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A note on p-adic q-ζ-functions II

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ABSTRACT

We show that p-adic q-ζ-function constructed by Koblitz [7] (see also Dąbrowski [4]) can be obtained as Γ-transform of some p-adic measure coming from Lubin–Tate formal group.

1. Introduction

Let p be odd rational prime. Let $\Theta_p : \mathbb{Z}_p \to \mathbb{Z}_p$ be the function defined by $\Theta_p(x) := \lim_{n\to\infty} x^{p^n}$ (p-adic limit). It is well known (Kubota–Leopoldt) that the function

$$L_p : k \mapsto \frac{1}{k} \lim_{r\to\infty} \frac{1}{p^r} \sum_{n=1}^{p^r} \left(\frac{n}{\Theta_p(n)}\right)^k, \quad k = 1, 2, \ldots,$$

can be continued to a meromorphic function on $\mathbb{Z}_p$ (the Kubota–Leopoldt p-adic zeta function) so that $L_p(k) = (1 - p^{k-1})\zeta(1-k)$ for $k \equiv 0 \pmod{p-1}$.

Note that a similar construction can be fulfilled in more general situations, especially for some Dirichlet series of type $\sum_n f'(n) f(n)^{-s}$, with $f(x) \in \overline{\mathbb{Q}}[x]$ (see [2]).

It is well known (due essentially to Iwasawa) that the Kubota–Leopoldt p-adic zeta function can be constructed as Γ-transform of some p-adic measure coming from Lubin–Tate formal group.

In order to explain many well-known p-adic interpolation functions in a unified manner, Shiratani and Imada [10] introduced the numbers $B_n(F, h)$ by

$$\frac{Xh'(e_F(X))}{\lambda'(e_F(X))h(e_F(X))} = \sum_{n=0}^{\infty} \frac{B_n(F,h)}{n!} X^n,$$

and constructed a p-adic interpolating function $\zeta_p(s, F, h)$. Here $F(X,Y) \in \mathbb{Z}_p[[X,Y]]$ is any Lubin–Tate formal group, and $\lambda_F(X)$ and $e_F(X)$ denote the logarithmic series and the exponential series of $F(X,Y)$ with $\lambda_F'(0) = e_F'(0) = 1$. Also, $h(X) \in \mathcal{O}((X))^\times$ denotes any meromorphic series with coefficients in the ring of integers of $\mathbb{C}_p$, with some unit as the leading term. In the special case of the formal multiplicative group $F(X,Y) = (X+1)(Y+1) - 1$, and if $h(X) = X$, we obtain $B_n(F,h) = B_n$ and the p-adic interpolating function coincides with the Kubota–Leopoldt p-adic zeta function. By choosing

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$F(X,Y)$ and $h(X)$ suitably, we can obtain the p-adic zeta function associated with elliptic curve with complex multiplication with good ordinary reduction at p.

In this note we show that p-adic q-ζ-function constructed by Koblitz [7] and Dąbrowski [4] can be obtained from [10, Theorem 9] by taking $F(X,Y)=(X+1)(Y+1)-1$ and choosing $h(X)$ suitably.

2. Results

Fix $q\in\mathbb{R}$, $0<q<1$. Let $f_q(x)=(q^{-x}-1)/(1-q)$ and define, for $\mathrm{Re}(s)>1$ the q-ζ function $\zeta_q(s)=\sum_{n=1}^{\infty}q^{-n}f_q(n)^{-s}$.

Proposition 1. *(See [11].) The function $\zeta_q(s)$ can be analytically continued to the whole complex plane, except for a simple pole at $s=1$.*

Consider the q-Bernoulli numbers $B_k(q)$ defined by

$$B_0(q)=\frac{q-1}{\log q},\quad \text{and}\quad \big(qB(q)+1\big)^k-B_k(q)=\begin{cases}1 & \text{if } k=1,\\ 0 & \text{if } k>1.\end{cases}$$

Note that $\lim_{q\to 1}B_k(q)=B_k$, the usual kth Bernoulli number.

Proposition 2. *(See [11].) If $k\geqslant 1$, then $\zeta_q(1-k)=\frac{(-1)^{k-1}}{k}B_k(q)$.*

Now fix an odd prime number p and assume, for simplicity, that $q\in\mathbb{Q}$, $0<q<1$, $|1-q|_p<1$. We will show that there exists continuous p-adic $\mathbb{C}_p$-valued function on $\mathbb{Z}_p$ which interpolates the values $\zeta_q(1-k)$, $k=1,2,\ldots$. More precisely, we prove the following result.

Theorem 1. *There exists a locally analytic function $\zeta_{p,q}(s)$ on $\mathbb{Z}_p$ such that for any positive integer $m\equiv 0 \pmod{p-1}$ we have $\zeta_{p,q}(1-m)=(1-p^{m-1})\frac{B_m(q)}{m}$.*

Proof. We have to determine a meromorphic series $h\in\mathcal{O}((X))^{\times}$ satisfying $B_k(F,h)=B_k(q)$ (for all $k\geqslant 1$) and $N_F h=h$, where N_F denotes the Coleman norm operator.

