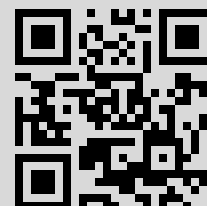


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Maslina Darus and K .Al Shaqsi

**ON HARMONIC UNIVALENT FUNCTIONS DEFINED BY
A GENERALIZED RUSCHEWEYH DERIVATIVES
OPERATOR**

(submitted by M. A. Malakhaltsev)

ABSTRACT. Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f = h + \bar{g}$ which are harmonic univalent and sense preserving in the unit disk \mathbf{U} . Al-Shaqsi and Darus[7] introduced a generalized Ruscheweyh derivatives operator denoted by D_{λ}^n where $D_{\lambda}^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]C(n, k)a_k z^k$, where $C(n, k) = \binom{k+n-1}{n}$. The authors, using this operators, introduce the class \mathcal{H}_{λ}^n of functions which are harmonic in \mathbf{U} . Coefficient bounds, distortion bounds and extreme points are obtained.

1. INTRODUCTION

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain \mathbb{C} if both u and v are real harmonic in \mathbb{C} . In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h + \bar{g}$, where h and g are analytic in \mathcal{D} . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in \mathcal{D} is that $|h'(z)| > |g'(z)|$ in \mathcal{D} . See Clunie and Sheil-Small (see [2]).

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Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbf{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1. \quad (1.1)$$

Observe that $\mathcal{S}_{\mathcal{H}}$ reduces to \mathcal{S} , the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

The class \mathcal{T} is defined as the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of all functions $f = h + \bar{g}$ where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = - \sum_{n=1}^{\infty} |b_n| z^n. \quad (1.2)$$

In 1984 Clunie and Sheil-Small [2] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{S}_{\mathcal{H}}$ and its subclasses such that Silverman [3], Silverman and Silvia [4] and, Jahangiri [5] studied the harmonic univalent functions.

We denote by \mathcal{H}_{λ}^n the class of all function of the form (1.1) that satisfy the condition

$$\Re(D_{\lambda}^n f(z))' > 0, \quad z \in \mathbf{U}. \quad (1.3)$$

where $D_{\lambda}^n f(z) = D_{\lambda}^n h(z) + \overline{D_{\lambda}^n g(z)}$, and D_{λ}^n denotes the operator introduced by Al-Shaqsi and Darus[7] and is given by

$$D_{\lambda}^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)] C(n, k) a_k z^k, \quad \lambda \geq 0, \quad (1.4)$$

where

$$C(n, k) = \binom{k+n-1}{n} = \frac{\prod_{j=1}^{k-1} (j+n)}{(k-1)!}, \quad k \geq 2. \quad (1.5)$$

Note that when $\lambda = 0$, we get Ruscheweyh differential operator (see[1]). Also note that the class $\mathcal{H}_{\lambda}^0 \equiv HP(\alpha)$ the class of harmonic univalent functions studied by Yalçın and Öztürk [6]. We further denote by $\mathcal{T}\mathcal{H}_{\lambda}^n$ the subclass of \mathcal{H}_{λ}^n , where $\mathcal{T}\mathcal{H}_{\lambda}^n = \mathcal{T} \cap \mathcal{H}_{\lambda}^n$.

2. COEFFICIENTS BOUNDS

Theorem 2.1. *Let $f = h + \bar{g}$ with h and g are given by (1.1). Let*

$$\sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)(|a_n| + |b_n|) \leq 2, \quad (2.1)$$

where $a_1 = 1$ and $\lambda \geq 0$. Then f is harmonic univalent sense preserving in \mathbf{U} and $f \in \mathcal{H}_{\lambda}^n$.

