Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Journal of Functional Analysis 216 (2004) 362–387

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Power-law bounds on transfer matrices and quantum dynamics in one dimension–II

David Damanik[a,*], András Sütő[b], Serguei Tcheremchantsev[c]

[a]Department of Mathematics 253–37, California Institute of Technology, Pasadena, CA 91125, USA
[b]Research Institute for Solid State Physics, Hungarian Academy of Sciences, P.O.B. 49, H-1525 Budapest, Hungary
[c]UMR 6628 – MAPMO, Université d'Orleans, B.P. 6759, F-45067 Orleans Cédex, France

Received 16 September 2003; accepted 24 May 2004

Communicated by L. Gross

Abstract

We establish quantum dynamical lower bounds for a number of discrete one-dimensional Schrödinger operators. These dynamical bounds are derived from power-law upper bounds on the norms of transfer matrices. We develop further the approach from part I and study many examples. Particular focus is put on models with finitely or at most countably many exceptional energies for which one can prove power-law bounds on transfer matrices. The models discussed in this paper include substitution models, Sturmian models, a hierarchical model, the prime model, and a class of moderately sparse potentials.

MSC: 81Q10

Keywords: Schrödinger operators; Quantum dynamics

1. Introduction

Consider a discrete one-dimensional Schrödinger operator

$$[H_V\psi](n) = \psi(n-1) + \psi(n+1) + V(n)\psi(n) \tag{1}$$

* Corresponding author.
E-mail addresses: damanik@its.caltech.edu (D. Damanik), suto@szfki.hu (A. Sütő), serguei.tcherem@labomath.univ-orleans.fr (S. Tcheremchantsev).

0022-1236/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2004.05.007

in $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$ (with a Dirichlet boundary condition). We are interested in proving lower bounds on the spreading of an initially localized wavepacket under the dynamics governed by H_V. That is, if we consider the initial state ψ, we ask how fast $\psi(t) = \exp(-itH_V)\psi$ spreads out. One is normally interested in initial states that are well localized. In the present paper, we shall limit our attention to the case $\psi = \delta_1$.

A typical quantity that is considered to measure the spreading of $\psi(t)$ is the following: Define

$$\langle |X|_\psi^p \rangle(T) = \sum_n |n|^p a(n,T), \tag{2}$$

where

$$a(n,T) = \frac{1}{T}\int_0^{+\infty} e^{-2t/T} |\langle \delta_n, \psi(t)\rangle|^2 \, dt. \tag{3}$$

Clearly, the faster $\langle |X|_\psi^p \rangle(T)$ grows, the faster $\psi(t)$ spreads out, at least averaged in time. One typically wants to prove power-law lower bounds on $\langle |X|_\psi^p \rangle(T)$ and hence it is natural to define the following quantity: For $p > 0$, define the lower growth exponent $\beta_\psi^-(p)$ by

$$\beta_\psi^-(p) = \liminf_{T\to+\infty} \frac{\log \langle |X|_\psi^p \rangle(T)}{\log T}.$$

There are presently two distinct approaches to proving lower bounds for $\beta_\psi^-(p)$. The first goes back to works of Guarneri [13], Combes [3], and Last [24] and is based on a study of the Hausdorff dimension of the spectral measure μ_ψ associated with the pair (H,ψ). Namely, we have the following bound:

$$\beta_\psi^-(p) \geqslant p \cdot \dim_H(\mu_\psi). \tag{4}$$

The Jitomirskaya–Last extension [15,16] of Gilbert–Pearson theory [12] allows for a convenient way of investigating $\dim_H(\mu_\psi)$ and hence this approach has enjoyed some popularity (see, e.g., [5,20,35] for applications).

On the other hand, this bound clearly gives nothing in the case of a zero-dimensional spectral measure, for example, in the case of a pure point measure, there are a number of models where one expects (or can prove) pure point spectrum with strictly positive values for $\beta_\psi^-(p)$. An example is given by the random dimer model; studied, for example, in [2,11,17]. It is therefore desirable to have a way of proving lower bounds on the transport exponents which works for such models and, of course, whose input is easy to verify in concrete cases. Such an approach was developed in [8] (and employed in [17] to prove the conjectured dynamical lower bound for the random dimer model), and the present article is a continuation of that paper. The necessary input are power-law upper bounds on transfer matrices for certain energies. It may come as a surprise that dynamical bounds can be obtained if there is only one energy where one can exhibit a power-law bound for the transfer matrix. This is indeed necessary for models such as the random dimer model and related ones [7], where there are only a finite number of such energies.

Another advantage of the approach from [8] over bound (4) is the stability of its input with respect to perturbations of the potential V. It was noted in [8] that if its approach can be applied to a given model, then it can also be applied to all finitely supported perturbations of the given potential—and it gives the same dynamical bounds for the perturbed models. Such a stability is not true, in general, for bounds derived using (4). For example, it may happen that the addition of a finitely supported perturbation turns a given singular continuous spectral measure into a pure point measure; see [10] for many examples illustrating this phenomenon.

In [8], the general criterion was applied to three prominent models from one-dimensional quasicrystal theory, namely, the Fibonacci model, the period doubling model, and the Thue–Morse model. All these models can be generated by a substitution process. This allows one to study the growth of transfer matrix norms with the help of an associated dynamical system—the trace map—and this provides, in particular, a very convenient way of verifying the input to the general dynamical criterion.

In the present paper, we will prove a more general version of the dynamical result from [8], involving also the weight assigned by the spectral measure to the set of energies with power-law bounded transfer matrices. This gives stronger dynamical results in cases where such bounds hold for all energies in the spectrum, for example, models with Sturmian potentials. We shall also prove a stronger stability result. Namely, we will show that, for a fixed energy, the power-law bound is stable with respect to power-decaying perturbations. Here, the power-decay of the perturbation that we can allow depends on the transfer matrix power-law bound we start out with. Finally, we shall study a large number of examples and derive dynamical results for them by applying our main theorem, Theorem 1 below. The examples discussed in this paper include, in particular, generalizations of each of the three prominent substitution models studied in [8].

The organization of the paper is as follows. In Section 2, we prove our main theorem which derives quantum dynamical lower bounds from power-law bounds on transfer matrices. Section 3 discusses the stability of such power-law bounds on transfer matrices with respect to power-decaying perturbations of the potential. Section 4 deals with a class of models that are "sparse" in a certain sense and which includes a variety of substitution models (in particular, generalizations of Fibonacci, period doubling, and Thue–Morse), the prime Schrödinger operator, and moderately sparse models which were studied by Zlatoš [35]. The hierarchical model, which was studied in detail by Kunz et al. [23] from a spectral point of view, will then be considered in Section 5. Finally, we present results for Sturmian models (studied, e.g., in [1,5,14]; see also the reviews [4,33]) in Section 6.

2. A quantum dynamical lower bound derived from power-law transfer matrix bounds

In this section, we prove a more general version of the main result from [8]. The general idea of proof is the same and the result derives lower bounds on the dynamical quantity $\beta^-_{\delta_1}(p)$ from power-law bounds on transfer matrices. However, the result established in this section gives improved bounds in many cases, in particular, in the case of Sturmian potentials discussed later in the paper.

Recall the notion of a transfer matrix. Consider for some $E \in \mathbb{R}$, a solution ϕ of the difference equation

$$\phi(n+1) + \phi(n-1) + V(n)\phi(n) = E\phi(n). \tag{5}$$

Denote $\Phi(n) = (\phi(n+1), \phi(n))^{\mathrm{T}}$. The transfer matrix $T(n, m; E)$ is defined by requiring

$$\Phi(n) = T(n, m; E)\Phi(m)$$

for every solution ϕ of (5). It is straightforward to verify that for $n > m$

$$T(n, m; E) = T(V(n); E) \times \cdots \times T(V(m+1); E),$$

where

$$T(x; E) = \begin{pmatrix} E - x & -1 \\ 1 & 0 \end{pmatrix}$$

and similarly for $n < m$.

