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COMMON FIXED POINT THEOREMS FOR MULTI-VALUED MAPS*

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Abstract We establish some results on coincidence and common fixed points for a two-pair of multi-valued and single-valued maps in complete metric spaces. Presented theorems generalize recent results of Gordji et al [4] and several results existing in the literature.

Key words Coincidence point; common fixed point; multi-valued maps

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1 Introduction and Preliminaries

The Banach fixed-point theorem [1] (also known as the Banach contraction mapping theorem or the Banach contraction mapping principle) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces, and provides a constructive method to find those fixed points. Many generalizations of this famous theorem exist in the literature (cf. [1–8] and others).

In [7], Nadler extended the Banach fixed-point theorem from the single-valued maps to the set-valued contractive maps. Before presenting this important theorem, we start with introducing some notations.

Let (X,d) be a metric space. Denote by $CB(X)$ the collection of non-empty closed bounded subsets of X. For $A,B\in CB(X)$ and $x\in X$, define

$$D(x,A)=\inf_{a\in A} d(x,a)$$

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and

$$H(A,B)=\max\left\{\sup_{a\in A}D(a,B),\sup_{b\in B}D(b,A)\right\}.$$

It is seen that H is a metric on $CB(X)$. H is called the Hausdorff metric induced by d. It is well known that $(CB(X),H)$ is a complete metric space, whenever (X,d) is a complete metric space.

Definition 1.1 Let $T:X\to CB(X)$ be a multi-valued map. An element $x\in X$ is said to be a fixed point of T if $x\in Tx$.

The Nadler's fixed-point theorem [7] is the following.

Theorem 1.1 Let (X,d) be a complete metric space, and $T:X\to CB(X)$ be a multi-valued map satisfying

$$H(Tx,Ty)\le qd(x,y)\quad \text{for all}\quad x,y\in X,$$

where q is a constant such that $q\in[0,1)$. Then, T has a fixed point.

Recently, an extension of Theorem 1.1 was obtained by Gordji et al [4]. They proved the following result.

Theorem 1.2 Let (X,d) be a complete metric space, and T be a map from X into $CB(X)$ such that

$$H(Tx,Ty)\le \alpha d(x,y)+\beta[D(x,Tx)+D(y,Ty)]+\gamma[D(x,Ty)+D(y,Tx)]$$

for all $x,y\in X$, where $\alpha,\beta,\gamma\ge 0$ and $\alpha+2\beta+2\gamma<1$. Then, T has a fixed point.

Note that Theorem 1.2 generalizes also other known results in the literature [5, 8, 9].

In this article, we establish some results on coincidence and common fixed points for a two-pair of multi-valued and single-valued maps in complete metric spaces. Presented theorems generalize the result given by Theorem 1.2 and other existing results in the literature.

The following definitions will be used later.

Definition 1.2 An element $x\in X$ is said to be a coincidence point of $T:X\to CB(X)$ and $f:X\to X$ if $fx\in Tx$. We denote

$$C(f,T)=\{x\in X\mid fx\in Tx\},$$

the set of coincidence points of T and f.

Definition 1.3 Maps $f:X\to X$ and $T:X\to CB(X)$ are weakly compatible if they commute at their coincidence points, that is, if $fTx=Tfx$ whenever $fx\in Tx$.

Definition 1.4 (see [6]) Let $T:X\to CB(X)$ be a multi-valued map and $f:X\to X$ be a single-valued map. The map f is said to be T-weakly commuting at $x\in X$ if $ffx\in Tfx$.

Definition 1.5 An element $x\in X$ is a common fixed point of $T,S:X\to CB(X)$ and $f:X\to X$ if $x=fx\in Tx\cap Sx$.