Proof. For $|z_1| \leq |z_2| < 1$, we have by (2.1),

$$\begin{aligned} & |f(z_1) - f(z_2)| \\ & \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ & = \left| (z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k) \right| - \left| \sum_{k=1}^{\infty} b_k(z_1^k - z_2^k) \right| \\ & \geq |z_1 - z_2| \left(1 - |b_1| - \sum_{k=2}^{\infty} k|z_2|^{k-1} \right) \\ & \geq |z_1 - z_2| \left(1 - |b_1| - |z_2| \sum_{k=2}^{\infty} k[1 + \lambda(k - 1)]C(n, k)[|a_k| + |b_k|] \right) \\ & \geq |z_1 - z_2|(1 - |b_1|)(1 - |z_2|) > 0. \end{aligned}$$

Consequently, f is univalent in \mathbf{U} . We note that f is sense preserving in \mathbf{U} . This is because

$$\begin{aligned} |h'(z)| & \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\ & \geq 1 - \sum_{k=2}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|a_k||z| \\ & \geq \sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|b_k| > \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|. \end{aligned}$$

Now we show that $f \in \mathcal{H}_{\lambda}^n$. Using the fact that $\Re w > 0$ if and only if $|1 + w| \geq |1 - w|$, it suffices to show that

$$\left| 1 + (D_{\lambda}^n h(z))' + \overline{(D_{\lambda}^n h(z))'} \right| - \left| 1 - (D_{\lambda}^n h(z))' - \overline{(D_{\lambda}^n h(z))'} \right| =$$

$$\begin{aligned}
& \left| 2 + \sum_{k=2}^{\infty} k [1 + \lambda(k-1)] C(n, k) a_n z^{n-1} + \sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) b_n \bar{z}^{n-1} \right| \\
& - \left| - \sum_{k=2}^{\infty} k [1 + \lambda(k-1)] C(n, k) a_n z^{n-1} - \sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) b_n \bar{z}^{n-1} \right| \\
& \geq 2 - \sum_{k=2}^{\infty} k [1 + \lambda(k-1)] C(n, k) |a_n| |z|^{n-1} \\
& - \sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) |b_n| |z|^{n-1} - \sum_{k=2}^{\infty} k [1 + \lambda(k-1)] C(n, k) |a_n| |z|^{n-1} \\
& \quad - \sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) |b_n| |z|^{n-1} \\
& = 2 - 2 \sum_{k=2}^{\infty} k [1 + \lambda(k-1)] C(n, k) |a_n| |z|^{n-1} \\
& \quad - 2 \sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) |b_n| |z|^{n-1} \geq \\
& 2 \left\{ 1 - \left(\sum_{k=2}^{\infty} k [1 + \lambda(k-1)] C(n, k) |a_n| + \sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) |b_n| \right) \right\} \\
& \qquad \qquad \qquad \geq 0, \text{ by (2.1).}
\end{aligned}$$

The harmonic mappings

$$\begin{aligned}
f(z) &= z + \sum_{k=2}^{\infty} \frac{x_k}{k [1 + \lambda(k-1)] C(n, k)} z^k \\
& \quad + \sum_{k=1}^{\infty} \frac{\bar{y}_k}{k [1 + \lambda(k-1)] C(n, k)} \bar{z}^k
\end{aligned}$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.2) are in \mathcal{H}_λ^n because

$$\sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) (|a_k| + |b_k|) = 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f = h + \bar{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f \in \mathcal{H}_\lambda^n$. We next show that the condition (2.1) is also necessary for

functions in \mathcal{TH}_λ^n .

Theorem 2.2. *Let $f = h + \bar{g}$ with h and g are given by (1.2). Then $f \in \mathcal{TH}_\lambda^n$ if and only if*

$$\sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)(|b_k| + |\bar{b}_k|) \leq 2, \quad (2.2)$$

where $a_1 = 1$ and $\lambda \geq 0$.