With this notation at hand we can now state:

Theorem 1. *The following statements hold*:

(a) *Suppose that for some* $K > 0$, $C > 0$, $\alpha > 0$, *the following condition holds*: *For any* $N > 0$ *large enough, there exists a non-empty Borel set* $A(N) \subset \mathbb{R}$ *such that* $A(N) \subset [-K, K]$ *and*

$$\|T(n, m; E)\| \leqslant CN^{\alpha} \quad \forall E \in A(N), \ \forall n, m : |n| \leqslant N, |m| \leqslant N \tag{6}$$

(*resp., with* $1 \leqslant n \leqslant N$, $1 \leqslant m \leqslant N$ *in the case of* $\ell^2(\mathbb{N})$). *Let* $N(T) = T^{1/(1+\alpha)}$ *and let, for* $j = 1, 2$, $B_j(T)$ *be the* j/T*-neighborhood of the set* $A(N(T))$:

$$B_j(T) = \{E \in \mathbb{R} : \exists E' \in A(N(T)), |E - E'| \leqslant j/T\}.$$

Denote by $F(z)$ *the Borel transform of the spectral measure of the state* $\psi = \delta_1$*:*

$$F(E + i\varepsilon) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - (E + i\varepsilon)}.$$

Then for the initial state $\psi = \delta_1$ *and all* $T > 1$ *large enough, the following bound holds with a suitable constant* $\tilde{C} > 0$*:*

$$P(T) \equiv \sum_{n:|n| \geqslant N(T)} a(n, T) \geqslant \frac{\tilde{C}}{T} N^{1-2\alpha}(T) \int_{B_2(T)} dE\,(1 + \mathrm{Im}^2 F(E + i\varepsilon)). \qquad (7)$$

In particular,

$$P(T) \geqslant \frac{\tilde{C}}{T} N^{1-2\alpha}(T)(|B_1(T)| + \mu(B_1(T))), \qquad (8)$$

where $|B|$ *denotes the Lebesgue measure. This gives the following bound for the time-averaged moments:*

$$\langle |X|^p_{\delta_1} \rangle(T) \geqslant \frac{\tilde{C}}{T} N^{p+1-2\alpha}(T)(|B_1(T)| + \mu(B_1(T))). \qquad (9)$$

(b) *Suppose that there exists a set* $A \subset [-K, K]$ *of positive measure* $\mu(A) > 0$ *such that*

$$\|T(n, m; E)\| \leqslant C(|n|^\alpha + |m|^\alpha)$$

for all $E \in A,\ n, m$*. Then*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p - 3\alpha}{1 + \alpha}. \qquad (10)$$

(c) *Assume that*

$$\|T(n, m; E_0)\| \leqslant C(E_0)(|n|^\alpha + |m|^\alpha)$$

for some E_0*, uniformly in* n, m*, then*

$$\langle |X|^p_{\delta_1} \rangle(T) \geqslant C T^{\frac{p-3\alpha}{1+\alpha}} (T^{-1} + \mu([E_0 - T^{-1}, E_0 + T^{-1}])). \qquad (11)$$

Assume moreover that E_0 *is an eigenvalue* (*possible only if* $\alpha > \frac{1}{2}$), *so that there exists* $\psi \in \ell^2, \ \psi \neq 0$ *such that* $H\psi = E_0\psi$. *Suppose that* $\psi(1) \neq 0$ (*this is always true in the case of* $\ell^2(\mathbb{N})$). *Then*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p+1-2\alpha}{1+\alpha}. \tag{12}$$

Proof. As in [8] we shall consider the case of $\ell^2(\mathbb{Z})$, because for $\ell^2(\mathbb{N})$, the proof is similar but simpler. The main part of the proof is virtually identical with that of [8]. For the sake of completeness we shall briefly recall the main lines.

The starting point is the Parseval equality:

$$a(n,T) \equiv \frac{1}{T}\int_0^\infty e^{-2t/T}|\langle\delta_n, \exp(-itH)\delta_1\rangle|^2\,dt$$
$$= \frac{\varepsilon}{2\pi}\int_{\mathbb{R}}|\langle\delta_n, R(E+i\varepsilon)\delta_1\rangle|^2\,dE,$$

where $R(z) = (H_V - zI)^{-1}$ and $\varepsilon = 1/T$. For $z = E + i\varepsilon, \ \varepsilon > 0$, we define $\phi = R(z)\delta_1, \ \Phi(n) = (\phi(n+1), \phi(n))^T$. For each $n > 1$, one has the inequality

$$||\Phi(n)|| \geqslant ||T(n,1;z)||^{-1}||\Phi(1)|| \tag{13}$$

and for each $n < 0$,

$$||\Phi(n)|| \geqslant ||T(n,0;z)||^{-1}||\Phi(0)||. \tag{14}$$

An upper bound for the norm of the transfer matrix with complex z is obtained using condition (6) and [8, Lemma 2.1]. Namely, let us fix some $T > 1, \ \varepsilon = 1/T$ and define $N \equiv N(T) = T^{1/(1+\alpha)}$. Then for every $E \in B_2(T)$ and $1 \leqslant n \leqslant N$

$$||T(n,1;E+i\varepsilon)|| \leqslant DN^\alpha, \tag{15}$$

where $D = C\exp(3C)$, and C is the constant from (6). A similar bound holds for negative values of n. Using bounds (13)–(15), one shows that for every $E \in B_2(T)$,

$$\sum_{n:|n|\geqslant N/2}|\langle\delta_n, R(E+i\varepsilon)\delta_1\rangle|^2 \geqslant cN^{1-2\alpha}(|\phi(0)|^2+|\phi(1)|^2+|\phi(2)|^2) \tag{16}$$

with uniform constant $c > 0$. It was shown in [8] that under the conditions of the theorem one always has

$$|\phi(0)| + |\phi(1)| + |\phi(2)| \geqslant c > 0$$

with uniform constant. What one can also observe (and this is a new point) is the fact that

$$\phi(1) = \langle R(z)\delta_1, \delta_1 \rangle = F(z),$$

where $F(z)$ is the Borel transform of the spectral measure corresponding to the pair (H, δ_1). Therefore, it follows from (16) that

$$\sum_{n:|n| \geqslant N/2} |\langle \delta_n, R(E+i\varepsilon)\delta_1 \rangle|^2 \geqslant cN^{1-2\alpha}(1 + \mathrm{Im}^2\, F(E+i\varepsilon)).$$

Integrating this bound over $E \in B_2(T)$, one proves (7). Next, one observes that $1 + \mathrm{Im}^2\, F(z) \geqslant 2\, \mathrm{Im}\, F(z)$. For any set S, denote by S_ε the ε-neighborhood of S. Following [19], one can see that

$$\begin{aligned} \int_{S_\varepsilon} \mathrm{Im}\, F(E+i\varepsilon)\, dE &= \int_{\mathbb{R}} d\mu(x) \int_{S_\varepsilon} \frac{\varepsilon\, dE}{(x-E)^2 + \varepsilon^2} \\ &\geqslant \int_S d\mu(x) \int_{-\varepsilon}^{\varepsilon} \frac{\varepsilon\, du}{u^2 + \varepsilon^2} \\ &= \frac{\pi}{2} \mu(S). \end{aligned}$$

Taking $S = B_1(T)$, we prove (8). Bound (9) immediately follows.

To prove part (b), one just takes $A(N) = A$ for every N. Since $\mu(B_1(T)) \geqslant \mu(A(N(T)) = \mu(A) > 0$, the result follows from bound (9).