Example 1.1 Consider $X=[0,+\infty)$ equipped with the metric $d(x,y)=|x-y|$ for every $x,y\in X$. Define $f:X\to X$ and $T:X\to CB(X)$ as

$$fx=\begin{cases}0 & \text{if}\ \ x\in[0,1),\\ 2x & \text{if}\ \ x\in[1,+\infty),\end{cases}\quad Tx=\begin{cases}\{x\} & \text{if}\ \ x\in[0,1),\\ [1,1+2x] & \text{if}\ \ x\in[1,+\infty).\end{cases}$$

We have

- $f1 = 2 \in [1,3] = T1$, that is, $x = 1$ is a coincidence point of f and T;
- $fT1 = [2,6] \neq [1,5] = Tf1$, that is, f and T are not weakly compatible mappings;
- $ff1 = 4 \in [1,5] = Tf1$, that is, f is T-weakly commuting at 1.

2 Main Results

The following lemma (see [2, 3]) plays an important role in the proof of our results.

Lemma 2.1 If $A, B \in CB(X)$ and $a \in A$, then for any fixed $h > 1$, there exists $b = b(a) \in B$ such that

$$d(a,b) \leq hH(A,B).$$

Our first result is the following.

Theorem 2.1 Let (X,d) be a complete metric space. Let $T, S : X \to CB(X)$ be a pair of multi-valued maps and $f, g : X \to X$ a pair of single-valued maps. Suppose that

$$H(Sx,Ty) \leq \alpha d(fx,gy) + \beta[D(fx,Sx) + D(gy,Ty)] + \gamma[D(fx,Ty) + D(gy,Sx)], \quad (1)$$

for each $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $0 < \alpha + 2\beta + 2\gamma < 1$. Suppose also that

(i) $SX \subseteq gX$, $TX \subseteq fX$,

(ii) $f(X)$ and $g(X)$ are closed.

Then, there exist points u and w in X, such that

$$fu \in Su, \quad gw \in Tw, \quad fu = gw \quad \textit{and} \quad Su = Tw.$$

Proof As $0 < \alpha + 2\beta + 2\gamma < 1$, there exists $r > 0$, such that

$$0 < \alpha + 2\beta + 2\gamma < \sqrt{r} < 1. \quad (2)$$

Let us denote

$$\lambda = \frac{\alpha + \beta + \gamma}{\sqrt{r} - (\beta + \gamma)}. \quad (3)$$

Clearly, from (2), it follows that

$$0 < \lambda < 1. \quad (4)$$

Let $x_0 \in X$ be arbitrary. Then, fx_0 and Sx_0 are well defined. From (i), there exists $x_1 \in X$, such that $gx_1 \in Sx_0$. Again from (i) and Lemma 2.1 with $h = 1/\sqrt{r}$, as $gx_1 \in Sx_0$, there exists $x_2 \in X$ such that $fx_2 \in Tx_1$ and

$$d(gx_1, fx_2) \leq \frac{1}{\sqrt{r}} H(Sx_0, Tx_1). \quad (5)$$

From (1) and (5), we obtain

$$\begin{aligned} d(gx_1, fx_2) \leq & \frac{\alpha}{\sqrt{r}} d(fx_0, gx_1) + \frac{\beta}{\sqrt{r}} [D(fx_0, Sx_0) + D(gx_1, Tx_1)] \\ & + \frac{\gamma}{\sqrt{r}} [D(fx_0, Tx_1) + D(gx_1, Sx_0)]. \end{aligned} \quad (6)$$

In contrast, we have

$$\begin{aligned}
&D(fx_0, Sx_0) \le d(fx_0, gx_1),\\
&D(gx_1, Tx_1) \le d(gx_1, fx_2),\\
&D(gx_1, Sx_0) = 0,\\
&D(fx_0, Tx_1) \le d(fx_0, fx_2) \le d(fx_0, gx_1) + d(gx_1, fx_2). \qquad (7)
\end{aligned}$$

