of B is supported at an element s of \mathcal{S} . A complex number λ exists such that the identity

$$b^- f = \lambda f(s)$$

holds for every element f of $\mathcal{C}(\mathcal{S})$. Since the set of hyperlinear elements of B is weakly compact, the set of products $b^- f$ with b a hyperlinear element of B is a compact subset of the complex plane for every function $f(s)$ of s in \mathcal{S} . It follows that only a finite number of elements s of \mathcal{S} support nonzero hyperlinear elements of B . Since the set B is then the convex span of a finite set of hyperlinear elements of \mathcal{P}^* , the hyperdisk of elements a of \mathcal{P} such that the real part of a^- maps B into the interval $(-1, 1)$ is an open subset. Every element of \mathcal{P}^* is a finite linear combination of hyperlinear elements. A similar argument shows that every bounded subset of \mathcal{P}^* is contained in the convex span of a finite set of hyperlinear elements of \mathcal{P}^* . It follows that every weakly closed and bounded subset of \mathcal{P}^* is weakly compact.

The Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is a locally hyperconvex space \mathcal{Q} in the topology of pointwise convergence on \mathcal{S}^\wedge . It will be shown that the space \mathcal{Q} has the hyperconvex Krein–Šmulyan property. If \mathcal{S}' is a finite subset of \mathcal{S}^\wedge , then the space $\mathcal{C}(\mathcal{S}')$ of all functions on \mathcal{S}' has a unique locally hyperconvex topology. A homomorphism of $\mathcal{C}(\mathcal{S})$ onto $\mathcal{C}(\mathcal{S}')$, which commutes with complex conjugation, is defined by restricting a function $f(s)$ of s in \mathcal{S}^\wedge to the function $f(s)$ of s in \mathcal{S}' . The topology of \mathcal{Q} is the weakest topology with respect to which the homomorphism is continuous for every finite subset \mathcal{S}' of \mathcal{S}^\wedge . An element of the conjugate dual space of $\mathcal{C}(\mathcal{S}')$ is identified with an element b of \mathcal{Q}^* such that b^- annihilates every function $f(s)$ of s in \mathcal{S} which vanishes on \mathcal{S}' . The space \mathcal{Q}^* is the union of the conjugate dual spaces of the Weierstrass algebras $\mathcal{C}(\mathcal{S}')$ taken over all finite subsets \mathcal{S}' of \mathcal{S}^\wedge . A convex subset of \mathcal{Q}^* is weakly closed if, and only if, it has a closed intersection with the conjugate dual space of $\mathcal{C}(\mathcal{S}')$ for every finite subset \mathcal{S}' of \mathcal{S}^\wedge . Since the Weierstrass

algebra $\mathcal{C}(\mathcal{S}')$ has the hyperconvex Krein–Šmulyan property for every finite subset \mathcal{S}' of \mathcal{S}^\wedge , a convex subset of \mathcal{Q}^* is weakly closed if it has a weakly compact intersection with every weakly compact hyperconvex set.

The inclusion of \mathcal{P} in \mathcal{Q} is a hyperlinear transformation which has a closed graph since it has a continuous inverse. Since \mathcal{P} is a primitive locally hyperconvex space and since \mathcal{Q} has the hyperconvex Krein–Šmulyan property, the inclusion of \mathcal{P} in \mathcal{Q} is continuous. Since the spaces \mathcal{P} and \mathcal{Q} then have the same topology, the completion \mathcal{S}^\wedge of \mathcal{S} is equal to \mathcal{S} . It follows that the hyperdisk topology of \mathcal{P} and \mathcal{Q} is the strongest locally hyperconvex topology. Since the topology of \mathcal{P} and \mathcal{Q} is also the weakest locally hyperconvex topology, it is the only locally hyperconvex topology. It has been shown that the topology of \mathcal{P} and \mathcal{Q} is a primitive locally hyperconvex topology which has the Krein–Šmulyan property.