Bound (11) of part (c) follows directly from (9), taking $A(N) = \{E_0\}$ for every N. Finally, to prove the second part of (c), we go back to (7) to obtain

$$\langle |X|_{\delta_1}^p \rangle(T) \geqslant \frac{C}{T} N^{p+1-2\alpha}(T) \int_{B_2(T)} \mathrm{Im}^2\, F(E+i\varepsilon)\, dE,$$

where $B_2(T) = [E_0 - 2\varepsilon, E_0 + 2\varepsilon]$. Under condition $\psi(1) \neq 0$, one has $\mu(\{E_0\}) > 0$. Thus,

$$\mathrm{Im}\, F(E+i\varepsilon) \geqslant \frac{c\varepsilon}{(E-E_0)^2 + \varepsilon^2}.$$

Integration over $B_2(T)$ yields (12). □

Remark. Part (b) of Theorem 1 remains true if

$$\|T(n,m;E)\| \leqslant C(E)(|n|^\alpha + |m|^\alpha) \tag{17}$$

for all n, m and $E \in A$ with $C(E) < \infty$ for μ-almost every E. To prove this, it is sufficient to take a smaller set $A' \subset A$ of positive measure where $C(E) \leqslant C < \infty$. Bound (10) should be compared with the well-known result of [15,16]: If (17) holds for some $\alpha \in [0, \frac{1}{2})$ on a set A of positive μ-measure, then the restriction of μ to A is $1 - 2\alpha$-continuous. In particular,

$$\beta^-_{\delta_1}(p) \geqslant p(1 - 2\alpha).$$

This bound is better than (10) for small p, but for p large enough, (10) is always better. Moreover, (10) holds also if $\alpha \geqslant \frac{1}{2}$.

3. Stability with respect to power-decaying perturbations

In this section, we discuss the stability of the crucial input to our dynamical bounds, power-law bounds on transfer matrices, with respect to perturbations of the potential. It is easy to see, and was noted in [8, Corollary 1.3], that finitely supported perturbations of the potential cannot destroy such a power-law bound. Here we strengthen this to stability with respect to power-decaying perturbations, where the allowed power depends on the bound we can prove for the unperturbed problem.

Theorem 2. *Assume that for some energy E and some constant C_1, the transfer matrices T associated with H_V satisfy*

$$\|T(n, m; E)\| \leqslant C_1 |n - m|^\alpha \quad \textit{for every } n, m \in \mathbb{Z} \textit{ with } nm \geqslant 0. \tag{18}$$

Assume further that, for some $\varepsilon > 0$, the perturbation W satisfies

$$|W(n)| \leqslant C_2 (1 + |n|)^{-1-2\alpha-\varepsilon} \quad \textit{for every } n \in \mathbb{Z}. \tag{19}$$

Then the transfer matrices T' associated with H_{V+W} satisfy

$$\|T'(n, m; E)\| \leqslant C_3 |n - m|^\alpha \quad \textit{for every } n, m \in \mathbb{Z} \textit{ with } nm \geqslant 0. \tag{20}$$

Proof. We present the proof in the special case where we assume (18) only for $n \geqslant 0$ and $m = 0$ and then prove (20) for $n \geqslant 0$ and $m = 0$. A slight variation of the argument below works for general $n, m \in \mathbb{Z}$ with $nm \geqslant 0$ (with a uniform constant C_3 in (20)).

Our strategy will be to work with solutions and employ a general perturbation method developed by Kiselev et al. [21].

Consider the unperturbed equation (5) and the perturbed equation

$$\psi(n + 1) + \psi(n - 1) + [V(n) + W(n)]\psi(n) = E\psi(n). \tag{21}$$

Note that the transfer matrix $T'(n,0;E)$ is given by

$$T'(n,0;E) = \begin{pmatrix} \psi_{\rm D}(n+1) & \psi_{\rm N}(n+1) \\ \psi_{\rm D}(n) & \psi_{\rm N}(n) \end{pmatrix},$$

where $\psi_{\rm D}, \psi_{\rm N}$ solve (21) and obey

$$\begin{pmatrix} \psi_{\rm D}(1) & \psi_{\rm N}(1) \\ \psi_{\rm D}(0) & \psi_{\rm N}(0) \end{pmatrix} = I.$$

Fix a complex reference solution ϕ of (5). For example, we could set $\phi = \phi_{\rm D} + i\phi_{\rm N}$, where $\phi_{\rm D}, \phi_{\rm N}$ solve (5) and have the same initial conditions as $\psi_{\rm D}, \psi_{\rm N}$. By (18) we have

$$|\phi(n)| \leqslant C|n|^{\alpha}. \tag{22}$$

Let ψ be one of the basic solutions $\psi_{\rm D}, \psi_{\rm N}$ of (21). Define $\rho(n)$ by

$$\begin{aligned} \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix} &= \frac{1}{2i}\left[\rho(n)\begin{pmatrix} \phi(n) \\ \phi(n-1) \end{pmatrix} - \overline{\rho(n)}\begin{pmatrix} \overline{\phi(n)} \\ \overline{\phi(n-1)} \end{pmatrix}\right] \\ &= \operatorname{Im}\left[\rho(n)\begin{pmatrix} \phi(n) \\ \phi(n-1) \end{pmatrix}\right]. \end{aligned}$$

Write $\phi(n)$ and $\rho(n)$ in polar coordinates,

$$\phi(n) = |\phi(n)|e^{i\gamma(n)}, \quad \rho(n) = R(n)e^{i\eta(n)}$$

and define

$$\theta(n) = \eta(n) + \gamma(n) \quad \text{and} \quad U(n) = -\frac{2W(n)}{\omega}|\phi(n)|^2,$$

where $i\omega$ is the Wronskian of $\overline{\phi}$ and ϕ, that is,

$$2i\,\operatorname{Im}(\phi(n+1)\overline{\phi(n)}) = i\omega \text{ for every } n.$$

Clearly, the assertion of the theorem follows if we can show that $R(n)$ remains bounded as $|n| \to \infty$. The key identity [21, Eq. (45)] is the following:

$$R(n+1)^2 = R(n)^2[1 + U(n)\sin(2\theta(n)) + U(n)^2\sin^2(\theta(n))]. \tag{23}$$

It follows from (19) and (22) that $U(n)$ is summable. Thus, boundedness of $R(n)$ follows from this and (23) (cf., e.g., [20, Lemma 3.5]). This concludes the proof. □

The theorem above implies the stability of the number α and of the sets $A(N)$, $B_1(T)$, A under suitable power-decaying perturbations of the potential. On the other hand, the measure of the sets $\mu(B_1(T))$, $\mu(A)$ and the Borel transform $F(z)$ may change after such a perturbation. In particular, it is possible that $\mu(A) = 0$ for the perturbed operator in part (b) of Theorem 1. Thus, bounds (10) and (12) are in general not stable. Of course, we still get a dynamical bound for the perturbed model. For example, we have the following consequence of Theorems 1 and 2.

Corollary 3.1. *Assume that for some energy E_0 and some constant C_1, the transfer matrices T associated with H_V satisfy $\|T(n,m;E_0)\| \leqslant C_1|n-m|^\alpha$ for every $n,m \in \mathbb{Z}$ with $nm \geqslant 0$. Assume further that, for some $\varepsilon > 0$, the perturbation W satisfies $|W(n)| \leqslant C_2|n|^{-1-2\alpha-\varepsilon}$ for every $n \in \mathbb{Z}$. Then we have for the operator H_{V+W},*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-1-4\alpha}{1+\alpha}$$

for every $p > 0$.

Proof. By Theorem 2, we have that the transfer matrices T' associated with H_{V+W} satisfy $\|T'(n,m;E_0)\| \leqslant C|n-m|^\alpha$ for every $n,m \in \mathbb{Z}$ with $nm \geqslant 0$. Then, an inspection of the proof of Theorem 1 shows that this suffices to prove bound (11) which yields

$$\langle |X|^p_{\delta_1} \rangle (T) \geqslant C T^{\frac{p-3\alpha}{1+\alpha}-1}$$

and the assertion of the corollary follows. More precisely, one can work independently on the two half-lines and hence needs bounds on $\|T'(n,m;E)\|$ only for the case where n,m have the same sign. □

4. A class of pseudo-sparse potentials

In this section, we study a class of "sparse" potentials which includes various substitution models and the prime model. These potentials are not all sparse in the standard sense, but the point is that the class we discuss contains sparse potentials, and also a number of other potentials that have been considered before and which can be studied within the same framework.

