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Operational methods, fractional operators and special polynomials

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Abstract

We use methods of operational nature to deal with families of partial differential equations of evolution type to treat problems involving fractional differential operators. We also discuss the properties of families of special polynomials or special functions (like the Riemann ζ function), naturally associated with the proposed formalism.

1. Introduction

It is well known that Hermite and Laguerre polynomials can be defined through the operational identities [1]

$$\begin{aligned} H_n(x,y) &= \exp\left(y\frac{\partial^2}{\partial x^2}\right)[x^n], \\ L_n(x,y) &= \exp\left(-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)\left[\frac{(-1)^n}{n!}x^n\right], \end{aligned} \tag{1}$$

which have been shown to play an important role in applications [2,3].

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The polynomials

$$H_n(x,y) = n!\sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!},$$
$$L_n(x,y) = n!\sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(n-r)!(r!)^2} \tag{2}$$

are slightly generalised forms of Hermite and Laguerre polynomials and are linked to the ordinary case by

$$H_n(x,y) = (-\mathrm{i})^n y^{n/2} H_n\left(\frac{\mathrm{i}x}{2\sqrt{y}}\right),$$
$$L_n(x,y) = y^n L_n\left(\frac{x}{y}\right). \tag{3}$$

Let us now concentrate on the first of Eq. (1), in particular on the exponential operator, which, according to a standard procedure [4], can be written in the following integral form:

$$\exp\left(y\frac{\partial^2}{\partial x^2}\right) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left(-\xi^2 + 2\sqrt{y}\xi\frac{\partial}{\partial x}\right) \mathrm{d}\xi. \tag{4}$$

The use of the above identity and the fact that

$$\exp\left(\lambda\frac{\partial}{\partial x}\right) f(x) = f(x+\lambda) \tag{5}$$

allows to conclude that the polynomials $H_n(x,y)$ satisfy the integral representation

$$H_n(x,y) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp(-\xi^2)[x+2\sqrt{y}\xi]^n \,\mathrm{d}\xi. \tag{6}$$

The above example shows that

(a) it is possible to define polynomials by means of an operational identity,
(b) such operational identity can in turn be used to derive an integral representation.

Methods employing the combined use of exponential operators and integral transforms provide a powerful tool for the solution of P.D.E. of evolution type. An appropriate example follows from the equation associated with the Black–Scholes financial model [5],

$$\frac{\partial}{\partial \tau}A = S^2\frac{\partial^2}{\partial S^2}A + \lambda S\frac{\partial}{\partial S}A - \lambda A, \quad A(S,0) = f(S), \tag{7}$$

which can be rewritten in the form

$$\frac{\partial}{\partial\tau}A = \left(S\frac{\partial}{\partial S} + \frac{\lambda - 1}{2}\right)^2 A - \left(\frac{\lambda + 1}{2}\right)^2 A \tag{8}$$

and which admits the formal solution

$$A(S,\tau) = \exp\left\{\left[\left(S\frac{\partial}{\partial S} + \frac{\lambda - 1}{2}\right)^2 - \left(\frac{\lambda + 1}{2}\right)^2\right]\right\} f(S). \tag{9}$$

According to the identity [4]

$$\exp\left(\lambda x\frac{\partial}{\partial x}\right) f(x) = f(\mathrm{e}^{\lambda}x) \tag{10}$$

and to the identity (4), which can be extended to any exponential of a quadratic operator, we obtain the solution of the Black–Scholes equation in terms of the integral transform

$$A(S,\tau) = \frac{\exp\left(-\left(\frac{\lambda+1}{2}\right)^2\tau\right)}{\sqrt{\pi}} \times \int_{-\infty}^{\infty} \exp[-\xi^2 + (\lambda - 1)\xi\sqrt{\tau}] f(\exp(2\xi\sqrt{\tau})S)\,\mathrm{d}\xi. \tag{11}$$

This last result shows that methods employing operational techniques can be used in a fairly wide context and allow noticeable flexibility.

In this paper we will introduce new families of special polynomials starting from a suitable operational definition. We will show that the point of view we develop is useful in different contexts including the theory of fractional derivatives.