This completes the proof of the theorem.

Cardinal numbers are constructed by a theorem of Georg Cantor, which states that no transformation maps a set onto the class of its subsets [4]. If a transformation T maps a set \mathcal{S} into the subsets of \mathcal{S} , then a subset \mathcal{S}_∞ of \mathcal{S} is constructed which does not belong to the range of T . The set \mathcal{S}_∞ is the set of elements s of \mathcal{S} for which no elements s_n of \mathcal{S} can be constructed for every nonnegative integer n so that s_0 is equal to s and so that s_n belongs to Ts_{n-1} when n is positive. An element s of \mathcal{S} then belongs to \mathcal{S}_∞ if, and only if, Ts is a subset of \mathcal{S}_∞ . It follows that \mathcal{S}_∞ is never equal to Ts for an element s of \mathcal{S} .

If γ is a given cardinal number, then a transformation T may map a set \mathcal{S} onto the class of its subsets of cardinality less than γ . A continuum of order γ is a set of least cardinality which has the same cardinality as the class of its subsets of cardinality less than γ . A parametrization of a continuum \mathcal{S} of order γ is an injective transformation J of \mathcal{S} onto the class of its subsets of cardinality less than γ such that no elements s_n of \mathcal{S} can be chosen for every nonnegative integer n so that s_n belongs to Js_{n-1} when n is positive. A continuum \mathcal{S} of order γ admits a parametrization since an injective transformation T exists of \mathcal{S} onto the class of its subsets of cardinality less than γ . Since \mathcal{S}_∞ is then a continuum of order γ , it has the same cardinality as \mathcal{S} . The restriction of T to \mathcal{S}_∞ is a parametrization of \mathcal{S}_∞ . If W is an injective transformation of \mathcal{S} onto \mathcal{S}_∞ , then a parametrization J of \mathcal{S} is defined so that Ja is always the set of elements b such that Wb belongs to TWa .

A parametrization J of a continuum \mathcal{S} of order γ is essentially unique. If an injective transformation T maps \mathcal{S} onto the class of its subsets of cardinality less than γ , then an injective transformation W of \mathcal{S} onto \mathcal{S}_∞ exists such that Ja is always the set of elements b such that Wb belongs to TWa . The construction of W is an application of the Zorn lemma. Consider the class \mathcal{C} of injective transformations W with domain contained in \mathcal{S} and with range contained in \mathcal{S}_∞ such that every element of Ja belongs to the domain of W whenever a belongs to the domain of W and such that Ja is always the set of elements b of \mathcal{S} such that Wb belongs to TWa . A transformation U of class \mathcal{C} is considered less than or equal to a transformation V of class \mathcal{C} if the graph of U is contained in the graph of V . A well-ordered subclass of \mathcal{C} always has an upper bound in \mathcal{C} . The graph of the upper bound is the union of the graphs of the members of the subclass. A maximal member of the class \mathcal{C} is an injective transformation W of \mathcal{S} onto \mathcal{S}_∞ such that Ja is always the set

of elements b of \mathcal{S} such that Wb belongs to TWa .

The empty set is a continuum whose order is the least cardinal number. A set which has only one element is a continuum whose order is one. A countably infinite set is a continuum whose order is the least infinite cardinal number. A construction is made of continua of greater order.

If \mathcal{S} is an infinite set and if γ is the least cardinal number greater than the cardinality of \mathcal{S} , then the class \mathcal{C} of all subsets of \mathcal{S} is a set which has the same cardinality as the class of its subsets of cardinality less than γ . For the cardinality of the class of all subsets of \mathcal{C} of cardinality less than γ is less than or equal to the cardinality of all transformations of \mathcal{S} into the set of all functions defined on \mathcal{S} with values in a set with two elements. The cardinality of the class of all subsets of \mathcal{C} of cardinality less than γ is then less than or equal to the cardinality of the set of all functions defined on the Cartesian product $\mathcal{S} \times \mathcal{S}$ with values in a set with two elements. Since \mathcal{S} is an infinite set, it has the same cardinality as the Cartesian product $\mathcal{S} \times \mathcal{S}$. The cardinality of the class of all subsets of \mathcal{C} of cardinality less than γ is then less than or equal to the cardinality of \mathcal{C} .