Let us consider the case where the potential V is defined on the half-line $\mathbb{N}$ and takes on two values $a, b \in \mathbb{R}$. We assume the following for n large enough, that is, for $n \geqslant N$:

(S1) Occurrences of b are always isolated, that is, if $V(n) = b$ for some n, then $V(n-1) = V(n+1) = a$.
(S2) The value a always occurs with odd multiplicity, that is, if $V(n) = V(n+k+1) = b$ and $V(n+j) = a$, $1 \leqslant j \leqslant k$, then k is odd.

Sparseness in this context refers to the b's being isolated and the results below holding for arbitrarily long gaps between consecutive b's. However, some of the concrete applications—for example the applications to substitution models—will not be sparse in a traditional sense.

We can prove the following.

Theorem 3. *Suppose* $V : \mathbb{N} \to \{a, b\} \subset \mathbb{R}$ *is a potential satisfying* (S1) *and* (S2) *above. We have for every* $p > 0$,

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2}.$$

Proof. Up to an initial piece, the transfer matrices are given by products of matrices of the following form:

$$T(a, E)^{2l+1} \quad \text{and} \quad T(b, E).$$

Let $E_0 = a$. Then

$$T(a, E_0)^{2l+1} = (T(a, E_0)^2)^l T(a, E_0) = (-I)^l T(a, E_0) = \pm T(a, E_0).$$

Up to sign, this gives rise to powers of

$$T(a, E_0) T(b, E_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a-b & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ a-b & -1 \end{pmatrix}.$$

Clearly, such powers satisfy a bound which is linear in the number of factors. Thus, the claim follows from (11). $\square$

Remark. We can apply Corollary 3.1 and obtain that the dynamical bound in Theorem 3 is stable with respect to perturbations W obeying $|W(n)| \leqslant C_2 n^{-3-\varepsilon}$ for some fixed $\varepsilon > 0$ and every $n \in \mathbb{N}$. Similarly, we have stability with respect to power-decaying perturbations for all the dynamical bounds that will be shown in this section and we will not make this explicit for each one of them.

Let us now discuss the case where the a's occur with even multiplicities. That is, we assume for n large enough,

(S3) The value a always occurs with even multiplicity, that is, if $V(n) = V(n+k+1) = b$ and $V(n+j) = a$, $1 \leqslant j \leqslant k$, then k is even.

In this case we can prove a dynamical bound even without assuming the sparseness condition (S1). However, we need that $|a-b|$ is not too large. Namely, we have the following result:

Theorem 4. *Suppose* $V : \mathbb{N} \to \{a, b\} \subset \mathbb{R}$ *is a potential satisfying* (S3) *above.*

(a) *If* $|a-b| < 2$, *then for every* $p > 0$,

$$\beta^-_{\delta_1}(p) \geqslant p - 1.$$

(b) *If* $|a-b| = 2$, *then for every* $p > 0$,

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2}.$$

Proof. The argument proceeds in a way similar to the proof above. Again, up to an initial piece, the transfer matrices are given by products of matrices of the following form:

$$T(a, E)^{2l} \quad \text{and} \quad T(b, E).$$

Again, let $E_0 = a$. Then

$$T(a, E_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and hence

$$T(a, E_0)^{2l} = (T(a, E_0)^2)^l = (-I)^l = \pm I.$$

On the other hand, $T(b, E_0)$ is elliptic when $|a-b| < 2$ and parabolic when $|a-b| = 2$. Thus, in the former case, products of matrices of the form $T(a, E)^{2l}$ or $T(b, E)$ remain bounded, while in the latter case such products satisfy a bound which is linear in the number of factors. The claim thus follows from (11). □

Let us note that a result like part (a) of Theorem 4 is implicitly contained in [17], where mainly random polymer models are studied.

It is clear that whole-line analogs of the above theorems hold. In this case, we need (S1) and (S2) or (S3) to hold for $|n|$ large enough.

More importantly, these results cover a variety of seemingly very different cases: First consider the period doubling Hamiltonian, which was already discussed in [8]. On the alphabet $A = \{a, b\} \subseteq \mathbb{R}$, consider the period doubling substitution $S(a) = ab$, $S(b) = aa$. Iterating on a, we obtain a one-sided sequence

$$u = abaaabababaaabaaab\ldots$$

which is invariant under the substitution process. Define the associated subshift Ω_{pd} to be the set of all sequences over A which have all their finite subwords occurring in u. Here, we can consider either one- or two-sided sequences. This does not matter for the results in this paper, but we remark that for substitution models, one generally considers the two-sided case. For $\omega \in \Omega_{\text{pd}}$, we define the potential V_ω by $V_\omega(n) = \omega_n$. It is easy to check that each V_ω satisfies (S1) and (S2) (even for every $n \in \mathbb{Z}$) and hence an application of Theorem 3 allows us to recover [8, Theorem 3]. However, we can prove a more general result. Consider, for example, substitutions of the form

$$S(a) = a^{2k-1}b, \quad S(b) = a^{2l}, \quad k, l \geqslant 1. \tag{24}$$

The case $k = 1, l = 1$ corresponds to the period doubling case. The potentials generated by a substitution of form (24) (by generating a one-sided fixed point and passing to the associated subshift, as in the period doubling case above) are easily seen to obey (S1) and (S2). On the other hand, substitutions of the form

$$S(a) = a^{2k}b, \quad S(b) = a^{2l}, \quad k, l \geqslant 1 \tag{25}$$

give rise to potentials satisfying (S3) and hence Theorem 4 applies in these cases. Thus we may state the following:

Corollary 4.1. (a) *Let S be a substitution of form (24), Ω the associated subshift, and for $\omega \in \Omega$, let $V_\omega(n) = \omega_n$, $n \in \mathbb{Z}$. Then, for every $\omega \in \Omega$, the potential V_ω gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \textit{for every } p > 0.$$

(b) *Let S be a substitution of form (25), Ω the associated subshift, and for $\omega \in \Omega$, let $V_\omega(n) = \omega_n$, $n \in \mathbb{Z}$. Then, for every $\omega \in \Omega$, the potential V_ω gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant p - 1 \quad \textit{for every } p > 0 \textit{ if } |a-b| < 2$$

and

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \textit{for every } p > 0 \textit{ if } |a-b| = 2.$$

Consider the following class of substitutions:

$$S(a) = a^m b^n, \quad S(b) = a. \tag{26}$$

The case $m = n = 1$ gives rise to the Fibonacci substitution. Hence, the substitutions in (26) are usually called generalized Fibonacci substitutions. If $n = 1$, the resulting potentials are Sturmian and will be discussed in this more general context in a later section. Here, we restrict our attention to the case $n \geqslant 2$. These substitutions and the associated Schrödinger operators were studied, for example, in [22,32,34].

If n is even, it is easily seen that each V_ω satisfies (S3) with the roles of a and b interchanged, that is, b's always occur with even multiplicity. Thus, we can derive a dynamical bound for the associated operators by applying Theorem 4.

If n is odd, the model satisfies neither (S2) nor (S3) but we can nevertheless employ a similar argument. As a warmup, let us consider the case $n = 3$ (the special case $m = 1$, $n = 3$ is usually called the nickel mean substitution). Then the transfer matrices are given by products of matrices of the following form:

$$T(a,E) \quad \text{and} \quad T(b,E)^3.$$

Let $E_0 = b + 1$. Then

$$T(b, E_0) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and hence

$$T(b, E_0)^3 = -I.$$

This would allow us to prove bounds on $\beta^-_{\delta_1}(p)$ in the same way as in the proof of Theorem 4.

