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ON A CLASS OF ANALYTIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY CONVOLUTION

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Abstract. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n > 0$ be a fixed analytic function defined on $\Delta = \{z; |z| < 1\}$. In the present investigation, we introduce the class of functions $f = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ satisfying

$$\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha \quad (z \in \Delta; 1 < \alpha < 3/2)$$

and obtain the coefficient inequality, coefficient estimate, distortion theorem, and a closure theorem. Also we consider a radius problem. Our result contains several new results as special cases.

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Key words. Starlike function, Ruscheweyh derivative, Salagean derivative, convolution, positive coefficients, coefficient inequality, distortion theorem, radius problem.

1. INTRODUCTION AND DEFINITIONS

Let T be the class of all analytic univalent functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0; z \in \Delta = \{z; |z| < 1\}).$$

A function $f(z) \in T$ is called a function with negative coefficients. The subclass of T consisting of starlike functions of order α, denoted by $TS^*(\alpha)$, is studied by Silverman [6]. Several other class of starlike functions with negative coefficients were studied; e.g., see [1]. For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution (or Hadamard product) of f and g, denoted by $f * g$ or $(f * g)(z)$, is defined to be function $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be a fixed analytic function in Δ with $b_n > 0$, $(n \geq 2)$. Using convolution, Ali *et al.* [2] (see also [4]) have studied a more general class of multivalent functions which includes the class $TS_g(\alpha)$ defined by

$$TS_g^*(\alpha) = \left\{ f \in T : \Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) > \alpha \quad (0 \leq \alpha < 1; z \in \Delta) \right\}.$$

Ravichandran and Sivaprasad Kumar [5] have studied a similar class of meromorphic functions. Note that several well-known subclasses of functions are special cases of the class $TS_g^*(\alpha)$ for suitable choices of $g(z)$. When $g(z) = z/(1-z)$, the class $TS_g^*(\alpha)$ is the class $TS^*(\alpha)$ of starlike functions with negative coefficients of order α introduced and studied by Silverman [6]. When

$g(z) = z/(1-z)^2$, the class $TS_g^*(\alpha)$ is the class of convex functions with negative coefficients of order α introduced and studied by Silverman [6]. The class $T_\lambda(\alpha)$ studied by Ahuja [1] is a special case of $TS_g^*(\alpha)$ when $g(z) = z/(1-z)^{\lambda+1}$. Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ with $f(0) = 0 = f'(0) - 1$. The class $M(\alpha)$ defined by

$$M(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \quad (1 < \alpha < 3/2; z \in \Delta) \right\}$$

was investigated by Uralegaddi *et al.* [7]. A subclass of $M(\alpha)$ was recently investigated by Owa and Srivastava [3].

In this paper, we introduce a more general class $PM_g(\alpha)$ of analytic function with positive coefficient motivated by $M(\alpha)$ and the earlier work of Ali *et al.* [2]. For the newly defined class $PM_g(\alpha)$, we obtain the coefficient inequality, coefficient estimate, distortion theorem, and a closure theorem. Also we compute the radius of starlikeness of order β and the radius of convexity of order β for the functions in the class $PM_g(\alpha)$. Our result contains several results as special cases.

DEFINITION 1. Let P be the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0) \tag{2}$$

be a fixed analytic function in Δ. Define the class $PM_g(\alpha)$ by

$$PM_g(\alpha) = \left\{ f \in P : \Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha \quad (1 < \alpha < 3/2; z \in \Delta). \right\}$$

When $g(z) = z/(1-z)$, the class $PM_g(\alpha)$ reduces to the subclass $PM(\alpha) = P \cap M(\alpha)$. When $g(z) = z/(1-z)^{\lambda+1}$, the class $PM_g(\alpha)$ reduces to the following class $P_\lambda(\alpha)$

$$P_\lambda(\alpha) = \left\{ f \in P : \Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) < \alpha, \quad (\lambda > -1, 1 < \alpha < 3/2; z \in \Delta) \right\},$$

where D^λ denotes the Ruscheweyh derivative of order λ. When $g(z) = z + \sum_{n=2}^{\infty} n^m z^n$, the class of function $PM_g(\alpha)$ reduces to the class $PM_m(\alpha)$ where

$$PM_m(\alpha) = \left\{ f \in P : \Re\left(\frac{z(\mathcal{D}^m f(z))'}{\mathcal{D}^m f(z)}\right) < \alpha \quad (1 < \alpha < 3/2; m \geq 0; z \in \Delta) \right\},$$

where $\mathcal{D}^m$ denotes the Salagean derivative of order m. Also we have

$$PM(\alpha) \equiv P_0(\alpha) \equiv PM_0(\alpha).$$

2. COEFFICIENT INEQUALITIES

Throughout the paper, we assume that the function $f(z)$ is given by the equation (1) and $g(z)$ is given by (2). We first prove a necessary and sufficient condition for functions to be in the class $PM_g(\alpha)$ in the following:

THEOREM 1. *A function* $f \in PM_g(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nb_n \leq \alpha - 1 \quad (1 < \alpha < 3/2). \tag{3}$$

Proof. If $f \in PM_g(\alpha)$, then (3) follows from

$$\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) < \alpha$$

by letting $z \to 1-$ through real values. To prove the converse, assume that (3) holds. Then by making use of (3), we obtain

$$\left|\frac{z(f*g)'(z)-(f*g)(z)}{z(f*g)'(z)-(2\alpha-1)(f*g)(z)}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_nb_n}{2(\alpha-1)-\sum_{n=2}^{\infty}[n-(2\alpha-1)]a_nb_n} \leq 1$$

or, equivalently, $f \in PM_g(\alpha)$. □

COROLLARY 1. *A function* $f \in P_\lambda(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nB_n(\lambda) \leq \alpha - 1 \quad (1 < \alpha < 3/2),$$

where

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}. \tag{4}$$

COROLLARY 2. *A function* $f \in PM_m(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_nn^m \leq \alpha - 1 \quad (1 < \alpha < 3/2).$$

Our next Theorem gives an estimate for the coefficient of functions in the class $PM_g(\alpha)$.

THEOREM 2. *If* $f \in PM_g(\alpha)$, *then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)b_n}$$

with the equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)b_n}z^n.$$

Proof. Let $f \in PM_g(\alpha)$. By making use of the inequality (3) for $f \in PM_g(\alpha)$, we have

$$(n-\alpha)a_n b_n \leq \sum_{n=2}^{\infty}(n-\alpha)a_n b_n \leq \alpha - 1$$

or $a_n \leq \frac{\alpha-1}{(n-\alpha)b_n}$. Clearly for

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)b_n}z^n \in PM_g(\alpha),$$

we have $a_n = \frac{\alpha-1}{(n-\alpha)b_n}$. □

COROLLARY 3. *If $f \in P_\lambda(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)B_n(\lambda)}$$

with the equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)B_n(\lambda)}z^n,$$

where $B_n(\lambda)$ is given by (4).

COROLLARY 4. *If $f \in PM_m(\alpha)$, then*

$$a_n \leq \frac{\alpha-1}{(n-\alpha)n^m}$$

with the equality only for functions of the form

$$f_n(z) = z + \frac{\alpha-1}{(n-\alpha)n^m}z^n.$$

3. GROWTH THEOREM

We now prove the growth theorem for the functions in the class $PM_g(\alpha)$.

THEOREM 3. *If $f \in PM_g(\alpha)$, then*

$$r - \frac{\alpha-1}{(2-\alpha)b_2}r^2 \leq |f(z)| \leq r + \frac{\alpha-1}{(2-\alpha)b_2}r^2, \quad |z| = r < 1,$$

provided $b_n \geq b_2$. The result is sharp for

$$f(z) = z + \frac{\alpha-1}{(2-\alpha)b_2}z^2.$$

Proof. By making use of the inequality (3) for $f \in PM_g(\alpha)$ together with

$$(2-\alpha)b_2 \leq (n-\alpha)b_n,$$

we obtain

$$b_2(2-\alpha)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty}(n-\alpha)a_n b_n \leq \alpha - 1$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha - 1}{(2-\alpha)b_2}. \tag{5}$$

By using (5) for the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in PM_g(\alpha)$, we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} a_n r^n \quad (|z| = r) \\ &\leq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leq r + r^2 \frac{\alpha - 1}{(2-\alpha)b_2} \end{aligned}$$

and similarly we have

$$|f(z)| \geq r - r^2 \frac{\alpha - 1}{(2-\alpha)b_2}. \qquad \square$$

COROLLARY 5. *If $f \in P_\lambda(\alpha)$, then*

$$r - \frac{\alpha - 1}{(2-\alpha)(\lambda+1)} r^2 \leq |f(z)| \leq r + \frac{\alpha - 1}{(2-\alpha)(\lambda+1)} r^2 \quad (|z| = r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha - 1}{(2-\alpha)(\lambda+1)} z^2.$$

COROLLARY 6. *If $f \in PM_m(\alpha)$, then*

$$r - \frac{\alpha - 1}{(2-\alpha)2^m} r^2 \leq |f(z)| \leq r + \frac{\alpha - 1}{(2-\alpha)2^m} r^2 \quad (|z| = r).$$

The result is sharp for

$$f(z) = z + \frac{\alpha - 1}{(2-\alpha)2^m} z^2.$$

4. CLOSURE THEOREMS

Let the functions $F_k(z)$ be given by

$$F_k(z) = z + \sum_{n=2}^{\infty} f_{n,k} z^n \quad (k = 1, 2, ..., m). \tag{6}$$

We shall now prove the following closure theorems for the class $PM_g(\alpha)$.