From (6) and (7), we obtain

$$\begin{aligned}
d(gx_1, fx_2) &\le \frac{\alpha}{\sqrt{r}}\, d(fx_0, gx_1) + \frac{\beta}{\sqrt{r}}\,[d(fx_0, gx_1) + d(gx_1, fx_2)]\\
&\quad + \frac{\gamma}{\sqrt{r}}\,[d(fx_0, gx_1) + d(gx_1, fx_2)]\\
&= \left(\frac{\alpha}{\sqrt{r}} + \frac{\beta}{\sqrt{r}} + \frac{\gamma}{\sqrt{r}}\right) d(fx_0, gx_1) + \left(\frac{\beta}{\sqrt{r}} + \frac{\gamma}{\sqrt{r}}\right) d(gx_1, fx_2).
\end{aligned}$$

Hence,

$$[\sqrt{r} - (\beta + \gamma)]d(gx_1, fx_2) \le (\alpha + \beta + \gamma)d(fx_0, gx_1).$$

Then, from (3),

$$d(gx_1, fx_2) \le \lambda d(fx_0, gx_1).$$

Again, from (i) and Lemma 2.1, as $fx_2 \in Tx_1$, there exists $x_3 \in X$ such that $gx_3 \in Sx_2$ and

$$d(fx_2, gx_3) \le \frac{1}{\sqrt{r}}\, H(Sx_2, Tx_1). \qquad (8)$$

By (1) and (8), we obtain

$$\begin{aligned}
d(fx_2, gx_3) &\le \frac{\alpha}{\sqrt{r}}\, d(fx_2, gx_1) + \frac{\beta}{\sqrt{r}}\,[D(fx_2, Sx_2) + D(gx_1, Tx_1)]\\
&\quad + \frac{\gamma}{\sqrt{r}}\,[D(fx_2, Tx_1) + D(gx_1, Sx_2)]. \qquad (9)
\end{aligned}$$

In contrast, we have

$$\begin{aligned}
&D(fx_2, Sx_2) \le d(fx_2, gx_3),\\
&D(gx_1, Tx_1) \le d(gx_1, fx_2),\\
&D(fx_2, Tx_1) = 0,\\
&D(gx_1, Sx_2) \le d(gx_1, gx_3) \le d(gx_1, fx_2) + d(fx_2, gx_3). \qquad (10)
\end{aligned}$$

Similarly as above, from (9) and (10), we obtain

$$d(fx_2, gx_3) \le \lambda d(gx_1, fx_2).$$

Continuing this process, we can construct a sequence $\{y_n\}$ in X, such that $y_0 = gx_1$ and, for each $n \in \mathbb{N}$,

$$y_{2n} = gx_{2n+1} \in Sx_{2n}, \quad y_{2n+1} = fx_{2n+2} \in Tx_{2n+1}, \qquad \text{for each} \quad n \in \mathbb{N}, \qquad (11)$$

and

$$d(y_{2n}, y_{2n+1}) = d(gx_{2n+1}, fx_{2n+2}) \leq \lambda d(gx_{2n+1}, fx_{2n}),$$
$$d(y_{2n-1}, y_{2n}) = d(fx_{2n}, gx_{2n+1}) \leq \lambda d(gx_{2n-1}, fx_{2n}).$$

Therefore, we have

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n), \quad \text{for all} \quad n \geq 1. \tag{12}$$

From (12), we get, by induction,

$$d(y_n, y_{n+1}) \leq \lambda^n d(y_0, y_1), \quad \text{for all} \quad n \in \mathbb{N}. \tag{13}$$

Now, we shall show that $\{y_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. We need to show that there is a positive integer $n_0 = n_0(\varepsilon)$ such that

$$d(y_n, y_{n+p}) < \varepsilon \quad \text{for every} \quad n \geq n_0, \quad \text{uniformly on} \quad p \in \mathbb{N}. \tag{14}$$

By the triangular inequality,

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}).$$

Thus, from (13), we obtain

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \lambda^n d(y_0, y_1) + \lambda^{n+1} d(y_0, y_1) + \cdots + \lambda^{n+p-1} d(y_0, y_1) \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{p-1}) d(y_0, y_1) \\ &\leq \lambda^n (1 + \lambda + \cdots + \lambda^{p-1} + \cdots) d(y_0, y_1). \end{aligned}$$