Let us now turn to the case of a general odd $n \geqslant 3$. Here, we can extend the above idea and prove a result which applies to the substitutions in (26) with n odd but which is much more general. Denote

(S4) There is some odd $k \geqslant 3$ such that the value b always occurs with a multiplicity which is a multiple of k, that is, if $V(n) = V(n+l+1) = a$ and $V(n+j) = b$, $1 \leqslant j \leqslant l$, then $l = mk$ for some $m \in \mathbb{N}$.

Then, we can prove the following.

Theorem 5. *Suppose* $V : \mathbb{N} \to \{a, b\} \subset \mathbb{R}$ *is a potential satisfying* (S4). *Then there is a set* $\mathcal{E} \subset \mathbb{R}$ *of cardinality* $k - 1$ *such that for every* $E \in \mathcal{E}$, we have

(a) *If* $|a - E| < 2$, *then for every* $p > 0$,

$$\beta^-_{\delta_1}(p) \geqslant p - 1.$$

(b) *If* $|a - E| = 2$, *then for every* $p > 0$,

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2}.$$

Proof. In this case, the transfer matrices are given by products of matrices of the following form:

$$T(a, E) \quad \text{and} \quad T(b, E)^k.$$

It suffices to exhibit $k - 1$ energies E_0 with

$$T(b, E_0)^k = \pm I. \tag{27}$$

This can be seen as follows: The matrix $T(b, E)^k$ is the monodromy matrix of the constant potential $V(n) = b$, regarded as a k-periodic potential. This gives rise to an operator with $k - 1$ gaps. However, since the operator with this potential has spectrum $[b - 2, b + 2]$, all these gaps are degenerate. Every degenerate gap corresponds to an energy where the monodromy matrix is equal to $\pm I$, hence there are exactly $k - 1$ energies E_0 for which we have (27). □

Putting everything together, we obtain the following result for the models generated by substitutions from (26):

Corollary 4.2. *Let S be a substitution of form* (26), Ω *and the* V_ω*'s as above.*

(a) *If* $n \geqslant 2$ *is even, then for every* $\omega \in \Omega$, *the potential* V_ω *gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant p - 1 \text{ for every } p > 0 \text{ if } |a - b| < 2$$

and

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \text{for every } p > 0 \text{ if } |a - b| = 2.$$

(b) *If* $n \geqslant 3$ *odd, then* $T(b, E_0)^n = \pm I$ *has* $n-1$ *solutions* $E_0 \in \mathbb{R}$ *and for each such solution* E_0*, we have that for every* $\omega \in \Omega$*, the potential* V_ω *gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant p - 1 \quad \text{for every } p > 0 \text{ if } |a - E_0| < 2$$

and

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \text{for every } p > 0 \text{ if } |a - E_0| = 2.$$

The final substitution model we consider is the following:

$$S(a) = a^m b^n, \quad S(b) = b^n a^m. \tag{28}$$

The case $m = n = 1$ gives rise to the Thue–Morse substitution. Hence, the substitutions in (28) are usually called generalized Thue–Morse substitutions. They were considered, for example, in [34]. If at least one of m, n is even, (S3) holds and we can apply Theorem 4. In the remaining case, where both m and n are odd (and at least one is $\geqslant 3$), (S4) holds and we can apply Theorem 5. Thus, for models generated by generalized Thue–Morse substitutions, we obtain the following dynamical bounds:

Corollary 4.3. *Let* S *be a substitution of form* (28), Ω *and the* V_ω*'s as above.*

(a) *If at least one of* m, n *is even, then for every* $\omega \in \Omega$*, the potential* V_ω *gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant p - 1 \quad \text{for every } p > 0 \text{ if } |a - b| < 2$$

and

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \text{for every } p > 0 \text{ if } |a - b| = 2.$$

(b) *If we have* $m \geqslant 3$ *odd, then* $T(b, E_0)^m = \pm I$ *has* $m-1$ *solutions* $E_0 \in \mathbb{R}$ *and for each such solution* E_0*, we have that for every* $\omega \in \Omega$*, the potential* V_ω *gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant p - 1 \quad \text{for every } p > 0 \quad \text{if } |b - E_0| < 2$$

and

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \text{ for every } p > 0 \text{ if } |b - E_0| = 2.$$

An analogous result holds if we have $n \geqslant 3$ *odd.*

(c) *If* $m = n = 1$, *then*

$$\beta^-_{\delta_1}(p) \geqslant p - 1 \text{ for every } p > 0.$$

Part (c) was proved in [8] and is stated for completeness. One might expect the bound $\beta^-_{\delta_1}(p) \geqslant p - 1$ to hold always. In fact, paper [34] claims, for every choice of m, n, a, b, the existence of an energy, where the transfer matrices remain bounded. However, the argument given in that paper is incomplete and it would be interesting to prove or disprove this claim.

Next, we consider the prime Schrödinger operator H_{prime} on $\ell^2(\mathbb{N})$ whose potential is given by

$$V_{\text{prime}}(n) = \begin{cases} a & \text{if } n \text{ is not prime}, \\ b & \text{if } n \text{ is prime}. \end{cases}$$

This operator was studied, for example, in [9,30]. Based on numerics and heuristics contained in these two papers, one may expect the following: On the one hand, for almost every energy E, there is an ℓ^2 solution to $H_{\text{prime}}\phi = E\phi$, that is, when one varies the boundary condition at the origin, one gets pure point spectrum for almost every boundary condition. On the other hand, the model displays non-trivial transport for every boundary condition. We will confirm the latter below (the proof discusses only the case of a Dirichlet boundary condition, but it readily extends to every other boundary condition). Let us briefly discuss the first point. It is natural to view V_{prime} as a sparse potential. In fact, this point of view was proposed in [9]. However, the current methods in the spectral analysis of models with sparse potentials (see, in particular, [20,28]) are clearly insufficient to conclude anything for the prime model. We regard this as an interesting problem and refer the reader also to [29] for further motivation to consider models of moderate sparseness.

Let us now turn to a dynamical result for the prime model. Clearly, (S1) and (S2) are satisfied for n large enough. Hence, we get:

Corollary 4.4. *For every* $a, b \in \mathbb{R}$, *the operator* H_{prime} *satisfies*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \text{for every } p > 0.$$

Finally, we discuss a model which is sparse in the standard sense. Namely, pick some integer $\gamma \geqslant 2$ and define $n_k = \gamma^k$ for $k \in \mathbb{N}$. Let $V_{\text{sparse}}(n) = b$ if $n = n_k$ for some k and $V_{\text{sparse}}(n) = a$ otherwise. Schrödinger operators with potentials of this kind were studied in [35]. Clearly, when γ is even, all n_k's are even, and when γ is odd, all n_k's are odd, so we have (S1) and (S2). Thus, Theorem 3 applies and we get

Corollary 4.5. *For every $a, b \in \mathbb{R}$ and $\gamma \in \mathbb{N} \setminus \{1\}$, the potential V_{sparse} gives rise to an operator satisfying*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-5}{2} \quad \textit{for every } p > 0.$$

This can be improved if $\gamma \gg e|a-b|$:

Proposition 4.6. *Let*

$$v = \frac{2\log\sqrt{2+(a-b)^2}}{\log\gamma}.$$

Then the potential V_{sparse} gives rise to an operator satisfying

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-1-4v}{1+v} \quad \textit{for every } p > 0.$$

Proof. Write $C(a,b) = \sqrt{2+(a-b)^2}$. Then

$$\|T(a, E=a)^{2l+1}T(b, E=a)\| = \left\| \begin{pmatrix} -1 & 0 \\ a-b & -1 \end{pmatrix} \right\| \leqslant C(a,b).$$