Proof. We first suppose that $f \in \mathcal{TH}_\lambda^n$, then by (1.3) we have

$$\begin{aligned} & \Re\{(D_\lambda^n h(z))' + \overline{(D_\lambda^n g(z))}'\} \\ &= \Re\left\{1 - \sum_{k=2}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|a_k|z^{n-1} \right. \\ & \quad \left. - \sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|b_k|\bar{z}^{n-1}\right\} \\ & > 0. \end{aligned}$$

If we choose z to be real and let $z \rightarrow 1^-$, we get

$$1 - \sum_{k=2}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|a_k| - \sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|b_k| \geq 0,$$

which is precisely the assertion (2.3) of Theorem 2.2.

Conversely, suppose that the inequality (2.3) holds true. Then we find from the definition (1.3) that

$$\begin{aligned} & \Re\{(D_\lambda^n h(z))' + \overline{(D_\lambda^n g(z))}'\} \\ &= \Re\left\{1 - \sum_{k=2}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|a_k|z^{n-1} \right. \\ & \quad \left. - \sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)|b_k|\bar{z}^{n-1}\right\} \\ & \geq 2 - \sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)(|b_k| + |\bar{b}_k|)\bar{z}^{n-1} \\ & > 2 - \sum_{k=1}^{\infty} k[1 + \lambda(k - 1)]C(n, k)(|b_k| + |\bar{b}_k|) \geq 0. \end{aligned}$$

provided that the inequality (2.3) is satisfied.

3. DISTORTION BOUNDS AND EXTREME POINTS.

In this section, we shall obtain distortion bounds for functions in \mathcal{TH}_λ^n and also provide extreme points for the class \mathcal{TH}_λ^n .

Theorem 3.1. *If $f \in \mathcal{TH}_\lambda^n$, for $\lambda \geq 0$ and $|z| = r > 1$, then*

$$|f(z)| \leq (1 + b_1)r + (1 - b_1)r + \frac{1 - |b_1|}{2(1 + \lambda)(n + 1)}r^2,$$

and

$$|f(z)| \geq (1 - b_1)r - (1 - b_1)r + \frac{1 - |b_1|}{2(1 + \lambda)(n + 1)}r^2.$$

Proof. We only prove the second inequality. The argument for first inequality is similar and will be omitted. Let $f \in \mathcal{TH}_\lambda^n$. Taking the absolute value of f , we obtain

$$\begin{aligned} |f(z)| &\geq (1 - b_1)r - \sum_{k=2}^{\infty} (|a_n| + |b_n|)r^n \\ &\geq (1 - b_1)r - \sum_{k=2}^{\infty} (|a_n| + |b_n|)r^2 \\ &= (1 - b_1)r - \frac{1}{2(1 + \lambda)(n + 1)} \sum_{k=2}^{\infty} 2(1 + \lambda)(n + 1)(|a_n| + |b_n|)r^2 \\ &\geq (1 - b_1)r - \frac{1}{2(1 + \lambda)(n + 1)} \sum_{k=2}^{\infty} k[1 + \lambda(k - 1)]C(n, k)(|a_n| + |b_n|)r^2 \\ &\geq (1 - b_1)r - \frac{1}{2(1 + \lambda)(n + 1)}[1 - |b_1|]r^2. \end{aligned}$$

The bounds given in Theorem 3.1 for the functions $f = h + \bar{g}$ of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

$$f(z) = z + |b_1|\bar{z} - \frac{1 - |b_1|}{2(1 + \lambda)(n + 1)}\bar{z}^2$$

and

$$f(z) = (1 - |b_1|)z - \frac{1 - |b_1|}{2(1 + \lambda)(n + 1)}z^2$$

for $|b_1| < 1$ show that the bounds given Theorem 3.1 are sharp.

The following covering result follows from the second inequality in Theorem 3.1.