THEOREM 4. *Let the function $F_k(z)$ defined by (6) be in the class $PM_g(\alpha)$ for every $k = 1, 2, ..., m$. Then the function $f(z)$ defined by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0)$$

belongs to the class $PM_g(\alpha)$, where $a_n = \frac{1}{m}\sum_{k=1}^{m} f_{n,k}$ $(n = 1, 2, \ldots)$.

Proof. Since $F_k(z) \in PM_g(\alpha)$, it follows from Theorem 1 that

$$\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,k} \leq \alpha - 1 \tag{7}$$

for every $k = 1, 2, ..., m$. Hence

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\alpha)g_n a_n &= \sum_{n=2}^{\infty}(n-\alpha)g_n\left(\frac{1}{m}\sum_{k=1}^{m} f_{n,k}\right)\\ &= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,k}\right)\\ &\leq \alpha - 1.\end{aligned}$$

By Theorem 1, it follows that $f(z) \in PM_g(\alpha)$. □

THEOREM 5. *The class $PM_g(\alpha)$ is closed under convex linear combination.*

Proof. Let the function $F_k(z)$, $k = 1, 2$, given by (6) be in the class $PM_g(\alpha)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1-\lambda)F_2(z) \quad (0 \leq \lambda \leq 1)$$

is also in the class $PM_g(\alpha)$. Since for $0 \leq \lambda \leq 1$

$$H(z) = z + \sum_{n=1}^{\infty}[\lambda f_{n,1} + (1-\lambda)f_{n,2}],$$

we observe that

$$\begin{aligned}&\sum_{n=2}^{\infty}(n-\alpha)g_n[\lambda f_{n,1} + (1-\lambda)f_{n,2}]\\ =\ &\lambda\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,1} + (1-\lambda)\sum_{n=2}^{\infty}(n-\alpha)g_n f_{n,2}\\ \leq\ &\alpha - 1.\end{aligned}$$

By Theorem 1, we have $H(z) \in PM_g(\alpha)$. □

THEOREM 6. *Let $F_1(z) = z$ and $F_n(z) = z + \frac{\alpha-1}{(n-\alpha)g_n}z^n$ for $n = 2, 3, \ldots$. Then $f(z) \in PM_g(\alpha)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty}\lambda_n F_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$.*

Proof. Let

$$\begin{aligned}f(z) &= \sum_{n=1}^{\infty}\lambda_n F_n(z)\\ &= z + \sum_{n=2}^{\infty}\frac{\lambda_n(\alpha-1)}{(n-\alpha)g_n}z^n.\end{aligned}$$

Then

$$\sum_{n=2}^{\infty} \frac{\lambda_n(\alpha-1)}{(n-\alpha)g_n} \frac{(n-\alpha)g_n}{(\alpha-1)} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

By Theorem 1, we have $f(z) \in PM_g(\alpha)$.

Conversely, let $f(z) \in PM_g(\alpha)$. From Theorem 2, we have

$$f_n \leq \frac{\alpha-1}{(n-\alpha)g_n} \quad \text{for} \quad n = 2, 3, \ldots.$$

Therefore we may take

$$\lambda_n = \frac{(n-\alpha)g_n f_n}{\alpha-1} \quad \text{for} \quad n = 2, 3, \ldots$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then $f(z) = \sum_{n=1}^{\infty} \lambda_n F_n(z)$. □

5. RADIUS PROBLEM

In this section, we find the radius of starlikeness of order β and the radius of convexity of order β for functions in the class $PM_g(\alpha)$.

THEOREM 7. *If $f \in PM_g(\alpha)$ $(1 < \alpha \leq 3/2)$, then f is starlike of order β $(0 \leq \beta < 1)$ in $|z| < r(\beta, \alpha, g)$ where*

$$r(\beta, \alpha, g) = \inf_{n \geq 2} \left[\frac{(1-\beta)(n-\alpha)}{(\alpha-1)(n-\beta)} b_n \right]^{1/(n-1)}.$$

Proof. It is enough to show that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} a_n |z|^{n-1} < 1 \tag{8}$$

which will imply that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \beta.$$

The inequality (8) follows if

$$\frac{n-\beta}{1-\beta} a_n |z|^{n-1} \leq \frac{n-\alpha}{\alpha-1} a_n b_n$$

and this proves the result. □

We have the following:

COROLLARY 7. *If $f \in PM_g(\alpha)$ $(1 < \alpha \leq 3/2)$, then f is convex of order β $(0 \leq \beta < 1)$ in $|z| < r(\beta, \alpha, g)$ where*

$$r(\beta, \alpha, g) = \inf_{n \geq 2} \left[\frac{(1-\beta)(n-\alpha)}{n(\alpha-1)(n-\beta)} b_n \right]^{1/(n-1)}.$$

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