A continuum of order γ exists. A partial ordering of a continuum is defined by a parametrization J of the continuum. The inequality $b < a$ for elements a and b of the continuum means that elements s_0, \dots, s_r of the continuum can be defined for some positive integer r so that s_0 is equal to a , so that s_n belongs to $J s_{n-1}$ when n is positive, and so that $J s_r$ is equal to b . If a cardinal number α is less than γ , then a continuum of order α is obtained as the set of elements a of the continuum of order γ such that the cardinality of Jb is less than α whenever b is less than or equal to a . The restriction of J to the continuum of order α is a parametrization of the continuum of order α .

A continuum of order γ exists for every cardinal number γ . The continuum is said to be regular if no set of cardinality γ is the union of a class of cardinality less than γ whose members are sets of cardinality less than γ . An equivalent condition is given using a parametrization J of the continuum. The cardinality of the set of elements of the continuum which are less than any given element s is less than γ . For otherwise a sequence of elements s_n of the continuum can be defined for every nonnegative integer n so that s_0 is equal to s , so that s_n belongs to $J s_{n-1}$ when n is positive, and so that the set of elements of the continuum which are less than s_n does not have cardinality less than γ . The inductive construction of s_n from s_{n-1} is made possible by regularity since the set of elements of the continuum which are less than an element a of the continuum is the union over the elements b of $J a$ of the set of elements of the continuum which are less than b . Since the cardinality of $J a$ is less than γ , the cardinality of the set of elements of the continuum which are less than b is at least γ for some element b of $J a$ if the cardinality of the set of elements of the continuum which are less than a is at least γ . A contradiction of the properties of a parametrization results.

An example of a regular continuum of order γ is obtained when γ is the least cardinal number which is greater than the cardinality of some infinite set. The cardinality of the continuum is then equal to the cardinality of the class of all subsets of the set. A topology is defined on a parametrized continuum.

A nonempty continuum \mathcal{S} acquires the structure of a commutative ring as a result of a parametrization J . The sum $a + b$ of elements a and b of the continuum is the element c of the continuum such that Jc is the set of elements of the union of Ja and Jb which do not belong to the intersection of Ja and Jb . The product ab of elements a and b of the continuum is the element c of the continuum such that Jc is the intersection of Ja and Jb . The origin of the continuum is defined as the element c of the continuum such that Jc is the empty set. The origin of the continuum is the zero element of the ring. The sum of every element of the ring with itself is equal to zero. The product of every element of the ring with zero is equal to zero. The ring contains no unit.

A generalization of convexity applies in a continuum with parametrization J . A subset of the continuum is said to be paraconvex if it contains an element c of the continuum whenever it contains elements a and b of the continuum such that Jc contains the intersection of Ja and Jb and is contained in the union of Ja and Jb . Examples of paraconvex sets whose complement is paraconvex are defined using an element c of the continuum. The set of elements s of the continuum such that c belongs to Jc is paraconvex. The set of elements s of the continuum such that c does not belong to Jc is paraconvex.

If C is a subset of a continuum with parametrization J , then the set of elements of the continuum which parametrize subsets of C is a subring. A homomorphism of the continuum onto the subring which leaves every element of the subring fixed is defined by taking a into b whenever Jb is the intersection of Ja with C . The inverse image under the homomorphism of every paraconvex subset is paraconvex.