Corollary 3.2. *If $f \in \mathcal{TH}_\lambda^n$, then*

$$\left\{ w : |w| < \frac{1}{2(1+\lambda)(n+1)} [(1-|b_1|)(2(1+\lambda)(n+1)-1)] \right\} \subset f(\mathbf{U}).$$

Theorem 3.3. *$f \in \mathcal{TH}_\lambda^n$ if and only if f can be expressed as*

$$f(z) = \sum_{k=1}^{\infty} (\gamma_k h_k + \mu_k g_k) \quad (3.1)$$

where $z \in \mathbf{U}$,

$$h_1(z) = z, \quad h_k(z) = z - \frac{1}{k[1+\lambda(k-1)]C(n,k)} z^k, \quad (k = 2, 3, \dots),$$

$$g_k(z) = z - \frac{1}{k[1+\lambda(k-1)]C(n,k)} \bar{z}^k, \quad (n = 1, 2, \dots),$$

$$\sum_{k=1}^{\infty} (\gamma_k + \mu_k) = 1, \quad \gamma_k \geq 0 \text{ and } \mu_k \geq 0.$$

In particular, the extreme points of \mathcal{TH}_λ^n are $\{h_k\}$ and $\{g_k\}$.

Proof. Note that for f we may write

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (\gamma_k h_k + \mu_k g_k) \\ &= \sum_{k=1}^{\infty} (\gamma_k + \mu_k) z - \sum_{k=2}^{\infty} \frac{1}{k[1+\lambda(k-1)]C(n,k)} \gamma_k z^k \\ &\quad - \sum_{k=1}^{\infty} \frac{1}{k[1+\lambda(k-1)]C(n,k)} \mu_k \bar{z}^k. \end{aligned}$$

Now the first part of the proof is complete, since by Theorem 2.2

$$\begin{aligned} &\sum_{k=2}^{\infty} k[1+\lambda(k-1)]C(n,k) \frac{\gamma_k}{k[1+\lambda(k-1)]C(n,k)} \\ &\quad - \sum_{k=1}^{\infty} k[1+\lambda(k-1)]C(n,k) \frac{\mu_k}{k[1+\lambda(k-1)]C(n,k)} \\ &= \sum_{k=1}^{\infty} (\gamma_k + \mu_k) - \gamma_1 = 1 - \gamma_1 \leq 1. \end{aligned}$$

Conversely, suppose that $f \in \mathcal{TH}_\lambda^n$. Then

$$\sum_{k=1}^{\infty} k [1 + \lambda(k-1)] C(n, k) (|a_k| + |b_k|) \leq 2.$$

Setting

$$\begin{aligned} \gamma_k &= k [1 + \lambda(k-1)] C(n, k) |a_k|, \quad 0 \leq \gamma_k \leq 1, \quad (k = 2, 3, \dots), \\ \mu_k &= k [1 + \lambda(k-1)] C(n, k) |b_k|, \quad 0 \leq \mu_k \leq 1, \quad (k = 1, 2, 3, \dots), \end{aligned}$$

and $\mu_1 = 1 - \gamma_1 - \sum_{k=2}^{\infty} (\gamma_k + \mu_k)$ we obtain

$$f(z) = \sum_{k=1}^{\infty} (\gamma_k h_k + \mu_k g_k) \text{ as required.}$$

REFERENCES

- [1] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49**, (1975), 109-115.
- [2] J. Clunie and T. Shell-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9**, (1984), 3-25.
- [3] H. Silverman, Harmonic univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51**, (1998), 283-289.
- [4] H. Silverman and E. M. Silvia, Subclasses of harmonic univalent functions, *New Zeal. J. Math.* **28**, (1999), 275-284.
- [5] J. M. Jahangiri, Harmonic functions starlike in the unite disk, *J. math. Anal. Appl.* **235**, (1999), 470-477.
- [6] S. Yalçın and M. Öztürk, A new subclass of complex harmonic functions, *Math. Ineq. Appl.* **7**, (2004), 55-61.
- [7] K.A.Shaqsi and M. Darus, On univalent functions with respect to K -symmetric points given by a generalised Ruscheweyh derivatives operator, *submitted*

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