The generating function for the numbers $B_k(q)$ is determined as the solution of the q-difference equation $F_q(t)=e^tF_q(qt)-t$. More precisely, we have the following result [11, Lemma 2].

Lemma 1. *We have $F_q(t)=t\sum_{n=0}^{\infty}q^{-n}e^{-f_q(n)t}$.*

Consider the series $G_q(t):=\frac{F_q(t)}{t}$. It is easy to check that $h(u):=e^{\int\frac{G_q(\log(1+u))}{1+u}\,du}$ satisfies the equation

$$\frac{te^th'(e^t-1)}{h(e^t-1)}=F_q(t),$$

and hence $B_k(F,h)=B_k(q)$ for $F(X,Y)=(X+1)(Y+1)-1$ and all positive integers k. Here we assume that 'the constant of integration' is zero.

We have to check that $N_F(h)=h$. By the construction of the operator N_F (see, for instance [3, Section 2.2]) it is enough to check the following identity:

$$\prod_{\zeta\in\mu_p}h\big(\zeta(1+T)-1\big)=h\big((1+T)^p-1\big).$$

Let $H(u):=\int\frac{G_q(\log(1+u))}{1+u}\,du$. It is plain that

$$H(u)=\log(1+u)+\log_{(q)}\left(1-\frac{1}{1+u}\right),$$

where

$$\log_{(q)}(1-T):=\sum_{n=1}^{\infty}\frac{q-1}{1-q^n}T^{\frac{q^{-n}-1}{1-q}}.$$

Here $T^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}(T-1)^k$. Note that $\lim_{q\to 1}\log_{(q)}(1-T)=\log(1-T)$.

We have the following useful result (equality of formal power series in X and Y):

Lemma 2. *We have*

$$\log_{(q)}\big((1-X)(1-Y)\big)=\log_{(q)}(1-X)+\log_{(q)}(1-Y).$$

Proof. Denote $a_n=\frac{q^{-n}-1}{1-q}$. We have to check the identity

$$\sum_{k=0}^{\infty}\binom{a_n}{k}\sum_{k_1+\cdots+k_4=k}\binom{k}{k_1,k_2,k_3,k_4}X^{k_1}Y^{k_2}(-XY)^{k_3}(-1)^{k_4}=X^{a_n}+Y^{a_n},$$

where the inner sum is over all integers $k_i\geqslant 0$ satisfying $k_1+\cdots+k_4=k$. The LHS is, of course, of the shape $\sum_{\alpha,\beta}a_{\alpha\beta}X^{\alpha}Y^{\beta}$, where α,β run over all non-negative integers. Taking $k_2=k_3=0$, we obtain

$$\sum_{k=0}^{\infty}\binom{a_n}{k}\sum\binom{k}{k_1,k_4}X^{k_1}(-1)^{k_4}=\sum_{k=0}^{\infty}\binom{a_n}{k}(X-1)^k=X^{a_n}.$$

Taking $k_1=k_3=0$, we obtain

$$\sum_{k=0}^{\infty}\binom{a_n}{k}\sum\binom{k}{k_2,k_4}Y^{k_2}(-1)^{k_4}=\sum_{k=0}^{\infty}\binom{a_n}{k}(Y-1)^k=Y^{a_n}.$$

In the remaining cases, the LHS gives no contribution (i.e. $a_{\alpha\beta}=0$ if $\alpha\beta>0$). To see this one can rewrite the LHS as a linear combination of special hypergeometric series and use their properties. Let us check that $a_{11}=0$. In this case

$$\sum_{k=0}^{\infty}(-1)^k\binom{k}{0,0,1,k-1}\binom{a_n}{k}=F(-a_n,2)-F(-a_n,1)$$

and

$$\sum_{k=0}^{\infty}(-1)^k\binom{k}{1,1,0,k-2}\binom{a_n}{k}=F(-a_n,3)-4F(-a_n,2)-6F(-a_n,1),$$

where we abbreviate $F(-a_n,m):=F(-a_n,m,1,1)$. Using [1, Section 2.8, formula (2.9)], we easily obtain $F(-a_n,m,1,1)=0$ for $m=1,2,3$. The general case is proved under the same lines. We omit the details. □

From Lemma 2 we easily deduce the identity

$$\sum_{\zeta\in\mu_p}H\big(\zeta(1+T)-1\big)=H\big((1+T)^p-1\big),$$

which implies $N_F(h)=h$. Applying Theorem 9 from [10], we are done. □

Remarks. (i) It would be interesting to use [10, Theorem 9] to prove variants of Theorem 1 for poly-Bernoulli numbers (and q-variants). The same for the numbers $C_r(i)$ defined in [5]. (ii) It would be interesting to interpret in this way other types of q-ζ functions (see, for instance, [6] or [8] and references therein). (iii) In [9], the author views the q-Bernoulli numbers in terms of the group which addition of X and Y is given by $X+Y+(q-1)XY$.

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