A Hausdorff topology for a continuum with parametrization J is said to be locally paraconvex if addition is continuous as a transformation of the Cartesian product of the continuum with itself into the continuum and if every open set is a union of paraconvex open sets. The discrete topology of the continuum is a locally paraconvex topology. A paraconvex set which is closed for every locally paraconvex topology is defined by an element s of the continuum and consists of the elements of the continuum which parametrize sets containing s . If C is a finite subset of the continuum, the canonical homomorphism of the continuum onto the subring of elements which parametrize subsets of C is continuous for every locally paraconvex topology when the subring is considered in the discrete topology. The weakest topology of the continuum with respect to which every such homomorphism is continuous is the weakest locally paraconvex topology of the continuum. If C is a subset of the continuum, the set of elements of the continuum which parametrize subset of C is compact with respect to the weakest locally paraconvex topology if, and only if, the cardinality of C is less than the order of the continuum.

The paraconvex Hahn–Banach theorem applies to a continuum with parametrization J when the continuum is considered in some locally paraconvex topology. A paradisk is a nonempty paraconvex subset of the continuum which is disjoint from the closure of every disjoint paraconvex set.

Theorem 10. *If A is a paradisk and if B is a nonempty disjoint paraconvex set, then a maximal paraconvex set which contains B and is disjoint from A is closed and has paraconvex complement.*

Proof of Theorem 10. Since A is a paradisk, the closure of B is disjoint from A . It will be shown that B is closed by showing that the closure of B is paraconvex. It needs to be shown that an element of the paraconvex span of u and v belongs to the closure of B if u and v belong to the closure of B . It can be assumed that the element of the paraconvex span of u and v is the origin. Then Ju and Jv are disjoint sets. It needs to be shown that every paraconvex open set Δ which contains the origin has a nonempty intersection with B .

A paraconvex open set which contains u is the set of elements of the continuum which parametrizes a subset of the union of Ju and Ja for some element a of Δ . Since u belongs to the closure of B , some subset of the union of Ju and Ja is parametrized by an element of B for some element a of Δ .

A paraconvex open set which contains v is the set of elements of the continuum which parametrize a subset of the union of Jv and Jb for some element b of Δ . Since v belongs to the closure of B , some subset of the union of Jv and Jb is parametrized by an element of B for some element b of Δ . Since Ju and Jv are disjoint sets and since B is paraconvex, a subset of the union of Ja and Jb is parametrized by an element of B . This completes the verification that Δ has a nonempty intersection with B .

It will be shown that the complement of B is paraconvex. If an element s of the continuum does not belong to B , a paraconvex set $B(s)$ is defined as the union of the paraconvex spans of s with elements of B . The verification that $B(s)$ is paraconvex reduces to a verification for pairs of elements a and b of B . The union of the paraconvex spans of s with elements of the paraconvex span of a and b needs to be shown paraconvex. If u belongs to the paraconvex span of s and a , then Ju contains the intersection of Js and Ja and is contained in the union of Js and Ja . If v belongs to the paraconvex span of s and b , then Jv contains the intersection of Js and Jb and is contained in the union of Js and Jb . If w belongs to the paraconvex span of u and v , then Jw contains the intersection of Js with the intersection of Ja and Jb and is contained in the union of Js with the union of Ja and Jb . If c is the element of the paraconvex span of a and b which parametrizes the union of Ja and Jb , then w belongs to the paraconvex span of s and c .

It remains to verify that the complement of B is paraconvex. If u belongs to the complement of B , an element a of B exists such that the paraconvex span of u and a contains an element a' of A . If v belongs to the complement of B , an element b of B exists such that the paraconvex span of v and b contains an element b' of A . It will be shown that every element of the paraconvex span of u and v belongs to the complement of B . It can be assumed that the element of the paraconvex span of u and v is the origin. Then Ju and Jv are disjoint sets. Since B is paraconvex, an element c of B exists which parametrizes the union of Ja and Jb . Since A is paraconvex, an element c' of A exists which parametrizes the intersection of Ja' and Jb' . Since Ja' is contained in the union of Ja and Ju , since Jb' is contained in the union of Jb and Jv , and since Ju and Jv are disjoint, Jc' is contained in Jc . Since c' belongs to the paraconvex span of c and the origin, the origin does not belong to B .