2. Fractional operators, integral transforms and new family of special polynomials

It is well known that one of the starting point of the theory of fractional operators, i.e. operators raised to a fractional power, is the identity [6, p. 218]:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} \exp(-at)t^{\nu-1}\,\mathrm{d}t. \tag{12}$$

It is therefore evident that

$$\left[\alpha - \frac{\partial^2}{\partial x^2}\right]^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} \exp(-\alpha t)t^{\nu-1}\exp\left(t\frac{\partial^2}{\partial x^2}\right) f(x)\,\mathrm{d}t. \tag{13}$$

In the case in which $f(x) = \exp(-x^2)$, the use of Eq. (4) yields

$$\left[\alpha - \frac{\partial^2}{\partial x^2}\right]^{-v} \exp(-x^2) = \frac{1}{\Gamma(v)} \int_0^\infty \frac{\exp(-\alpha t) t^{v-1}}{\sqrt{1+4t}} \exp\left(-\frac{x^2}{1+4t}\right) \mathrm{d}t. \tag{14}$$

Let us now consider the simpler case $f(x) = x^n$. According to Eq. (13) and Eq. (1), we find that

$$\left(1 - y\frac{\partial^2}{\partial x^2}\right)^{-v} x^n = \frac{1}{\Gamma(v)} \int_0^\infty \exp(-t) t^{v-1} H_n(x, yt)\, \mathrm{d}t. \tag{15}$$

The integral on the r.h.s. of Eq. (15) can be written in terms of a new family of special polynomials defined below

$${}_vH_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{(v)_r y^r x^{n-2r}}{r!(n-2r)!}, \tag{16}$$

where $(v)_r$ is the Pochhammer symbol. By introducing the notation

$$(vy)^r = (v)_r y^r \tag{17}$$

and the operators

$$\hat{Y}_v \quad \text{and} \quad \hat{\mathscr{D}}_{y,v} \tag{18}$$

such that

$$\begin{aligned} \hat{Y}_v[(vy)^r] &= (vy)^{r+1}, \\ \hat{\mathscr{D}}_{y,v}[(vy)^r] &= r(vy)^{r-1}. \end{aligned} \tag{19}$$

It is not difficult to realise that the polynomials (16) are an umbral image [7] of the ordinary Hermite polynomials and that they satisfy the recurrences

$$\begin{aligned} \frac{\partial}{\partial x}({}_vH_n(x,y)) &= n\,{}_vH_{n-1}(x,y), \\ \left[x + 2\hat{Y}_v \frac{\partial}{\partial x}\right] {}_vH_n(x,y) &= {}_vH_{n+1}(x,y) \end{aligned} \tag{20}$$

and the following "differential" equations as well:

$$\left[2\hat{Y}_v \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - n\right] {}_vH_n(x,y) = 0 \tag{21}$$

and

$$\hat{\mathscr{D}}_{y,v} H_n(x,y) = \frac{\partial^2}{\partial x^2}\, {}_vH_n(x,y). \tag{22}$$

It is therefore worth noting that the ${}_vH_n(x,y)$ can be derived from the operational rule

$$ {}_{\nu}H_n(x,y) = \exp\left(\hat{Y}_{\nu}\frac{\partial^2}{\partial x^2}\right)[x^n]. \tag{23} $$

Analogous results can be obtained for a new family of polynomials, which can be viewed as an umbral image of the Laguerre family.

We consider indeed the operational definition

$$ {}_{\nu}L_n(x,y) = \left(1 + y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)^{-\nu}\left[\frac{(-1)^n x^n}{n!}\right], \tag{24} $$

which leads to

$$ {}_{\nu}L_n(x,y) = \frac{1}{\Gamma(\nu)}\int_0^{\infty}\exp(-t)t^{\nu-1}L_n(x,yt)\,\mathrm{d}t = n!\sum_{r=0}^{n}\frac{(-1)^r(\nu)_{n-r}y^{n-r}x^r}{(r!)^2(n-r)!}. \tag{25} $$