For $d_{n,m} = \#\{m \leqslant k \leqslant n : V(k) = b\}$, we have $d_{n,m} \leqslant \log|n-m|/\log\gamma$ and hence

$$\|T(n,m; E=a)\| \leqslant C(a,b)^{d_{n,m}} \leqslant C(a,b)^{\log|n-m|/\log\gamma} = |n-m|^{\log C(a,b)/\log\gamma}.$$

This yields the assertion. □

5. A hierarchical model

The hierarchical model is defined through the potential

$$V(n) = \lambda f(\operatorname{ord} n), \tag{29}$$

where f is some real function and $\operatorname{ord} n$ is the number of factors 2 in the prime decomposition of n. Sequence (29) has some nice symmetries. Because $\operatorname{ord}(-n) = \operatorname{ord} n$ for all n and $\operatorname{ord}(l \cdot 2^m + k) = \operatorname{ord} k$ for $m \geqslant 1$, all l and $|k| < 2^m$, analogous identities hold for V. In particular,

$$V(l \cdot 2^m + k) = V(k) = V(-k) = V(l' \cdot 2^m - k) \tag{30}$$

for any l and l', $m \geqslant 1$ and $|k| < 2^m$. The Schrödinger operator with such a potential appeared first in works [26] and [31] with the special choice

$$f(m) = \sum_{k=0}^{m-1} R^k,$$

where R is a positive constant. The advantage of this choice is that in this case,

$$x_m = \mathrm{tr}\, M_m(E) \equiv \mathrm{tr}\, T(2^m, 0; E)$$

satisfies an autonomous difference equation [31],

$$x_{m+1} = x_m^2 - 2 + R x_m (x_m - x_{m-1}^2 + 2), \quad m \geqslant 1. \tag{31}$$

The above recurrence and symmetries (30) made it possible to obtain many rigorous results about the spectrum of the corresponding Schrödinger operator. A detailed mathematical study of this model was carried out by Kunz et al. [23]. Among other things, it was shown that for every $R > 0$, the spectrum is a Cantor set, and for $R \geqslant 1$, it is purely singular continuous. From the point of view of the present article, it is interesting that a countable infinite set of exceptional energies in the spectrum could be identified explicitly. The 2^m zeros E_{mk}, $1 \leqslant k \leqslant 2^m$, of $x_m(E)$ are simple and $x_m = 0$ implies $x_{m+1} = -2$ and $x_{m+l} = 2$ for $l > 1$; compare (31). From this it was possible to show that E_{mk}, for $m \geqslant 0$ and $1 \leqslant k \leqslant 2^m$, are lower (resp., upper) gap-edges in the spectrum of H_V if $\lambda > 0$ (resp., $\lambda < 0$) and they are dense in the spectrum. For the corresponding gap-edge states, the following result was obtained [23, Proposition 15].

Proposition 5.1. *Let $x_m(E) = 0$ and let ψ be a solution of $H\psi = E\psi$.*

(i) *If $\psi(0) = 0$, then $\psi(k + 2^{m+1}) = -\psi(k)$ for every integer k.*
(ii) *If $\psi(0) \neq 0$, then $\psi(2l \cdot 2^m) = (-1)^l \psi(0)$ and asymptotically, as $l \to \infty$,*

$$\psi((2l+1)2^m) - \psi(2^m) \asymp (-1)^{l+1} \lambda_m \psi(0) f_R(l) \tag{32}$$

where

$$f_R(l) = \begin{cases} \frac{2}{2-R} l, & R < 2 \\ l \cdot \log_2 l, & R = 2 \\ \left(\frac{2}{R}\right)^{\varepsilon_l} \frac{R^2}{2(R-1)(R-2)} l^{\log_2 R} & R > 2. \end{cases} \tag{33}$$

Here $\lambda_m = \lambda R^m x_{m-1}(E) \cdots x_0(E)$, $\varepsilon_l \in [0, 1)$ is the fractional part of $\log_2 l$ and $\asymp$ means equality in the leading order of l.

We use this proposition to prove the following theorem.

Theorem 6. *For every* $\lambda \neq 0$ *and* $R > 0$,

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-1-4\alpha}{1+\alpha},$$

where

$$\alpha = \alpha(R) = \max\{1, \log_2 R\}.$$

Proof. We apply Proposition 5.1 with $m = 0$ for which it provides the precise asymptotic form of the solutions. Because $x_0(E) = E$, these belong to $E = 0$. Let ψ_{D} and ψ_{N} be the two solutions defined by the initial values

$$\psi_{\mathrm{D}}(0) = \psi_{\mathrm{N}}(1) = 0, \quad \psi_{\mathrm{D}}(1) = \psi_{\mathrm{N}}(0) = 1. \tag{34}$$

According to part (i) of Proposition 5.1, ψ_{D} is a periodic solution with period 4, namely

$$\psi_{\mathrm{D}}(2l) = 0, \quad \psi_{\mathrm{D}}(2l+1) = (-1)^l. \tag{35}$$

On the other hand,

$$\psi_{\mathrm{N}}(2l) = (-1)^l, \quad \psi_{\mathrm{N}}(2l+1) \asymp (-1)^{l+1}\lambda f_R(l). \tag{36}$$

Eqs. (35) and (36) permit us to compute the asymptotic form of $T(n,m;0)$. Because of $V(-n) = V(n)$, it suffices to consider $n \geqslant m \geqslant 0$. In what follows, we use the simplified notation $T(n,m)$. Let $\Psi^i(n) = (\psi^i(n+1) \ \ \psi^i(n))^{\mathrm{T}}$ for $i = 0,1$. Then $T(n,0) = (\Psi_{\mathrm{D}}(n) \ \ \Psi_{\mathrm{N}}(n))$. The determinant of any transfer matrix being unity, the inverse is easy to compute. We find

$$T(n,m) = T(n,0)T(m,0)^{-1} \tag{37}$$

$$= \begin{pmatrix} \psi_{\mathrm{D}}(n+1) & \psi_{\mathrm{N}}(n+1) \\ \psi_{\mathrm{D}}(n) & \psi_{\mathrm{N}}(n) \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{N}}(m) & -\psi_{\mathrm{N}}(m+1) \\ -\psi_{\mathrm{D}}(m) & \psi_{\mathrm{D}}(m+1) \end{pmatrix}. \tag{38}$$

With the short-hand notation

$$F(l) = (-1)^l \psi_{\mathrm{N}}(2l+1),$$

Eqs. (35), (36), and (38) then yield

$$T(2l,2k) = (-1)^{k+l} \begin{pmatrix} 1 & F(l) - F(k) \\ 0 & 1 \end{pmatrix},$$

$$T(2l+1,2k) = (-1)^{k+l}\begin{pmatrix} 0 & -1 \\ 1 & F(l)-F(k) \end{pmatrix},$$

$$T(2l,2k+1) = (-1)^{k+l+1}\begin{pmatrix} F(l)-F(k) & -1 \\ 1 & 0 \end{pmatrix},$$

$$T(2l+1,2k+1) = (-1)^{k+l+1}\begin{pmatrix} -1 & 0 \\ F(l)-F(k) & -1 \end{pmatrix}. \tag{39}$$

All these matrices have the same norm. Denoting the Hilbert–Schmidt norm by $\|\cdot\|_2$, for $n = 2l, 2l+1$ and $m = 2k, 2k+1$, we have

$$\|T(n,m)\| \leqslant \|T(n,m)\|_2 = \sqrt{2 + [F(l)-F(k)]^2} \asymp \sqrt{2 + \lambda^2 [f_R(l) - f_R(k)]^2}.$$

Therefore,

$$\|T(n,m;0)\| \leqslant 2\lambda f_R(n/2)$$

for any n large enough and $m \leqslant n$. If $R \neq 2$, the assertion of the theorem obviously follows from the definition (33) of f_R and Theorem 1. If $R = 2$, we note that for any $\varepsilon > 0$,

$$\|T(n,m;0)\| \leqslant \lambda n^{1+\varepsilon}$$

if n is large enough. Therefore, by Theorem 1,

$$\beta^-_{\delta_1}(p) \geqslant \frac{p - 5 - 4\varepsilon}{2+\varepsilon}$$

for any $\varepsilon > 0$ and, thus, for $\varepsilon = 0$ as well. □

Remark. The proof shows that we can apply Corollary 3.1 and obtain that the dynamical bound in Theorem 6 is stable with respect to perturbations W obeying $|W(n)| \leqslant C_2 |n|^{-1-2\alpha-\varepsilon}$ for some fixed $\varepsilon > 0$ and every $n \in \mathbb{Z}$.