This completes the proof of the theorem.

A continuum \mathcal{S} with parametrization J is contained in a continuum whose order is the least cardinal number greater than the cardinality of \mathcal{S} and whose parametrization extends the parametrization of \mathcal{S} . The weakest locally paraconvex topology of \mathcal{S} is the subspace topology of the weakest locally paraconvex topology of the larger continuum. The completion of \mathcal{S} in the weakest locally paraconvex topology is the set of elements of the larger continuum which parametrize subsets of \mathcal{S} . The completion of \mathcal{S} in the weakest locally paraconvex topology is compact. The completion of the continuum in a locally paraconvex topology is a subring of its completion in the weakest locally paraconvex topology. The discrete topology of the continuum is its strongest locally paraconvex topology. The continuum is complete in its strongest locally paraconvex topology.

When a parametrized continuum is considered in a locally paraconvex topology, a locally paraconvex topology exists whose open sets are the unions of paradisks. For the existence of the paradisk topology it needs to be verified that the intersection of two paradisks is a paradisk if it is not empty. A preliminary verification is made. If A is a paradisk and if B is a paraconvex set, then the intersection of A with the closure of B is contained in the closure of the intersection of A and B . For if an element of A does not belong to the closure of the intersection of A and B , it belongs to a paraconvex open set C which is disjoint from the intersection of A and B . Since C is an open set, the intersection of C with the closure of B is contained in the closure of the intersection of C and B . Since the intersection of B and C is a paraconvex set which is disjoint from A , the closure of the intersection of B and C is disjoint from A . Since the intersection of C with the closure of B is disjoint from A , the intersection of A and C is disjoint from the closure of B .

If A and B are paradisks whose intersection is nonempty, and if C is a paraconvex set which is disjoint from the intersection of A and B , then the intersection of B and C is a paraconvex set which is disjoint from A . Since A is a paradisk, A is disjoint from the closure of the intersection of B and C . Since B is a paradisk, the intersection of B with the closure of C is contained in the closure of the intersection of B and C . Since the intersection of A and B is disjoint from the closure of C , the intersection of A and B is a paradisk.

A paraconvex set is closed for the given topology if, and only if, it is closed for the paradisk topology. Since every open set for the given topology is an open set for the paradisk topology, every closed set for the given topology is a closed set for the paradisk topology. If a paraconvex set B is closed for the paradisk topology and if an element a of the continuum does not belong to B , then a belongs to a paradisk A which is disjoint from B . Since A is disjoint from the closure of B for the given topology, a does not belong to the closure of B for the given topology. Neighborhoods of the origin for the paradisk topology induced by the parametrization are determined by subsets of the continuum whose cardinality is less than the order of the continuum. The neighborhood determined by the set C is the set of elements s of the continuum such that J_s is disjoint from C .

Uniqueness of the locally hyperconvex topology for the Weierstrass algebra of all functions on a discrete space removes a hypothesis in a theorem of Shiota [10].

Theorem 11. *The strongest locally hyperconvex topology of the Weierstrass algebra $\mathcal{C}(\mathcal{S})$*

of continuous functions on a complete uniform space \mathcal{S} is the hyperdisk topology of the topology of pointwise convergence on the space.