The use of the same operators as before allows the derivation of the following properties:

$$ \hat{\mathscr{D}}_{y,\nu}\,{}_{\nu}L_n(x,y) = -\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\,{}_{\nu}L_n(x,y) \tag{26} $$

and

$$ \left[\hat{Y}_{\nu}x\frac{\partial^2}{\partial x^2} + (\hat{Y}_{\nu} - x)\frac{\partial}{\partial x} + n\right]{}_{\nu}L_n(x,y) = 0. \tag{27} $$

It is now evident that the above procedure can be extended to any family of special polynomials. To give an example, we note that

$$ \left[1 - \sum_{s=2}^{m}x_s\frac{\partial^s}{\partial x_1^s}\right]^{-\nu}x^n = \frac{1}{\Gamma(\nu)}\int_0^{\infty}\exp(-t)t^{\nu-1}H_n^{(m)}(x_1, x_2t, \ldots, x_mt)\,\mathrm{d}t, \tag{28} $$

where [1] $(\{x\}_1^m = x_1, \ldots, x_m)$

$$ H_n^{(m)}(\{x\}_1^m) = n!\sum_{r=0}^{[\frac{n}{m}]}\frac{x_m^r H_{n-mr}^{(m-1)}(\{x\}_1^{m-1})}{r!(n-mr)!} \tag{29} $$

leads to the definition of the polynomial ${}_{\nu}H_n^{(m)}(\{x\}_1^m)$ with generating function

$$ \sum_{n=0}^{\infty}\frac{\xi^n}{n!}\left({}_{\nu}H_n^{(m)}(\{x\}_1^m)\right) = \frac{\exp(x_1\xi)}{\left[1 - \sum_{s=2}^{m}x_s\xi^s\right]^{\nu}}. \tag{30} $$

A particularly interesting case arises when the highest order of the derivative appearing in the fractional operator is 1, namely, when

$$ \left[1 - y\frac{\partial}{\partial x}\right]^{\nu}x^n = \frac{1}{\Gamma(\nu)}\int_0^{\infty}\exp(-t)t^{\nu-1}(x+yt)^n\,\mathrm{d}t. \tag{31} $$

It is evident that the corresponding polynomial

$$ {}_{v}H_{n}^{(1)}(x,y) = n!\sum_{r=0}^{n}\frac{(vy)^{r}x^{n-r}}{r!(n-r)!} \tag{32} $$

is an umbrual image of the ordinary binomials.

However, by setting $v = n+1$, $y = z/2$, we can make the identification

$$ {}_{n+1}H_{n}^{(1)}\left(1,\frac{z}{2}\right) = y_{n}(z), \tag{33} $$

where $y_n(z)$ are the simple Bessel polynomials, originally introduced by Krall and Frink (see [6, p. 419]; see also [8]).

The above example provides an indication of the implications offered by the method we have proposed. In the next concluding section, we will discuss further examples aimed at proving that the combined use of operational rules and integral representations may provide unsuspected links between apparently disconnected fields.

3. Concluding remarks

One of the advantages offered by the use of the integral representation in dealing with differential operators is the possibility of giving a meaning to apparently meaningless operations. This is indeed the case of the Riemann–Liouville definition of fractional derivative (see [9]; see also [6, Chapter 5]).

By taking advantage from the definition of the Euler Γ function, that is, Eq. (12) with $a = 1$, we can easily show that

$$ \left[\frac{\mathrm{d}}{\mathrm{d}x}\right]!f(x) = \int_{0}^{\infty}\exp(-t)t^{\mathrm{d}/\mathrm{d}x}f(x) = \int_{0}^{\infty}\exp(-t)f(x+\ell nt)\,\mathrm{d}t \tag{34} $$

and also (see Eq. (10))

$$ \left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]!f(x) = \int_{0}^{\infty}\exp(-t)f(xt)\,\mathrm{d}t. \tag{35} $$

Furthermore, the use of the identity (12) allows to conclude that

$$ \left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]^{-v}f(x) = \frac{1}{\Gamma(v)}\int_{0}^{\infty}t^{v-1}f(\exp(-t)x)\,\mathrm{d}t. \tag{36} $$