We note that instead of $m = 0$, we could have used Proposition 5.1 with any $m > 0$ and any zero of $x_m(E)$. This holds because of the following:

Theorem 7. *For any $\lambda \neq 0$, $R > 0$, $m \geqslant 0$, and $k \in \{1, 2, \ldots, 2^m\}$, there exists a positive number $C_{\lambda,R}(m, E_{mk})$ such that for any $n \geqslant n' \geqslant 0$,*

$$\|T(n,n';E_{mk})\| \leqslant C_{\lambda,R}(m,E_{mk}) f_R(2^{-m-1}n).$$

Proof. We fix $m > 0$ and a zero E_{mk} of x_m. From Eq. (38) it is clear that we have to bound the two particular solutions (34) of $H_V\psi = E_{mk}\psi$. According to Proposition 5.1, ψ_{D} is 2^{m+1}-antiperiodic and, thus, bounded. On the other hand,

$$\psi_{\mathrm{N}}(2l \cdot 2^m) = (-1)^l, \quad \psi_{\mathrm{N}}((2l+1)2^m) - \psi_{\mathrm{N}}(2^m) \asymp (-1)^{l+1}\lambda_m f_R(l). \tag{40}$$

Thus, the task is to bound $\psi_{\mathrm{N}}(n)$ in the intervals

$$2l \cdot 2^m < n < (2l+1)2^m \quad \text{and} \quad (2l+1)2^m < n < 2(l+1)2^m. \tag{41}$$

To proceed with the proof, let us recall Eq. (3.29) of [23], according to which

$$\psi_{\mathrm{D}}(2^m) = x_{m-1} \cdots x_0$$

for any energy. Thus, $\psi_{\mathrm{D}}(2^m) \neq 0$ in the present case ($E = E_{mk}$), for otherwise $x_i = 0$ for some $i < m$ would imply $|x_j| = 2$ for every $j > i$, contradicting $x_m = 0$. Then $u_0 := \psi_{\mathrm{D}}/\psi_{\mathrm{D}}(2^m)$ is a solution of the Schrödinger equation satisfying the boundary conditions $u_0(0) = 0$, $u_0(2^m) = 1$ and, according to Proposition 5.1, $u_0(k + 2^{m+1}) = -u_0(k)$ for any k. From the general theory of second-order difference (differential) equations, it follows that there exists a linearly independent solution u_1 with boundary values $u_1(1) = -1$, $u_1(2^m) = 0$ and that we can write ψ_{N} for $0 \leqslant n \leqslant 2^m$ in the form

$$\psi_{\mathrm{N}}(n) = \psi_{\mathrm{N}}(2^m)u_0(n) + \psi_{\mathrm{N}}(0)u_1(n).$$

Next, we observe that u_1 can be expressed in terms of u_0. Indeed, from Eq. (30) we can see that the sequence $V(1), \ldots, V(2^m - 1)$ is a palindrome,

$$V(2^{m-1} - k) = V(2^{m-1} + k), \quad k = 1, \ldots, 2^{m-1} - 1$$

and, hence,

$$u_1(n) = u_0(2^m - n), \quad n = 1, \ldots, 2^m - 1.$$

Furthermore, the translational symmetry of the potential,

$$(V(l \cdot 2^m + 1), \ldots, V((l+1)2^m - 1)) = (V(1), \ldots, V(2^m - 1)),$$

valid for any l, implies that the translates of u_0 and u_1 can be used to give ψ_{N} in each of intervals (41). Altogether we find

$$\psi_{\mathrm{N}}(n) = \psi_{\mathrm{N}}((2l+1)2^m)u_0(n - 2l \cdot 2^m) + \psi_{\mathrm{N}}(2l \cdot 2^m)u_0((2l+1)2^m - n)$$

if $2l \cdot 2^m \leqslant n \leqslant (2l+1)2^m$ and

$$\psi_{\mathrm{N}}(n) = \psi_{\mathrm{N}}(2(l+1)2^m)u_0(n-(2l+1)2^m) \\ +\psi_{\mathrm{N}}((2l+1)2^m)u_0(2(l+1)2^m - n)$$

if $(2l+1)2^m \leqslant n \leqslant 2(l+1)2^m$. Together with (40), in both intervals,

$$|\psi_{\mathrm{N}}(n)| \leqslant \frac{\max|\psi_{\mathrm{D}}|}{|\psi_{\mathrm{D}}(2^m)|}(|\psi_{\mathrm{N}}((2l+1)2^m)|+1).$$

Since $l \leqslant n/2^{m+1}$, we obtain that for n large enough

$$|\psi_{\mathrm{N}}(n)| \leqslant \frac{\max|\psi_{\mathrm{D}}|}{|\psi_{\mathrm{D}}(2^m)|}(|\lambda_m| f_R(2^{-m-1}n) + |\psi_{\mathrm{N}}(2^m)|+1).$$

Due to (38), the assertion of the theorem follows from this bound. □

6. Sturmian potentials

In this section, we discuss dynamical bounds for the standard one-dimensional quasicrystal model which is given by a Schrödinger operator on the whole line whose potential is given by

$$V(n) = \lambda v_{\omega,\theta}(n), \text{ where } v_\theta(n) = \chi_{[1-\omega,1)}(n\omega + \theta \bmod 1), \tag{42}$$

where $\lambda \neq 0$ is the coupling constant, $\omega \in (0,1)$ irrational is the rotation number, and $\theta \in [0,1)$ arbitrary is the phase. For more information on this family of operators, we refer the reader to the survey articles [4,33].

It is well known, and easy to see, that the spectrum of the operator $H_{\lambda,\omega,\theta}$ with potential V from (42) is independent of θ, that is, for every λ, ω, there is a set $\Sigma_{\lambda,\omega}$ with $\sigma(H_{\lambda,\omega,\theta}) = \Sigma_{\lambda,\omega}$ for every θ.

Consider the continued fraction expansion of ω,

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with uniquely determined $a_n \in \mathbb{N}$ (cf. [18]). The associated rational approximants p_k/q_k are defined by

$$p_0 = 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2},$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The number ω is said to have bounded density if

$$d(\omega) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k < \infty. \tag{43}$$

The set of bounded density numbers is uncountable but has Lebesgue measure zero. The following was shown in [6] (see also [14] for the case of zero phase):

Theorem 8. *Suppose ω is a bounded density number. For every λ, there is a constant C such that for every θ, every $E \in \Sigma_{\lambda,\omega}$, and every $n, m \in \mathbb{Z}$, we have*

$$\|T_{\lambda,\omega,\theta}(n, m; E)\| \leqslant C|n - m|^{\alpha(\lambda,\omega)}, \tag{44}$$

with

$$\alpha(\lambda, \omega) = D \cdot d(\omega) \cdot \log C_\lambda, \tag{45}$$

where D is some universal constant, C_λ is given by

$$C_\lambda = 2 + \sqrt{8 + \lambda^2} \tag{46}$$

and $d(\omega)$ is as in (43).

This yields the following.

Corollary 6.1. *Let ω be a bounded density number. Then, for every λ, θ, the operator $H_{\lambda,\omega,\theta}$ satisfies*

$$\beta^-_{\delta_1}(p) \geqslant \frac{p - 3\alpha(\lambda, \omega)}{1 + \alpha(\lambda, \omega)} \quad \text{for every } p > 0,$$

with $\alpha(\lambda, \omega)$ given by (45).