Proof of Theorem 11. A defining pseudo-metric for the uniform space is a function $\rho(a, b)$ of elements a and b of the space with nonnegative values which satisfies the identity

$$\rho(a, b) = \rho(b, a)$$

for all elements a and b of the space and which satisfies the inequality

$$\rho(a, c) \leq \rho(a, b) + \rho(b, c)$$

for all elements a, b , and c of the space. Continuity of a function $f(s)$ of s in \mathcal{S} means that for every element a of \mathcal{S} and every positive number ϵ a defining pseudo-metric ρ exists such that the inequality

$$|f(a) - f(b)| < \epsilon$$

holds whenever the inequality

$$\rho(a, b) < 1$$

is satisfied. The space of all continuous functions on a uniform space \mathcal{S} forms a Weierstrass algebra $\mathcal{C}(\mathcal{S})$. The space \mathcal{S} is a subspace of a Hausdorff space \mathcal{S}^\wedge such that every continuous function $f(s)$ of s in \mathcal{S} has a unique continuous extension as a function $f(s)$ of s in \mathcal{S}^\wedge and such that every homomorphism of the Weierstrass algebra $\mathcal{C}(\mathcal{S})$ onto the complex numbers is of the form f into $f(s)$ for a unique element s of \mathcal{S}^\wedge . It will be shown that every defining pseudo-metric $\rho(a, b)$ of a and b in \mathcal{S} admits an extension as a pseudo-metric $\rho(a, b)$ of a and b in \mathcal{S}^\wedge such that every element of the Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is continuous on the resulting uniform space \mathcal{S}^\wedge and such that \mathcal{S} is dense in \mathcal{S}^\wedge . It is sufficient to show that for every defining pseudo-metric ρ and for every element s of \mathcal{S}^\wedge an element a of \mathcal{S} exists such that the inequality

$$\rho(a, s) < 1$$

is satisfied.

A well-ordering of the space \mathcal{S} is assumed for the construction of functions from a given pseudo-metric ρ . An element b of the space is said to be generated by an element a of the space if a is the least element of the space which satisfies the inequality

$$\rho(a, b) < 1.$$

The inequality

$$\rho(a, b) \leq 1 - 2^{-n}$$

then holds when n is sufficiently large. The inequality

$$\rho(a', b) \geq 1$$

holds when a' is less than a . If an element s of the space satisfies the inequality

$$\rho(b, s) < 2^{-n-1},$$

then the inequality

$$\rho(a, s) < 1 - 2^{-n-1}$$

is satisfied and the inequality

$$\rho(a', s) > 1 - 2^{-n-1}$$

holds when a' is less than a . A function

$$\delta_n(a, s') = \inf \rho(s, s')$$

of s' is defined as a greatest lower bound taken over the elements s such that either the inequality

$$\rho(a, s) < 1 - 2^{-n-1}$$

is violated or the inequality

$$\rho(a', s) > 1 - 2^{-n-1}$$

is violated for some element a' less than a . The inequality

$$\delta_n(a, b) \geq 2^{-n-1}$$

then holds when b is generated by a and the inequality

$$\rho(a, b) \leq 1 - 2^{-n}$$

is satisfied. When a and a' are distinct generators, the set of elements s such that $\delta_n(a, s)$ is positive is disjoint from the set of elements s such that $\delta_n(a', s)$ is positive.

For every positive integer n the sum

$$\sum \delta_n(a, s)$$

taken over all generators a is a continuous function of s in \mathcal{S} which has a unique continuous extension as a function of s in \mathcal{S}^\wedge . The sum has a positive limit in the limit of large n for every element s of \mathcal{S}^\wedge . If $k(a)$ is a function of generators a , then the sum

$$\sum k(a)\delta_n(a, s)$$

taken over all generators a is a continuous function of s in \mathcal{S} which has a unique continuous extension as a function of s in \mathcal{S}^\wedge . The Weierstrass algebra of all functions on the discrete space of generators admits a unique locally hyperconvex topology. Since the taking of function values at an element s of \mathcal{S}^\wedge is a hyperlinear functional on the algebra, it determines a generator a such that

$$\delta_n(a, s) > 0$$

and hence such that

$$\rho(a, s) < 1.$$

This completes the proof of the theorem.