If we set $f(x) = x/(1-x)$ in Eq. (36) we find that

$$ \left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]^{-v}\left(\frac{x}{1-x}\right) = \frac{1}{\Gamma(v)}\int_{0}^{\infty}\frac{t^{v-1}x}{\mathrm{e}^{t}-x}\,\mathrm{d}t = \sum_{n=1}^{\infty}\frac{x^{n}}{n^{v}} = \zeta(x,v), \quad |x|<1. \tag{37} $$

According to the previous result, it becomes clear that fractional forms of the operator $x\,\mathrm{d}/\mathrm{d}x$ can be used as an operational definition of the Riemann ζ-function [8]. A further example in this direction is provided by

$$\left(\alpha + x\frac{\partial}{\partial x}\right)^{-v} f(x) = \frac{1}{\Gamma(v)}\int_0^\infty \exp(-\alpha t)t^{v-1} f(\exp(-t)x)\,\mathrm{d}t \tag{38}$$

and if $f(x) = \exp(x)$, we end up with

$$\left(\alpha + x\frac{\partial}{\partial x}\right)^{-v} \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{[\alpha + n]^v n!}. \tag{39}$$

Eq. (39) suggests further consequences, too. By replacing $f(x)$ with $L_n(x, y)$, we find that

$$\left(\alpha + x\frac{\partial}{\partial x}\right)^{-v} L_n(x, y) = n!\sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(\alpha + r)^v (r!)^2 (n-r)!}. \tag{40}$$

Denoting by $L_n(x, y; \alpha, v)$ the polynomials defined on the r.h.s. of Eq. (40), we find the recurrences

$$\begin{aligned} &\frac{\partial}{\partial y} L_n(x, y; \alpha, v) = nL_{n-1}(x, y; \alpha, v), \\ &-\frac{\partial}{\partial y} x\frac{\partial}{\partial x} L_n(x, y; \alpha, v) = nL_{n-1}(x, y; \alpha + 1, v). \end{aligned} \tag{41}$$

The previous examples are only a few concerning the techniques one can exploit once dealing with fractional derivative operators. It is therefore worth considering further possibilities involving different methods as those associated with Laplace transform techniques (see also [6, Chapter 4]).

We consider therefore the "differential equations"

$$\sqrt{\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1}\, f(x) = S(x), \tag{42}$$

where $S(x)$ denotes a known function.

The formal solution of Eq. (42) can be cast in the form

$$f(x) = \frac{1}{\sqrt{\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1}} S(x). \tag{43}$$

By recalling from the theory of Laplace transforms that [10]

$$\frac{1}{\sqrt{A^2 + 1}} = \int_0^\infty J_0(t)\mathrm{e}^{-At}\,\mathrm{d}t, \tag{44}$$

and on replacing A with $\alpha\,\mathrm{d}/\mathrm{d}x$, we find that

$$f(x) = \int_0^\infty J_0(t) e^{-\alpha t (d/dx)} S(x) \, dt = \int_0^\infty J_0(t) S(x - \alpha t) \, dt. \tag{45}$$

The solution of Eq. (42) in the form (45) holds only if the integral is convergent and can be viewed as a kind of convolution of $S(x)$ on the 0th-order cylindrical Bessel function.

As a final example, we will consider the solution of the fractional diffusive equation

$$\begin{aligned} &\frac{\partial}{\partial y} f(x,y) = -\frac{\partial^{1/2}}{\partial x^{1/2}} f(x,y), \\ &f(x,0) = g(x), \end{aligned} \tag{46}$$

which can be treated using an identity valid within the framework of the Laplace transform theory [10], namely

$$e^{-y\sqrt{d}} = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(y^2/4t) - td}}{t\sqrt{t}} \, dt. \tag{47}$$

By replacing d with $\partial/\partial x$ and by proceeding as before, we find the solution of Eq. (47) as

$$e^{-y(d/dx)^{1/2}} g(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-y^2/4t}}{t\sqrt{t}} g(x - t) \, dt. \tag{48}$$

This result can be viewed as the analogue of the Gauss transform for the solution of the heat diffusion equation.

In this paper it has been shown that methods of operational nature can provide a fairly useful tool to solve a large body of problems including fractional propagation equations. Further applications will be discussed elsewhere.

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