Hence, we obtain

$$d(y_n, y_{n+p}) \leq \frac{\lambda^n}{1-\lambda} d(y_0, y_1) \quad \text{for all} \quad n \in \mathbb{N} \tag{15}$$

uniformly on $p \in \mathbb{N}$. As $0 < \lambda < 1$, it follows that $\lambda^n \to 0$ as $n \to \infty$. Thus there is a positive integer n_0, such that

$$\frac{\lambda^n}{1-\lambda} d(y_0, y_1) < \varepsilon \quad \text{for all} \quad n \geq n_0. \tag{16}$$

From (15) and (16), we get (14). Thus, we have proved that $\{y_n\}$ is a Cauchy sequence.

Now, as (X, d) is complete, $\{y_n\}$ converges to some $y \in X$. Therefore,

$$\lim_{n \to +\infty} y_n = \lim_{n \to +\infty} gx_{2n+1} = \lim_{n \to +\infty} fx_{2n+2} = y. \tag{17}$$

As $y_{2n} = gx_{2n+1}; y_{2n+1} = fx_{2n+2}$; and $f(X)$ and $g(X)$ are closed, then, $y \in f(X)$ and $y \in g(X)$. So, there exist $u, w \in X$, such that $fu = y$ and $gw = y$. Thus, we have proved that

$$fu = gw. \tag{18}$$

From the contraction type condition (1) and (11), we obtain

$$\begin{aligned} D(fu, Su) &\leq d(fu, fx_{2n+2}) + D(fx_{2n+2}, Su) \\ &\leq d(fu, fx_{2n+2}) + H(Su, Tx_{2n+1}) \\ &\leq d(fu, fx_{2n+2}) + \alpha d(fu, gx_{2n+1}) + \beta[D(fu, Su) + D(gx_{2n+1}, Tx_{2n+1})] \\ &\quad + \gamma[D(fu, Tx_{2n+1}) + D(gx_{2n+1}, Su)] \\ &\leq d(fu, fx_{2n+2}) + \alpha d(fu, gx_{2n+1}) + \beta[D(fu, Su) + d(gx_{2n+1}, fx_{2n+2})] \\ &\quad + \gamma[d(fu, fx_{2n+2}) + D(gx_{2n+1}, Su)]. \end{aligned}$$

Letting $n \to +\infty$ in the above inequality and using (17) and (18), we obtain

$$D(fu, Su) \le (\beta + \gamma) D(fu, Su).$$

As $\beta + \gamma < 1$, it implies that $D(fu, Su) = 0$. Hence, as Su is closed,

$$fu \in Su. \tag{19}$$

Similarly, we can prove that

$$D(gw, Tw) \le (\beta + \gamma) D(gw, Tw).$$

Hence,

$$gw \in Tw. \tag{20}$$

Now, we have to prove that

$$Su = Tw. \tag{21}$$

Using (1), (18), (19), and (20), we obtain

$$\begin{aligned} H(Su, Tw) &\le \alpha d(fu, gw) + \beta[D(fu, Su) + D(gw, Tw)] + \gamma[D(fu, Tw) + D(gw, Su)] \\ &= \alpha \cdot 0 + \beta[0 + 0] + \gamma[D(gw, Tw) + D(fu, Su)] \\ &= 0. \end{aligned}$$

Hence, $Su = Tw$. Thus, by (18), (19), (20), and (21), we have proved that

$$fu \in Su, \quad gw \in Tw, \quad fu = gw \quad \text{and} \quad Su = Tw. \tag{22}$$

Example 2.1 Let $X = [0, \infty)$ be the Euclidean space with the usual metric. Define S, T, f, and g on X as follows :

$$Sx = x^2 + 7/64, \quad Tx = x^3 + 7/64, \quad fx = 8x^2 \quad \text{and} \quad gx = 8x^3.$$

Then,

$$d(Sx, Ty) = |x^2 - y^3| \le \frac{8|x^2 - y^3|}{4} = \frac{d(fx, gy)}{4}.$$

Thus, (1) holds for all $x, y \in X$. Also, the other hypotheses (i) and (ii) are satisfied. It is seen that $S(1/8) = f(1/8) = 1/8$ and $T(1/4) = g(1/4) = 1/8$. Therefore, S and f have the coincidence at the point $u = 1/8$, T and g at the point $w = 1/4$, and $S(1/8) = T(1/4)$.