Since $\mu(\Sigma_{\lambda,\omega}) = 1$, this is an immediate consequence of (10). This bound is better than the corresponding result in [8] (which follows from (9), bounding from below $|B_1(T)|$). One should stress that as opposed to all the other examples discussed earlier, the dynamical bound in Corollary 6.1 is not stable with respect to perturbations of the potential. This is due to the fact that $\mu(\Sigma_{\lambda,\omega})$ may vanish for the perturbed measure. However, by Corollary 3.1, we have the following result:

Corollary 6.2. *Let ω be a bounded density number and let λ be arbitrary. If $\alpha(\lambda,\omega)$ is given by* (45) *and W satisfies*

$$|W(n)| \leqslant C_2(1+|n|)^{-1-2\alpha(\lambda,\omega)-\varepsilon} \quad \text{for every } n \in \mathbb{Z}$$

for some $\varepsilon > 0$, then, for every θ, the operator $H_{\lambda,\omega,\theta} + W$ satisfies

$$\beta^-_{\delta_1}(p) \geqslant \frac{p-1-4\alpha(\lambda,\omega)}{1+\alpha(\lambda,\omega)} \quad \text{for every } p > 0.$$

As in the case $\omega = (\sqrt{5}-1)/2$ and $\theta = 0$, studied in [8], it is possible to improve this lower bound somewhat by exhibiting a suitable set $A(N)$ (stable under perturbation), studying its Lebesgue measure, and applying (9). The set $A(N)$ will again be given by the spectra of suitable periodic approximants, and the Lebesgue measure can again be bounded through a fine analysis of the trace map, akin to what is done in [8,19,27]; compare also [25]. We leave the details to the interested reader.

Acknowledgments

D.D. was supported in part by NSF Grant DMS-0227289. A.S. acknowledges support by OTKA through Grant T 042914.

References

[1] J. Bellissard, B. Iochum, E. Scoppola, D. Testard, Spectral properties of one-dimensional quasi-crystals, Comm. Math. Phys. 125 (1989) 527–543.

[2] S. de Bièvre, F. Germinet, Dynamical localization for the random dimer Schrödinger operator, J. Statist. Phys. 98 (2000) 1135–1148.

[3] J.M. Combes, Connections between quantum dynamics and spectral properties of time-evolution operators, in: W.F. Ames, E.M. Harrel II, J.V. Herod (Eds.), Differential Equations with Applications to Mathematical Physics, Academic Press, Boston, 1993, pp. 59–68.

[4] D. Damanik, Gordon-type arguments in the spectral theory of one-dimensional quasicrystals, in: M. Baake, R.V. Moody (Eds.), Directions in Mathematical Quasicrystals, CRM Monograph Series 13, AMS, Providence, RI, 2000, pp. 277–305.

[5] D. Damanik, R. Killip, D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, III. α-continuity, Comm. Math. Phys. 212 (2000) 191–204.

[6] D. Damanik, D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, II. The Lyapunov exponent, Lett. Math. Phys. 50 (1999) 245–257.

[7] D. Damanik, R. Sims, G. Stolz, Localization for discrete one-dimensional random word models, J. Funct. Anal. 208 (2004) 423–445.

[8] D. Damanik, S. Tcheremchantsev, Power-law bounds on transfer matrices and quantum dynamics in one dimension, Comm. Math. Phys. 236 (2003) 513–534.

[9] C.R. de Oliveira, G.Q. Pellegrino, Delocalization in the prime Schrödinger operator, J. Phys. A 34 (2001) L239–L243.

[10] R. del Rio, S. Jitomirskaya, Y. Last, B. Simon, Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations and localization, J. Anal. Math. 69 (1996) 153–200.

[11] D.H. Dunlap, H.-L. Wu, P.W. Phillips, Absence of localization in a random-dimer model, Phys. Rev. Lett. 65 (1990) 88–91.
[12] D.J. Gilbert, D.B. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, J. Math. Anal. Appl. 128 (1987) 30–56.
[13] I. Guarneri, Spectral properties of quantum diffusion on discrete lattices, Europhys. Lett. 10 (1989) 95–100.
[14] B. Iochum, L. Raymond, D. Testard, Resistance of one-dimensional quasicrystals, Physica A 187 (1992) 353–368.
[15] S. Jitomirskaya, Y. Last, Power-law subordinacy and singular spectra, I. Half-line operators, Acta Math. 183 (1999) 171–189.
[16] S. Jitomirskaya, Y. Last, Power-law subordinacy and singular spectra, II. Line operators, Comm. Math. Phys. 211 (2000) 643–658.
[17] S. Jitomirskaya, H. Schulz-Baldes, G. Stolz, Delocalization in random polymer models, Comm. Math. Phys. 233 (2003) 27–48.
[18] A.Ya. Khinchin, Continued Fractions, Dover Publications, Mineola, 1997.
[19] R. Killip, A. Kiselev, Y. Last, Dynamical upper bounds on wavepacket spreading, Amer. J. Math. 125 (2003) 1165–1198.
[20] A. Kiselev, Y. Last, B. Simon, Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators, Comm. Math. Phys. 194 (1998) 1–45.
[21] A. Kiselev, C. Remling, B. Simon, Effective perturbation methods for one-dimensional Schrödinger operators, J. Differential Equations 151 (1999) 290–312.
[22] M. Kolàř, M.K. Ali, One-dimensional generalized Fibonacci tilings, Phys. Rev. B 41 (1990) 7108–7112.
[23] H. Kunz, R. Livi, A. Sütő, Cantor spectrum and singular continuity for a hierarchical Hamiltonian, Comm. Math. Phys. 122 (1989) 643–679.
[24] Y. Last, Quantum dynamics and decompositions of singular continuous spectra, J. Funct. Anal. 142 (1996) 406–445.
[25] Q.-H. Liu, Z.-Y. Wen, Hausdorff dimension of spectrum of one-dimensional Schrödinger operator with Sturmian potentials, Potential Anal. 20 (2004) 33–59.
[26] R. Livi, A. Maritan, S. Ruffo, The spectrum of a one-dimensional hierarchical model, J. Statist. Phys. 52 (1988) 595–608.
[27] L. Raymond, A constructive gap labelling for the discrete Schrödinger operator on a quasiperiodic chain, preprint, 1997.
[28] C. Remling, A probabilistic approach to one-dimensional Schrödinger operators with sparse potentials, Comm. Math. Phys. 185 (1997) 313–323.
[29] C. Remling, Sparse potentials spectral type, Millennium-prize problems, author's homepage: http://www.mathematik.uni-osnabrueck.de/staff/phpages/remlingc/mill/allg.htm.
[30] C.S. Ryu, I. Kim, G.Y. Oh, M.H. Lee, Localized and extended states in a deterministically aperiodic chain, Phys. Rev. B 49 (1994) 14991–14995.
[31] T. Schneider, D. Wurtz, A. Politi, M. Zanetti, Schrödinger problem for hierarchical heterostructures, Phys. Rev. B 36 (1987) 1789–1792.
[32] M. Severin, M. Dulea, R. Riklund, Periodic and quasiperiodic wavefunctions in a class of one-dimensional quasicrystals: an analytical treatment, J. Phys.: Condens. Matter 1 (1989) 8851–9958.
[33] A. Sütő, Schrödinger difference equation with deterministic ergodic potentials, in: F. Axel, D. Gratias (Eds.), Beyond Quasicrystals (Les Houches, 1994), Springer, Berlin, 1995, pp. 481–549.
[34] R.B. Tao, Extended states in aperiodic systems, J. Phys. A 27 (1994) 5069–5077.
[35] A. Zlatoš, Sparse potentials with fractional Hausdorff dimension, J. Funct. Anal. 207 (2004) 216–252.