A construction of invariant subspaces is an application of the hyperconvex Hahn–Banach theorem. A Banach space is a complete locally convex space whose topology is determined by a distinguished disk, called the unit disk. The unit disk is centered at the origin and is invariant under multiplication by numbers of absolute value one. Every neighborhood of the origin contains the image of the unit disk under multiplication by some positive number. Every Banach space is a primitive locally convex space which has the Krein–Šmulyan property. A Banach space \mathcal{H} is said to be reflexive if its conjugate dual space \mathcal{H}^* is a Banach space in the disk topology constructed from the weak topology induced by \mathcal{H} . The weak closure of the unit disk of \mathcal{H}^* is then weakly compact. The unit disk of \mathcal{H}^* is chosen so that its weak closure is the set of elements b of \mathcal{H}^* such that the real part of b^- maps the unit disk of \mathcal{H} into the interval $(-1, 1)$. Then \mathcal{H} is the conjugate dual space of \mathcal{H}^* . The topology of \mathcal{H} is the disk topology of the weak topology induced by \mathcal{H}^* .

The conjugate dual space \mathcal{H}^* of \mathcal{H} is treated as a Hausdorff space in the weak topology induced by \mathcal{H} . A corresponding Weierstrass algebra $\mathcal{C}(\mathcal{S})$ is the space of all continuous functions on \mathcal{S} . A module $\mathcal{H}^*(\mathcal{S})$ over the Weierstrass algebra is the set of all functions $f(s)$ of s in \mathcal{S} with values in \mathcal{H}^* such that the function $c^- f(s)$ of s in \mathcal{S} belongs to $\mathcal{C}(\mathcal{S})$ for every element c of \mathcal{H} . The adjoint of a continuous linear transformation of \mathcal{H} into itself is an example of an element of $\mathcal{H}^*(\mathcal{S})$ which is characterized by linearity as a transformation of \mathcal{H}^* into itself.

A continuous linear transformation ab^- of the Banach space \mathcal{H} into itself is defined by

$$(ab^-)c = a(b^-c)$$

for every element c of \mathcal{H} when a is an element of \mathcal{H} and b is an element of \mathcal{H}^* . The adjoint is the transformation ba^- of \mathcal{H}^* into itself defined by

$$(ba^-)c = b(a^-c)$$

for every element c of \mathcal{H}^* .

Theorem 12. *If the identity transformation does not belong to the closure of an algebra of continuous linear transformations of a reflexive Banach space \mathcal{H} into itself in the weak topology induced by the trace class, then a nontrivial subspace of the Banach space, which is a common invariant subspace for the elements of the algebra, and a nontrivial subspace of the conjugate dual space \mathcal{H}^* , which is a common invariant subspace for the adjoints of elements of the algebra, exist which are orthogonal to each other with respect to the pairing between \mathcal{H} and \mathcal{H}^* .*

REFERENCES

1. L. de Branges, *The Stone-Weierstrass theorem*, Proceedings of the American Mathematical Society **10** (1959), 822-824.

2. ———, *Vectorial topology*, Journal of Mathematical Analysis and Applications **69** (1979), 443–454.
3. ———, *A construction of invariant subspaces*, Mathematische Nachrichten **163** (1993), 163–175.
4. ———, *The Cantor construction*, Journal of Mathematical Analysis and Applications **76** (1980), 623–630.
5. ———, *Vector lattice topology*, Journal of Mathematical Analysis and Applications **77** (1980), 451–464..
6. ———, *Consequences of the Cantor construction*, Journal of Mathematical Analysis and Applications **98** (1984), 198–210.
7. ———, *Nevanlinna factorization and the Bieberbach conjecture*, preprint (2004).
8. ———, *Riemann zeta functions*, preprint (2004).
9. S. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations and Operator Theory **1** (1978), 310–333.
10. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.
11. J. Glimm, *A Stone–Weierstrass theorem for C^* -algebras*, Annals of Mathematics **72** (1960), 216–243.
12. F. Hausdorff, *Grundzüge der Mengenlehre*, Veit Verlag, Leipzig, 1914.

Department of Mathematics
Purdue University
Lafayette IN 47907-2067