If $f = g$ in Theorem 2.1, then, we obtain the following coincidence result.

Theorem 2.2 Let (X, d) be a complete metric space. Let $T, S : X \to CB(X)$ be multi-valued maps and $f : X \to X$ be a single-valued map satisfying, for each $x, y \in X$,

$$H(Sx, Ty) \le \alpha d(fx, fy) + \beta[D(fx, Sx) + D(fy, Ty)] + \gamma[D(fx, Ty) + D(fy, Sx)], \tag{23}$$

where $\alpha, \beta, \gamma \ge 0$ and $0 < \alpha + 2\beta + 2\gamma < 1$. If fX is a closed subset of X and $TX \cup SX \subseteq fX$, then, f, T, and S have a coincidence in X. Moreover, if f is both T-weakly commuting and S-weakly commuting at each $z \in C(f, T)$, and $ffz = fz$, then, f, T, and S have a common fixed point in x.

Proof If $f = g$ in Theorem 2.1, we obtain that there exist points u and w in X such that

$$fu \in Su, \quad fw \in Tw, \quad fu = fw \quad and \quad Su = Tw.$$

As $u \in C(f, T)$, f is T-weakly commuting at u and $ffu = fu$. Set $v = fu$. Then, we have $fv = v$ and $v = ffu \in T(fu) = Tv$. Now, since also $u \in C(f, S)$, then f is S-weakly commuting at u, and so we obtain $v = fv = ffu \in S(fu) = Sv$. Thus, we have proved that $v = fv \in Tv \cap Sv$, that is, v is a common fixed point of f, T and S.

If $f = g = I_X$ (I_X being the identity map on X) in Theorem 2.1, then, we obtain the following common fixed-point result.

Corollary 2.1 Let (X, d) be a complete metric space. Let $T, S : X \to CB(X)$ be multi-valued maps satisfying, for each $x, y \in X$,

$$H(Sx, Ty) \le \alpha d(x, y) + \beta[D(x, Sx) + D(y, Ty)] + \gamma[D(x, Ty) + D(y, Sx)],$$

where $\alpha, \beta, \gamma \ge 0$ and $0 < \alpha + 2\beta + 2\gamma < 1$. Then, there exists a point z in X, such that $z \in Sz \cap Tz$ and $Sz = Tz$.

Remark 1 If $S = T$ in Corollary 2.1, then, we obtain Theorem 1.2 of Gordji et al [4].

Remark 2 If in Theorem 2.1: (i) $\beta = \gamma = 0$ and $S = T$; $f = g = I_X$, then, we obtain Theorem of Nadler [7]; (ii) if $S = T$ and $f = g = I_X$, then, we obtain the results of Reich [8, 9].

If S and T in Corollary 2.1 are single-valued maps, then, we obtain the following result.

Corollary 2.2 Let (X, d) be a complete metric space. Let $T, S : X \to X$ be single-valued maps satisfying, for each $x, y \in X$,

$$d(Sx, Ty) \le \alpha d(x, y) + \beta[d(x, Sx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Sx)],$$

where $\alpha, \beta, \gamma \ge 0$ and $0 < \alpha + 2\beta + 2\gamma < 1$. Then, S and T have a common fixed point in X, that is, there exists $z \in X$ such that $z = Sz = Tz$.

Remark 3 If $S = T$ in Corollary 2.2, then, we obtain the result of Hardy and Rogers [5].

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